On Aggregate Control of Clustered Consensus Networks

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Abstract—We address a consensus control problem for networks that have multiple dense areas with sparse interconnection structure. The sparsity pattern in such networks naturally gives rise to a time-scale separation in its dynamics, whereby nodes inside an area synchronize over a fast time-scale while the areas themselves synchronize over a slow time-scale. Our goal is to design state-feedback controllers at the network nodes that predominantly shape the closed-loop response of the slow dynamics. The specific design objective in this case is posed as maximizing the speed of convergence of the slow dynamics. A sparsity-promoting graph design problem is formulated for achieving this purpose. A critical observation is that every area-coordinator needs to design only one aggregate control law to satisfy this objective for the slow system. Applying results from singular perturbation theory, we show that when these individual controllers are implemented on the actual network model, the closed-loop response is close to that obtained from the approximate models, provided that the clustering is strong. The design procedure is demonstrated by a simulation example.

Index Terms—Large-scale systems; Singular perturbation; Consensus networks; Sparse networks; Area aggregation

I. INTRODUCTION

Using simple models that sufficiently capture the different dynamics of the system has proven to be beneficial for the purpose of analysis and control. This is especially true for large-scale networks with complex dynamics. One of the fundamental tools for model-order reduction of such networks is singular perturbation theory that exploits the time-scale separation properties of the network models arising due to clustering. Agents inside a cluster are tightly connected while the clusters themselves are sparsely connected. Examples of such networks include small-world networks [1], wireless sensor networks [2] and power systems networks [3]. An early result that developed linear singular perturbation models for these systems appeared in [4], and recently for the nonlinear counterpart in [5].

The majority of the results that pertain to these types of systems, however, address model reduction of open-loop systems. Very limited efforts have been spent to study how one can take advantage of the time separation properties of these large networks in the design of closed-loop control systems. In this paper, we address this problem in the context of control of clustered consensus networks. Our main goal is to design state-feedback controllers to shape the closed-loop response of the slow dynamics. This objective can be easily achieved by a simple centralized pole placement type design. We aim, however, to achieve the same goal by decoupling the control design for the slow and fast dynamics. This can be achieved by distributing the design to different area-level coordinators, each of which employs a simplified model of the system. We follow a multi-area modeling approach, first proposed in [6], to approximately represent the system through different relatively simple models that capture the dynamics of the network from the prospective of each clustered area. We then follow a methodology based on singular perturbation theory to design a composite control in both the slow and fast time-scales to approximately assign desired eigenvalues to both the slow and fast dynamics. Accordingly, the control design problem for each area is formulated as solving two sub-problems; namely, a slow sub-problem and a fast sub-problem. The slow sub-problem is solved using a graph-theoretic approach combined with employing tools from convex optimization literature. The controller is ensured to promote sparsity in the feedback gain matrix so as to reduce communication overhead. Related literature to this result, although in a different context, is reported in [7], [8]. An important contribution for this part of the design is that every coordinator only needs to design a single control law for each agent inside an area, irrespective of the number of agents. This reduces the computational effort for the control design, and scales well with the increasing network size as long as the number of areas remain fairly constant. The fast sub-problem is, thereafter, solved by a high-gain feedback that is consistent with the slow pole placement. Thanks to this multi-area modeling approach, the problem, thus, reduces to a state feedback control for a number of totally-uncoupled reduced-order linear systems associated with each area. The control design for the slow sub-problem, however, still needs cooperation of all the coordinators. Finally, we prove that by combining the solutions of the two sub-problems in one composite control and applying it to the actual model, we achieve a closed-loop performance that is close to the desired response provided that the clustering is strong.

Our work compliments the control designs reported in the consensus literature [9], [10], by showing how singular perturbation theory can not only be employed for open-loop analysis of consensus but also for its closed-loop control, especially aggregate control of its slow time-scale dynamics.

The remainder of the paper is organized as follows. Section II describes the mathematical model of clustered networks and includes the problem statement. Sections III, IV and V present the main result of the paper, where Section III presents the general structure of the control strategy and Sections IV and V describe in details the design procedure.
Section VI includes a simulation example that demonstrates the design procedure. Finally, Section VII includes some concluding remarks.

II. MODELING AND PROBLEM STATEMENT

A. System Modeling

In this work, we consider a network of \( n \) agents that has \( r \) internally dense and sparsely connected areas. The dynamics of each agent is given by

\[
\dot{x}_i^\alpha = \sum_{\beta=1}^{m_\alpha} a_{ij}^\beta (x_j^\beta - x_i^\alpha) + b_{ia} u_{ai}, \quad \text{for } i = 1, \ldots, m_\alpha, \tag{1}
\]

where \( \alpha = 1, 2, \ldots, r \) and \( \beta \neq \alpha \) when \( j = i \). \( x_i^\alpha \) is the state of the \( \beta \)-th agent in area \( \alpha \) and \( b_{ia} \neq 0 \). Furthermore, \( a_{ij}^\beta = 1 \) if node \( i \) in area \( \alpha \) and node \( j \) in area \( \beta \) are connected, \( a_{ij}^\beta = 0 \) otherwise. Equivalently, system (1) can be represented in a matrix form as

\[
x = Ax + Bu, \tag{2}
\]

where \( x \in \mathbb{R}^n \) is the vector of agent states, \( u \in \mathbb{R}^n \) is the control input, \( B = \text{diag}(b_{11}, \ldots, b_{m_1 \times 1}, \ldots, b_{m_r \times 1}) \), \( m_\alpha \) is the number of agents in area \( \alpha \) and \( M = \text{diag}(m_1, m_2, \ldots, m_r) \). The \( n \times n \) matrix \( A = A^I + A^E \) represents the Laplacian of the network, where \( A^I \) is a block diagonal matrix that represents the internal connections and \( A^E \) is a sparse matrix that represents the external connections.

System (2) can represent swarms of robotic agents or groups of biological systems. We also envision that the results in this paper can be extended in a straightforward manner to multi-dimensional agents.

It is shown in [4] that systems that have dense and sparse connections exhibit two time-scale properties. For these systems, the states within each dense area move in a fast time-scale relative to the states that represent the aggregate motion of the areas. More specifically and following the development in [4], we define the slow (aggregate) state \( w^\alpha \) and the fast state \( x^\alpha \) as the average of the states and the relative motion of each state to a reference state in the area \( \alpha \), respectively. Accordingly, we use the transformation

\[
\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} C \\ G \end{bmatrix} x, \tag{3}
\]

where \( C = M_a^{-1} U^T, M_a = \text{diag}(m_{a1}, m_{a2}, \ldots, m_{ar}), \) \( U = \text{diag}(U_1, U_2, \ldots, U_r), U_{ai} = 1 \) (\( m_{ai} > 1 \)), where \( 1 \) is a column vector of all ones, and \( G = \text{diag}(G_1, G_2, \ldots, G_r) \), where

\[
G_{\alpha} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 1 \end{bmatrix} (|m_{ai} - 1| \times m_{ar}). \tag{4}
\]

The inverse of transformation (3) is given by

\[
x = [U G^+] [w \ z]^T, \tag{5}
\]

where \( G^+ = G^T (GG^T)^{-1} = \text{diag}(G_1^+, G_2^+, \ldots, G_r^+) \). Applying the transformation (3) to (2) we get

\[
\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u, \tag{6}
\]

where

\[
\begin{align*}
\bar{A}_{11} &= CA^E U, & \bar{A}_{12} &= CA^E G^+, & \bar{A}_{21} &= GA^E U, \\
\bar{A}_{22} &= G(A^I + A^E) G^+ & \bar{B}_1 &= CB & \bar{B}_2 &= GB.
\end{align*}
\]

It is shown in [4] (see also [11]) that the \( \infty \)-norm of the matrices in (7) satisfy

\[
||\bar{A}_{11}|| \in O(c^I \delta), \quad ||\bar{A}_{12}|| \in O(c^E \delta), \quad ||\bar{A}_{21}|| \in O(\beta \delta) \quad \text{and} \quad ||\bar{A}_{22}|| \in O(c^I) \tag{7}
\]

where \( d = c^E / c^I \), \( c^I = \min\{c_{ai}^I\} \), \( c^E = \max\{c_{ai}^E\} \), \( c_{ai}^I \) and \( c_{ai}^E \) denote the number of internal and external connections of node \( i \) in area \( \alpha \), respectively, \( \delta = \bar{c}^E / (\bar{c}^I) \), \( \bar{c}^E = \max\{\bar{c}_{ai}^E\} \), \( \bar{c}_{ai}^E \) denotes the total number of external connections in area \( \alpha \), and \( m \) is the minimum number of nodes of all areas. We can also show that

\[
||\bar{B}_1|| = ||CB|| \leq ||C|| ||B|| \leq 1 / m ||B|| = ||B|| \leq \bar{c}^E \delta \leq \bar{c}^E b \in O(c^I \delta)
\]

where, by assumption, \( c^I \geq 1, \bar{c}^E \geq 1 \), and \( ||B|| = b \) for some positive number \( b \). The area-parameter \( \delta \) and the node-parameter \( d \) are assumed to satisfy \( 0 \leq \delta \ll 1 \) and \( 0 < d \ll 1 \cdot \)

We now use the scaling

\[
A_{11} = \bar{A}_{11} / c^I \delta, \quad A_{12} = \bar{A}_{12} / c^I \delta, \quad A_{21} = \bar{A}_{21} / c^E \delta, \quad A_{22} = \bar{A}_{22} / c^I \delta,
\]

\[
B_1 = \bar{B}_1 / c^E \delta, \quad B_2 = \bar{B}_2 / c^I \delta,
\]

so that the norms of the above matrices are all \( O(1) \). Using this scaling makes the model (6) singularly perturbed by the parameter \( \delta \) and in a fast time-scale \( t_f \). Consequently, we redefine the time-scale by \( t = \delta c^I t_f \), so the system model takes the standard singular perturbation form

\[
\dot{w} = A_{11} w + A_{12} z + B_1 u, \quad \dot{z} = d A_{21} w + A_{22} z + B_2 u, \tag{8}
\]

where \( w \in \mathbb{R}^r \) and \( z \in \mathbb{R}^{|(a-r) \times 1|} \). It can be shown that the matrix \( A_{22} \) is non-singular [11].

B. Problem Statement

The main objective of our work is to design \( u(t) \) in (8) as a state feedback control using \( w(t) \) and \( z(t) \), specifically to shape the closed-loop dynamics of \( w(t) \). The motivation for this arises from the fact that controllers inside an area that typically employ only local feedback have the highest participation only in the fast variables, while the slow variables do not get influenced much. In this case, we consider the problem of maximizing the rate of convergence of \( w(t) \) to a slow consensus equilibrium.

From the structure of (8), it is clear that the objective can be easily achieved by a simple centralized pole placement type design. Our objective, however, is not to pursue a centralized approach, but rather to achieve the same goal by distributing the design to different area-level coordinators, each of which employs a simplified model of the system. The derivation of this model from the perspective of each area is described next.
C. Multi-Area Representation

Looking closer at the structure of the system matrices of (8), we can see that

\[
A_{12} = [A_{o1} \quad A_{o2} \ldots \quad A_{or}]_{r \times (n-r)},
\]
\[
A_{21} = [A'_{io} \quad A'_{i2} \ldots \quad A'_{ir}]',
\]
\[
A_{22} = A_{22o} + dA_{22d},
\]
\[
A_{22o} = \text{block-diag}(\tilde{A}_{i1}, \ldots, \tilde{A}_{ir})_{(n-r) \times (n-r)},
\]
\[
A_{22d} = \begin{bmatrix}
\tilde{A}_{i1} & \tilde{A}_{i2} & \ldots & \tilde{A}_{ir} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{A}_{r1} & \tilde{A}_{r2} & \ldots & \tilde{A}_{rr}
\end{bmatrix}_{(n-r) \times (n-r)},
\]
\[
B_1 = \text{block-diag}(\tilde{B}_1, \ldots, \tilde{B}_r)_{r \times m}
\]
\[
B_2 = \text{block-diag}(\tilde{B}_{11}, \ldots, \tilde{B}_{rr})_{(n-r) \times m}.
\]

The sizes of the sub-matrices that make up

\[
A_{1i} : r \times (m_{ai} - 1),
\]
\[
A_{io} : (m_{ai} - 1) \times r
\]
\[
\tilde{A}_{ii} : (m_{ai} - 1) \times (m_{ai} - 1),
\]
\[
\tilde{A}_{jj} : (m_{ai} - 1) \times (m_{ai} - 1)
\]
\[
\tilde{B}_{ii} : (m_{ai} - 1) \times m_{ai}.
\]

Considering the structure of the system matrices, we can express (8) in the singular perturbation form

\[
\dot{w} = A_{11}w + \sum_{j=1}^{r} A_{oj}z_j + \sum_{j=1}^{r} B_{oj}u_j, \quad w(0) = w_o, \quad (9)
\]
\[
\dot{z}_i = -A_{ii}z_i + B_{ii}u_i + \delta \dot{z}_i, \quad z_i(0) = z_{io}, \quad (10)
\]

where \(i = 1, \ldots, r\), \(z_i \in \mathbb{R}^{(m_{ai}-1)}\), \(u_i \in \mathbb{R}^{m_{ai}}\), \(\tilde{A}_i = \tilde{A}_{ii} + d\tilde{A}_{ii}\),
\[
B_{oi} = \begin{bmatrix}
B_i \quad 0_2 \\
0_2 \quad 0_2 \\
\vdots & \vdots & \ddots & \vdots \\
0_2 & 0_2 & \ldots & 0_2
\end{bmatrix}_{1 \times m_{ai}}
\]

1 \times m_{ai} row vector of zeros.

It is worth noting that system (9)-(10) is a special case of the one considered in [6], where the fast subsystems are strongly coupled with the slow subsystem and all the fast subsystems have the same singular perturbation parameter \(\delta\) and the same coupling parameter \(d\).

If we now assume that the decision maker in the \(k^{th}\) area neglects the weak coupling parameter, i.e. \(d = 0\) that is associated with \(\tilde{A}_i\) and \(\tilde{A}_{ij}\) for \(i \neq j\), and the fast dynamics of all other subsystems, i.e. \(\delta = 0\) for all \(z_i\) dynamics and \(i \neq j\), but retain the exact model of his own subsystem, then we get the following model of the \(k^{th}\) area

\[
\dot{w} = A_{11}w + \sum_{j=1}^{r} A_{oj}z_j + \sum_{j=1}^{r} B_{oj}u_j, \quad w(0) = w_o, \quad (11)
\]
\[
\dot{z}_k = -A_{kk}z_k + B_{kk}u_k, \quad 0 = \tilde{A}_{i}z_i + B_{ii}u_i, \quad i \neq k, i = 1, \ldots, r \quad (12)
\]

Now, if \(\tilde{A}_i\) is nonsingular, then the substitution of

\[
z_i = -A_i^{-1}B_i u_i
\]

into (11) results in the \(k^{th}\) simplified model

\[
\dot{w}_k = A_{11}w_k + A_{oi}z_k + B_{oi}u_k + \sum_{j \neq k} B_{oj}u_j, \quad w(0) = w_o, \quad (14)
\]
\[
\dot{z}_k = -A_k z_k + B_{kk}u_k, \quad z_k(0) = z_{ko}, \quad (15)
\]

where \(B_{kj} = B_{oj} - A_{oj}\tilde{A}_i^{-1}B_j\) and \(w_k \in \mathbb{R}^r\), \(z_k \in \mathbb{R}^{(m_{ai}-1)}\) and the subscript \(k\) refers to the states being perceived by the \(k^{th}\) coordinator. It is worth mentioning that the reason we considered the matrices \(\tilde{A}_k\) instead of \(\tilde{A}_{kk}\) in the fast subsystems is that they may enhance the system performance especially in the case when \(d\) is not sufficiently small [4].

System (14)-(15) can be considered as an effective model of the entire network from the point of view of the coordinator of the area \(k\). This model is simpler than the original model (2) or the singular perturbation model (8). Primarily, this simplification manifests itself in two points: first, the controller now does not have to consider the effect of the fast dynamics of the other areas. Secondly, for each area \(k\), system (14)-(15) has dimension less than that of (2) by \(n-r-m_{ak} - 1\), which is the total dimension of the fast dynamics of all areas except the \(k^{th}\) area.

III. Multi-Area Control

In this section, we will follow a multi-model approach to design \(r\) unique controllers for \(r\) aggregate areas in order to improve the consensus performance of the network (2). Consider the simplified \(k^{th}\) area system from (14)-(15). Since it clearly exhibits a two-time scale structure, we will use the standard singular perturbation approach so that each coordinator can solve two separate sub-problems for the fast and slow subsystems of (14)-(15). Accordingly, we first need to design a slow control \(u_k\) for the slow sub-problem. To get the slow time-scale model, we set \(\delta = 0\) in (14)-(15). As a result, the reduced model is given by

\[
\dot{w}_s = A_{11}w_s + \sum_{k=1}^{r} B_{ks}u_{ks}, \quad w_s(0) = w_o, \quad (16)
\]
\[
\dot{z}_s = -\tilde{A}_k^{-1}B_{kk}u_k, \quad (17)
\]

where \(B_{kk} = B_{ok} - A_{ok}\tilde{A}_k^{-1}B_{kk}\).

We also need to design a fast control \(u_{kf}\) for the fast subsystem of (14)-(15), which can be obtained by using the time-scale \(\tau = t/\delta\) and then setting \(\delta = 0\) to get

\[
\frac{d z_{kf}}{d\tau} = \tilde{A}_k z_{sf} + B_{kk}u_{kk}, \quad z_{kf}(0) = z_{ks} + \tilde{A}_k^{-1}B_{kk}u_k(0), \quad (18)
\]

where we used the change of variables \(z_{sf} = z_k - z_{ks}\) and \(u_{sf} = u_k - u_{ks}\).

The composite control is designed as

\[
u_k = u_{ks} + u_{kf} = G_{ks}w_s + G_{kf}z_{sf}, \quad (19)
\]

It is important to mention here that a realizable control requires (19) to be expressed in terms of the actual states \(w\) and \(z_k\) instead of the system states \(w_s\) and \(z_{sf}\). This can be achieved by replacing \(w_s\) by \(w\) and \(z_{sf}\) by \(z_k - z_{ks}\), so that the
composite control (19), in view of (17), takes the realizable feedback form \[ u_k = [(I + G_k f \tilde{A}^{-1} G_k w + G_k z) w + G_k z_k]. \quad (20) \]

Since \( u_k \) is designed only from the prospective of the \( k \)th coordinator, we refer to (19) as the decoupled design. The specific approach for designing \( u_k \) and \( u_{kf} \) will be provided in Sections IV and V, respectively.

Let the perturbation parameters \( \delta \) and \( d \) be ordered as components of the vector \( \Omega \) in a set \( \mathbb{R}^2 \). Recall that \( \delta \) is strictly positive parameter while \( d \), for the sake of generality, can be negative, positive or zero. Therefore, the set \( \Omega \) is defined in such a way so that it restricts the values of the perturbation parameters accordingly. We next state a theorem that shows that if \( u_k \) designed by the \( k \)th coordinator, using the approximates models (16) and (18), is implemented in the actual model (9)-(10) then, for small values of \( \Omega \) in \( \mathbb{R} \), the resulted closed-loop eigenvalues are close to those of the decoupled design for \( k = 1, 2, \ldots, r \).

**Theorem 1:** If the slow pairs \((A_{11}, B_{kk})\) and the fast system pairs \((\bar{A}_k, B_{kk})\) for \( k = 1, 2, \ldots, r \) are each controllable, and \( G_k \) and \( G_{kf} \) are designed to assign distinct (stable) eigenvalues \( \lambda_{i}, i = 1, \ldots, r \) and \( \lambda_{j}, j = 1, \ldots, m_k - 1 \), to the matrices \( A_{11} + B_{kk}G_k \) and \( \bar{A}_k + B_{kk}G_{kf} \) respectively, then there exists a positive scalar \( \sigma \) such that, for all \( \Omega \in \mathbb{R} \), \( 0 < \| \Omega \| \leq \sigma \), the closed-loop system has \( r \) small eigenvalues \( \{ \lambda_{1}', \lambda_{2}', \ldots, \lambda_{n}' \} \) and \( n - r \) large eigenvalues \( \{ \lambda_{r+1}', \ldots, \lambda_{n}' \} \), which are approximated by

\[
\begin{align*}
\lambda_{i}' &= \lambda_{i}(A_{11} + \sum_{k=1}^{r} B_{kk}G_k + O(\| \Omega \|)), \quad i = 1, \ldots, r, \\
\lambda_{j}' &= |\lambda_{j}(\bar{A} + B_{kk}G_k) + O(\| \Omega \|)|/\delta, \quad i = r + j, \quad j = 1, \ldots, m_k - 1, \\
&\vdots \\
\lambda_{l}' &= |\lambda_{l}(\bar{A} + B_{kk}G_{kf}) + O(\| \Omega \|)|/\delta, \quad i = r + m_k - 1 + l - 1, \quad l = 1, \ldots, m_k - 1.
\end{align*}
\]

**Proof:** See Appendix.

According to Theorem 1, we can follow a two time-scale design procedure to get an \( O(\epsilon) \) approximate eigenvalues assignment of the singularly perturbed linear system (9)-(10). In more details, we can design \( G_k \) for \( k = 1, \ldots, r \), to place the \( r \) small eigenvalues of \( A_{11} + B_{kk}G_k \) separately, we can design each \( G_{kf} \) to place the \( m_k - 1 \) large eigenvalues of \( \bar{A}_k + B_{kk}G_{kf} \); then we can use the composite feedback control (20).

**IV. SLOW SUB-PROBLEM CONTROL**

**A. Aggregate Area Control**

The fact that the fast states reach consensus in a faster time relative to the reduced system indicates that all the agents within each area reach consensus relatively quickly. This motivates us to design the slow controls for each agent in each area to be the same. This means we set

\[ b_{kk} u_{ki} = \bar{u}_k, \quad \forall \ k = 1, \ldots, r, \quad \text{and} \ q = 1, \ldots, m_k. \quad (21) \]

Recall that the slow control input vector for each area \( k \) is \( u_k = [u_{k1} u_{k2} \ldots u_{km_k}]' \) and notice that \( B_{kk} = G_k R_k \), where \( R_k = \text{diag}(b_{1k}, \ldots, b_{mk}) \) and \( G_k \) is defined in (4). Now using (21), we get \( b_{kk} u_{ki} = b_{kk} \bar{u}_k = \ldots = b_{km_k} \bar{u}_k = \bar{u}_k \). This places \( u_k \) in the null space of \( B_{kk} \), i.e. \( B_{kk} u_k = 0 \). Based on the previous argument we can consider a single (aggregate) control input for each area. This leads to

\[ B_{ok} u_k - A_{ok} \bar{A}_{kk} B_{kk} u_k = B_{ok} \bar{u}_k. \]

The summation of \( B_{ok} u_k \), for \( k = 1, \ldots, r \), represents a vector whose elements represent the average of all the controls within each area. Therefore, considering (21), system (16) can now be given by

\[ \dot{\bar{w}}_s = A_{11} \bar{w}_s + B_{sa} \bar{u}_s, \quad (22) \]

where \( \bar{w}_s = [\bar{u}_{i1} \bar{u}_{i2} \ldots \bar{u}_{ir}]' \) and \( B_{sa} \) is a \( r \times r \) identity matrix, which can be significantly smaller in size than \( B_{kk} \) described in (16).

**Remark 1:** The result of Theorem 1 is applicable if the slow sub-problem (22) is controlled through the aggregate control \( u_s \).

Remark 1 suggests that using the relatively simple model (22) to design a control \( u_s \) is equivalent to designing a control for the slow subsystem (16). We then can use (21) to obtain the implementable control. Notice that because \( u_k \) is now in the null space of \( B_{kk} \), the composite control (20) becomes

\[ u_k = G_k \bar{w}_s + G_{kf} z_k. \quad (23) \]

**B. Sparsity-Promoting Control Design**

In this part of the work we will adopt a graph theoretic approach and utilize convex optimization techniques to achieve the desired goal of enhancing the consensus performance of the inter-area dynamics with the controller using as few communication links as possible.

We begin by first noticing that indeed the aggregate model (22) can be represented by a connected (undirected) graph \( G = (V, E) \), where \( V \) is a collection of nodes and \( E \) is the corresponding set of edges. This graph has a weighted Laplacian matrix \( L = -A_{11} = N^{-1} E W E^T \), where \( E \) is the incidence matrix \(^1\), \( N \succ 0 \) is the diagonal node weighting matrix and \( W \succeq 0 \) is the diagonal edge weighting matrix. It is worth mentioning that for sparse networks, it is typical that unlike the overall graph of the network, the aggregate graph can be a weighted one \(^4\).

Define the slow state vector as \( \bar{w}_s = [\bar{w}_{s1} \bar{w}_{s2} \ldots \bar{w}_{sr}]' \) and consider a control input for each aggregate-node as

\[ \bar{u}_{ks} = -\sum_{j \in Vk} K_{k(j)}(\bar{w}_{ks} - \bar{w}_{js}) = -K_k \bar{w}_s, \quad k = 1, \ldots, r. \quad (24) \]

where \( G_k \) denotes the neighbors of node \( k \) and \( K_k \) is a row vector of dimension \( 1 \times r \). Applying the control (24) to the aggregate model (22) leads to the closed-loop system

\[ \dot{\bar{w}}_s = -(L + K) \bar{w}_s \equiv -\bar{L} \bar{w}_s, \quad (25) \]

\(^1\) The Incidence Matrix \( E \) is an \( r \times n \) matrix, where \( r \) is the number of nodes and \( h \) is the number of edges, with each \( s \) column, \( s = 1 \ldots h \), represents an edge of the graph linking nodes \( v_i \) and \( v_j \) in \( V \), with \( [E]_{is} = 1, [E]_{js} = -1 \), and \( E_{ls} = 0 \), for \( l \neq i, j \).
where $K = [K_1' \ldots K_r']'$ is the control gain matrix we need to design.

It is a well known fact that the eigenvalues of a Laplacian matrix $L$ for a connected graph satisfy $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$. It is also known that enlarging the second smallest eigenvalue of the Laplacian matrix (also known as algebraic connectivity) increases the speed of the network consensus [9]. Consequently, to enhance the performance of the closed-loop system, we need to design the control matrix $K$ such that the closed loop matrix $\bar{L} = N^{-1}EWE^T$ is a weighted Laplacian, where $\bar{W} \succeq 0$ is the corresponding diagonal edge weighting matrix, with $L$ having some desired positive eigenvalues $(\lambda_{1sd}, \lambda_{2sd}, \ldots, \lambda_{rsd})$ that satisfy $0 = \lambda_{1sd} < \lambda_2 < \lambda_{2sd} \leq \cdots \leq \lambda_{rsd}$. Equivalently, this problem can be formulated as how to determine the edge weights $\bar{W}$ to maximize $\lambda_{2sd}$ starting from an initial value $\bar{L} = L$. Towards this end, we can maximize the weighted algebraic connectivity of the graph by following the convex program [13]

$$\begin{align*}
& \text{maximize} \quad \mu, \bar{W} \\
& \text{subject to} \quad Q^T(EWE^T - \mu N)Q \succeq 0 \quad (26)
\end{align*}$$

where $0 < \varepsilon < 1$ and $Q \in \mathbb{R}^{\ell \times (\ell-1)}$ is a full rank matrix whose columns are orthogonal.

We now would like to design a diagonal and sparse $K$ while simultaneously solve the above optimization problem. To do that, we first notice from (22) and the definition of the aggregate Laplacian matrix that the closed-loop system can be written as

$$N\bar{w}_f = -[EWE' + NK]w_k. \quad (27)$$

Clearly, (27) represents the aggregate system with the corresponding aggregate inertia. Therefore, making the diagonal matrix $NK$ sparse implies that we are minimizing the number of required communication links between the areas. Towards solving this problem we observe that:

$$NK = EWE^T - EW E^T = E(\bar{W} - W)E^T \quad (28)$$

Therefore, the sparsification of $NK$ is equivalent to

$$\begin{align*}
& \text{minimize} \quad \sum_k \beta_k |\bar{w}_k - w_k| \\
& \text{subject to} \quad Q^T(EWE^T - \mu N)Q \succeq 0 \quad (29)
\end{align*}$$

where $\beta_k$ is a weighting factor and $\bar{w}_k$ and $w_k$ are the elements of the diagonal matrices $\bar{W}$ and $W$, respectively. Statement (29) indicates that we can achieve sparsity for $NK$ by maximizing the number of zeros elements in the matrix $\bar{W} - W$. It can be shown that this can be accomplished by considering a re-weighted $\ell_1$-minimization of $||\bar{W} - W||_1$ [14], [15]. This is a convex problem, and hence, can be easily solved by linear programming. A certain a-priori choice of the $\ell_1$-weights can be chosen to force the solution towards some desired graph topology. More specifically, assigning large weights to certain edges means that these edges are not desirable, whereas small weights promote the existence of those edges.

By drawing inspiration from [14] and recalling that $w_k$ are given, we can combine the two objective functions (26) and (29) into a convex sum

$$\begin{align*}
& \text{minimize} \quad (1 - \alpha) \sum_k \beta_k |\bar{w}_k - w_k| - \alpha \mu \\
& \text{subject to} \quad Q^T(EWE^T - \mu N)Q \succeq 0 \quad (30)
\end{align*}$$

where the weighting factor $\alpha \in [0, 1]$ is a tuning parameter for the relative emphasis on each term in the objective function.

V. Fast Sub-problem Control

Once the slow sub-problem is solved as described in Section IV, we would know the maximum desired eigenvalue $\lambda_{rds}$ of the new closed-loop slow subsystem. Consequently, the fast sub-problem for the $k^{th}$ area now becomes how to design $u_{kf}$ so that the closed-loop fast subsystem has some desired eigenvalues, which are faster than those of the closed-loop slow subsystem. More precisely, we need to design $G_{kf}$ for (18) so that the closed-loop matrix $A_k + B_{wk}G_{kf}$ has eigenvalues $\lambda_{kd1}, \lambda_{kd2}, \ldots, \lambda_{kfd(m_k-1)}$, which satisfy

$$\sigma \times \lambda_{rds} \leq \lambda_{kd1} \leq \lambda_{kd2} \leq \cdots \leq \lambda_{kfd(m_k-1)},$$

where $\sigma > 1$ is some desired number that is chosen so that we maintain the gap between the slow and fast eigenvalues of the closed-loop system. Finally, we combine the slow and fast controls using (23) to finish the design procedure.

VI. Simulation Example

We consider a network of twenty agents distributed over four dense areas as shown in Fig. 1(a). Each agent has a control input and unity inertia. The structure of the network is such that $d = 1/3, \delta = 1/4$. We will start the design procedure by considering the system model in the form (6). The open-loop eigenvalues of the system are $(-0.261, -0.599, -0.701, -2.645, -3.274, -3.382, -4.183, -4.407, -4.643, -5, -5, -5, -5.243, -5.618, -5.683, -5.853).$
The open-loop eigenvalues clearly show that the system exhibit two time-scale structure, with four slow eigenvalues and sixteen fast eigenvalues. The goal is to enhance the speed of the network consensus. To accomplish this goal, we will follow the design procedure described in Section III. More specifically, we will first design a slow control by solving (30). This will result in four slow controls $\bar{u}_{s1}, \bar{u}_{s2}, \bar{u}_{s3}$ and $\bar{u}_{s4}$ for the four aggregate areas that share sparse feedback network topology and collectively maximize the algebraic connectivity of the graph. The aggregate controls are related to the individual control inputs of the agents by

\[
\begin{align*}
\bar{u}_{s1} &= u_{1i}, \ i = 1, \ldots , 5, \\
\bar{u}_{s2} &= u_{2i}, \ i = 1, \ldots , 5 \\
\bar{u}_{s3} &= u_{3i}, \ i = 1, \ldots , 4, \\
\bar{u}_{s4} &= u_{4i}, \ i = 1, \ldots , 6.
\end{align*}
\]

We solve the convex optimization problem for this example by using the software program CVX [16]. For the purpose of illustration, we choose to encourage eliminating the link between areas I and III. This was done by choosing the $\ell_1$-weights as $\bar{\beta}_2 = 10000, \bar{\beta}_1 = \bar{\beta}_3 = \bar{\beta}_5 = 1$ and $\alpha = 0.8, \varepsilon = 0.01$. As a result, solving (30) results in the closed-loop eigenvalues $(0, -36.405, -44.935, -82.593)$. The corresponding aggregate control gain matrix is

\[
NK = \begin{bmatrix}
197.3342 & -98.6667 & -0.0009 & -98.6667 \\
-98.6667 & 197.3342 & -98.6667 & 0 \\
-0.0009 & -98.6667 & 197.3342 & -98.6667 \\
98.6667 & 0 & -98.6667 & 197.3342
\end{bmatrix}.
\]

Notice the resulted weak link between nodes 1 and 3. This link can be eliminated with minimal affect to the resulted eigenvalues. We can do that by zeroing the $(1,3)$ and $(3,1)$ entries of the $NK$ matrix and adding $-0.0009$ to the $(1,1)$ and $(3,3)$ entries, respectively.

Furthermore, we observer from Fig. 1(c) that the connectivity of the graph will not be lost if we eliminate another link between any two neighboring areas. However, by doing that we expect the observability of the graph to be weakened, and hence, the performance will not be as good as in the case of eliminating only one link. To examine this point further, we choose to also eliminate the link between the areas II and III. We can do that by changing the previous value of $\bar{\beta}_4$ to $\bar{\beta}_4 = 10000$ and solving (30) in a similar way as before. The resulted closed-loop slow eigenvalues are $(0, -0.615, -1.329, -1.408)$. Comparing these eigenvalues to the previous case and to the open-loop ones clearly indicates that there is a trade-off between maximizing the algebraic connectivity of the graph and the number of links that can be eliminated.

We will finish the design procedure for the case when only the link between areas I and III is eliminated. Accordingly, we design a fast control gain matrix $G_f$ for each area so that the eigenvalues of the fast subsystem will be faster than the those assigned by the slow control. Thus, we choose to assign the fast eigenvalues for areas I to IV to be, respectively, $2/\delta, 2.2/\delta, 2.5/\delta$ and $2.8/\delta$ multiples of $-82.593$ and then each subsequent eigenvalue is larger by $1/\delta$ than the previous. To accomplish this step, we use the Matlab command place. Based on the previous design steps, the closed loop eigenvalues are found to be $(0, -36.404, -44.932, -82.591, -660.747, -664.731, -668.744, -672.741, -726.827, -730.828, -734.826, -738.815, -825.970, -829.959, -833.927, -925.049, -929.049, -933.049, -937.049, -941.053)$. Notice that all the closed-loop eigenvalues remain close to the desired ones.

Fig. 2 shows the open-loop response of the system and Fig. 3 shows the closed-loop response including the control effort. It should be noted that the faster we choose the fast eigenvalues, the faster the consensus would be reached, however, that would be on the expense of the initial value of the control effort.

VII. Conclusions

We tackled the problem of consensus control of clustered networks. We used a multi-modeling approach that allows each decision maker of each clustered area to assume a unique and relatively simple model of the entire network. We then took advantage of the system’s two time-scale properties to design control strategies for each area by solving two separate problems in the two time-scales. The main goal of the paper is to shape the transient performance of the slow dynamics. The design of a controller for this problem
is conducted in an aggregate way, i.e. one control for each area, and is focused on increasing the speed of convergence while simultaneously promoting sparsity of the feedback matrix. The control in the fast time-scale is designed in such a way that it preserves the time-separation property of the network. We prove that following the proposed design methodology results in a close performance to the desired one provided that the clustering is strong. Finally, we presented a simulation example that demonstrated the design procedure and showed that there is a trade-off between enhancing the closed-loop performance and eliminating as much communication links between the areas as possible.

**APPENDIX**

**Proof of Theorem 1:**

Without loss of generality and for ease of presentation, we only consider the case of two coordinators, i.e. \( k = 1, 2 \). Applying the control (20) to the actual system modeled in the singular perturbation form (9)-(10), we get the closed-loop system

\[
\dot{w} = [A_{11} + B_{01}\tilde{G}_{1x} + B_{02}\tilde{G}_{2w}]w + \sum_{j=1}^{2} [A_{0j} + B_{0j}G_{jj}]z_j \tag{31}
\]

\[
\delta \dot{z}_1 = d[A_{10} + B_{11}\tilde{G}_{1z}w + d\tilde{A}_{11} + \tilde{A}_{11} + B_{11}G_{11}]z_1 \tag{32}
\]

\[
\delta \dot{z}_2 = d[A_{20} + B_{22}\tilde{G}_{2z}]w + d\tilde{A}_{21}z_1 + [d\tilde{A}_{22} + \tilde{A}_{22} + B_{22}G_{22}]z_2. \tag{33}
\]

For convenience, we rewrite the closed loop system (31)-(33) as

\[
\begin{bmatrix}
\dot{w} \\
\delta \dot{z}_1 \\
\delta \dot{z}_2
\end{bmatrix} =
\begin{bmatrix}
F_{11} & F_{12} & F_{13} \\
F_{21} & F_{22} & F_{23} \\
F_{31} & F_{32} & F_{33}
\end{bmatrix}
\begin{bmatrix}
w \\
z_1 \\
z_2
\end{bmatrix} \tag{34}
\]

where

\[
F_{11} = A_{11} + B_{01}\tilde{G}_{1x} + B_{02}\tilde{G}_{2w},
\]

\[
F_{12} = A_{01} + B_{01}G_{1f},
\]

\[
F_{13} = A_{02} + B_{02}G_{2f},
\]

\[
F_{21} = d[A_{10} + B_{11}\tilde{G}_{1z}],
\]

\[
F_{22} = d\tilde{A}_{11} + \tilde{A}_{11} + B_{11}G_{11},
\]

\[
F_{23} = d\tilde{A}_{21},
\]

\[
F_{31} = d[A_{20} + B_{22}\tilde{G}_{2z}],
\]

\[
F_{32} = d\tilde{A}_{22} + \tilde{A}_{22} + B_{22}G_{22}.
\]

We now need to transform the system matrix in (34) to a block diagonal one so that we transform the system into a slow and fast parts. For this purpose, we use the nonsingular transformation [6]

\[
\begin{bmatrix}
y \\
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
I_0 - \sum_{i=1}^{2} \delta M_i L_i & -\delta M_1 & -\delta M_2 \\
L_1 & I_1 & 0 \\
L_2 & 0 & I_2
\end{bmatrix}
\begin{bmatrix}
w \\
z_1 \\
z_2
\end{bmatrix} \tag{35}
\]

where \( I_0, I_1 \) and \( I_2 \) are the identity matrices of the appropriate dimensions and \( L_1, L_2, M_1 \) and \( M_2 \) satisfy the matrix algebraic equations

\[
P_1 \triangleq F_{22}L_1 - F_{21} + F_{23}L_2 - \delta L_1[F_{11} - F_{12}L_1 - F_{13}L_2] = 0, \tag{36}
\]

\[
P_2 \triangleq F_{23}L_2 - F_{31} + F_{32}L_1 - \delta L_2[F_{11} - F_{12}L_1 - F_{13}L_2] = 0, \tag{37}
\]

\[
P_3 \triangleq M_1[F_{22} + \delta L_1F_{12}] + \delta M_2L_2F_{12} - F_{12} + M_2F_{32} - \delta [F_{11} - F_{12}L_1 - F_{13}L_2]M_1 = 0, \tag{38}
\]

\[
P_4 \triangleq M_2[F_{33} + \delta L_2F_{13}] + \delta M_1L_1F_{13} - F_{13} + M_1F_{23} - \delta [F_{11} - F_{12}L_1 - F_{13}L_2]M_2 = 0. \tag{39}
\]

Using (35), the closed-loop system in the new coordinates takes the form

\[
\begin{bmatrix}
y \\
\delta v_1 \\
\delta v_2
\end{bmatrix} =
\begin{bmatrix}
E_0 & 0 & 0 \\
0 & E_1 & d\tilde{A}_{12} + \delta L_1F_{13} \\
0 & d\tilde{A}_{21} + \delta L_2F_{12} & E_2
\end{bmatrix}
\begin{bmatrix}
y \\
v_1 \\
v_2
\end{bmatrix} \tag{40}
\]

where \( E_0, E_1, 0_0 \) and \( 0_1 \) are zero matrices with the appropriate dimensions and \( E_0 = F_{11} - F_{12}L_1 - F_{13}L_2, \ E_1 = F_{22} + \delta L_1F_{12}, \) and \( E_2 = F_{33} + \delta L_2F_{13}. \) It can be seen that the operator \( P_1, P_2, P_3, P_4, \mathcal{E} = [P_1, P_2, P_3, P_4] \) is analytic in its arguments and its partial derivatives with respect to \( L_1, M_1, L_2, M_2, \) respectively, exist and are invertible. Thus, by the implicit function theorem \( L_1, L_2, M_1 \) and \( M_2 \) are analytic in \( \mathcal{E} \) at \( \mathcal{E} = 0. \) Therefore, the matrices and initial conditions of the transformed system (34) can be uniformly approximated by the matrices and initial conditions of the sub-problems (18) and (16), that is

\[
\begin{align*}
\dot{y} &= [A_{11} + \sum_{k=1}^{2} B_{ks}G_{ks} + O(||\mathcal{E}||)]y, \quad y(0) = w_0 + O(||\mathcal{E}||), \tag{41} \\
\delta \dot{v}_1 &= [(\tilde{A}_1 + B_{11}G_{1f}) + O(||\mathcal{E}||)]v_1 + O(||\mathcal{E}||)v_2, \tag{42} \\
\delta \dot{v}_2 &= [(\tilde{A}_2 + B_{22}G_{2f}) + O(||\mathcal{E}||)]v_2 + O(||\mathcal{E}||)v_1, \tag{43}
\end{align*}
\]

We can see from the continuous dependence of (41) on its right hand side and initial conditions that \( y(t) \to w_0 \) as \( ||\mathcal{E}|| \to 0. \) Similarly, we can see from (42) and (43) and in the fast time-scale \( \tau = t/\delta \) the uniform convergence of \( v_1(\tau) \) and \( v_2(\tau) \) to \( z_{1f}(\tau) \) and \( z_{2f}(\tau), \) respectively.

Complete controllability of the pairs \((A_{11}, B_{ks})\) and \((\tilde{A}_k, B_{ks}), k = 1, 2,\) implies the existence of gain matrices \( G_{ks} \) and \( \tilde{G}_{ks} \) which arbitrarily assign corresponding eigenvalues to the matrices \( A_{11} + \sum_{k=1}^{2} B_{ks}G_{ks} \) and \( \tilde{A}_k + B_{ks}G_{ks}, \) respectively [17]. Noticing that the closed-loop eigenvalues \( \lambda_{k}^{c}, i = 1, \ldots, n \) are precisely those of (41)-(43) and the fact that the eigenvalues of \( A_{11} + B_{11}G_{11} + B_{22}G_{22}, \tilde{A}_1 + B_{11}G_{1f} \) and \( \tilde{A}_2 + B_{22}G_{2f} \) are \( O(||\mathcal{E}||) \) regular perturbations of those of (41)-(43) and the application of [12, Ch.2, Th.3.1] completes the proof. □

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