Abstract—In this paper we consider the problem of localizing unknown disturbances in large power systems using a measurement-based graph theoretic method. We first show that the aggregate electro-mechanical model of the power system, under certain assumptions, can be analyzed analogously to a generic \( n \)-th-order asymmetric networked dynamic system. Considering that the network exhibits a clustering structure leading to slow and fast eigenvalues, we propose an input localization method based on the properties of the weak nodal domains corresponding to the first \( p \) dominant slow eigenvalues. In particular, we prove that for systems defined over \( p \)-area complete graphs the input localization is unique. We provide simulation results to illustrate our algorithm.

Index Terms—Disturbance localization, power systems, identification, algebraic graph theory.

I. INTRODUCTION

Research on networked dynamic systems has progressed tremendously over the past two decades, largely due to advancements in sensor and communication technology. Today, network analysis can be found in applications ranging from social networks [1], coordination of autonomous and unmanned vehicles [2], and most recently power system analysis and control [3]. The advantage of network analysis in power systems is that it allows for monitoring and controlling the grid from a global, or wide-area, perspective. Interest in such analysis of wide-area power system dynamics has received significant impetus following the Northeast Blackout of 2003. In particular, the development of Wide-Area Measurement Systems (WAMS) technology [4], [5] and the installation of sophisticated measurement devices called Phasor Measurement Units (PMUs) across the US grid have been key facilitators for wide-area visualization and situational awareness. In the current state-of-art, the primary applications of WAMS include real-time monitoring [6], state estimation [7], and power flow control [8].

An unanswered question of particular interest to system operators is how PMU measurements, beyond monitoring, can also be used to localize potentially dangerous disturbances, especially when knowledge about the detailed system models may not be available. Recent findings from PMU data analysis for the US Eastern Interconnect, for example, indicate that such input detection is highly dependent on the underlying geometry of the system. The natural question is whether these geometrical, or graph-theoretic properties of the power system network can be used for detecting a disturbance rapidly in real-time. In this paper we address this pertinent problem by developing a novel measurement-based input localization technique for a power system using algebraic graph theory. Specifically, we assume that the power system is composed of \( p \) coherent areas [9], and that the wide-area dynamics are dominated by \( p - 1 \) slow oscillation modes. We show that by analyzing the relationship between the residues of the system transfer function, which can be identified from PMU data, and the discrete nodal domains of the \( p - 1 \) slow modes, one can develop a table of localization keys that relates the disturbance location to the measurement node. We prove that this method provides a unique localization for systems defined over \( p \)-area complete graphs. Our approach is more theoretically concrete compared to the simulation-based triangulation method for input detection proposed in [10]. It also captures the graph-theoretic interpretation of the system dynamics compared to the ellipsoid algorithm proposed in [11].

The remaining sections of the paper are organized as follows: Section II reviews the necessary graph-theoretic results, including the asymmetric graph Laplacian and discrete nodal domain theorem. Section III presents the swing oscillation model for an \( n \)-machine power system, and reviews system identification using PMU data. Section IV develops the input localization method by relating the residues of the identified transfer function to the discrete nodal domains. Section V illustrates the results with simulations of a five-area power system model. Section VI concludes the paper.

II. PRELIMINARIES

We first review pertinent graph-theoretic results including the asymmetric graph Laplacian and discrete nodal domain theorem, which will be used in Section IV.

A. Graph theory

The node- and edge- weighted graph \( G(n,m) = (V,E) \) has vertex set (or node set) \( V = \{1,2,\ldots,n\} \) and edge set \( E \subseteq V \times V \). If there is an edge \( ij \in E \), then vertices \( i,j \in V \) are adjacent, which is also written \( i \sim j \). The graph is undirected i.e., \( i \sim j \Leftrightarrow j \sim i \). The set of vertices adjacent to vertex \( i \) is \( N_i = \{ j \in V | j \sim i \} \), often called the neighborhood of \( i \). Each vertex \( i \in V \) has a node weight \( m_i > 0 \), and each edge \( ij \in E \) has an edge weight \( w_{ij} = w_{ji} > 0 \). A graph \( G = (V',E') \) is a subgraph of \( G \) if \( V' \subseteq V \) and \( E' \subseteq E \). If \( G' \) contains all edges \( ij \in E \) such that \( i,j \in V' \), then \( G' \) is an induced sub-graph, and is written \( G[V'] \) [12].

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1T. R. Nudell is with the Department of Electrical Engineering, North Carolina State University, Raleigh, NC, USA trnudell at ncsu dot edu

2A. Chakrabortty is with the Faculty of Electrical Engineering, North Carolina State University, Raleigh, NC, USA aranya.chakrabortty at ncsu dot edu
The node- and edge- weighted graph Laplacian can be defined in terms of the node- and edge-weight matrices of the graph [13], or equivalently in terms of degree and adjacency matrices as
\[
\mathcal{L}_m(G) = M^{-1} B W B^T = D_m - A_m, \tag{1}
\]
where the diagonal node weight matrix and edge weight matrix are defined as \(M = \text{diag}[m_i], \ i \in V, \ W = \text{diag}[w_{ij}],\) respectively; the \(n \times m\) oriented incidence matrix \(B\) has columns \(k = 1, \ldots, m\) that represent the edges \(ij \in E\) with entries \([B]_{jk} = 1, [B]_{jk} = -1, \forall i < j\) and zero otherwise; the degree matrix and adjacency matrix are defined, respectively, as
\[
D_m = \text{diag} \left( \sum_{j \in \mathcal{N}_i} w_{ij} \right),
\]
\[
[A_m]_{ij} = \begin{cases} w_{ij}/m_i & \text{if } j \in \mathcal{N}_i, \\ 0 & \text{otherwise}. \end{cases} \tag{3}
\]

We use the subscript \(m\) to distinguish the asymmetric matrices \(\mathcal{L}_m, D_m, A_m\) in (1), (2), (3) from their symmetric counterparts. For the remainder of the paper we will use the terms ‘node- and edge-weighted’ and ‘asymmetric’ interchangeably. The symmetric Laplacian \(\mathcal{L}\) has the following well-known properties [14]. First, \(\mathcal{L}\) is positive semi-definite with eigenvalues \(0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m\). The number of zero eigenvalues is the number of connected components of \(G\). If the graph is connected there is exactly one zero eigenvalue, with the corresponding right eigenvector \(r_1 = 1/\sqrt{n}1\), called the Perron vector. We remark that because of the special type of asymmetry that arises in \(\mathcal{L}_m\) all of these properties hold with the exception that the left and right Perron vectors, respectively, must be generalized to account for the asymmetry as
\[
\ell_1 = \frac{1}{\sqrt{\text{tr}(M)}} M1; \quad r_1 = \frac{1}{\sqrt{\text{tr}(M)}} 1. \tag{4}
\]

B. Discrete Nodal Domain Theorem

Discrete nodal domain theorem is a discrete version of Courant’s nodal domain theorem, which provides an upper bound on the number of nodal domains corresponding to \(k^{th}\) eigenfunction of the Laplace operator on continuous bounded manifolds. The differences between the discrete and continuous versions of the theorem arise because discrete eigenfunctions, equivalently eigenvectors, may change sign in a discontinuous manner. In [15] the authors provide complete proofs of the two discrete analogs, strong and weak discrete nodal domain theorems. These results are further developed in [16], [17]. We reproduce their main results, which we will use in Section IV, as follows.

Discrete nodal domains are maximal connected induced sign graphs of the connected graph \(G = (V,E)\). Associate the vertices \(V = \{1,2,\ldots,n\}\) with the real vector \(x = [x_1, x_2, \ldots, x_n]^T\), then

- a positive (negative) strong nodal domain of the vector \(x\) on \(V\) is a maximal connected induced sub-graph of \(G\) on vertices \(i \in V\) such that \(x_i > 0 (x_i < 0)\),
- a positive (negative) weak nodal domain of the vector \(x\) on \(V\) is a maximal connected induced sub-graph of \(G\) on vertices \(i \in V\) such that \(x_i \geq 0 (x_i \leq 0)\), and
- two different strong (weak) nodal domains \(S_1, S_2\) are adjacent if there exist vertices \(i \in S_1\) and \(j \in S_2 \setminus S_1\) such that \(ij \in E\).

By definition, if two different strong (weak) nodal domains are adjacent they have opposite sign. Strong and weak nodal domains differ in the role of zero vertices, or where \(x_i = 0\). Zero vertices separate strong nodal domains and join weak nodal domains, i.e., no zero vertices belong to any strong nodal domains, but each zero vertex belongs to exactly one weak positive nodal domain and one weak negative nodal domain. The following theorem provides upper bounds on the number of nodal domains of the eigenvectors of \(\mathcal{L}\).

**Discrete Nodal Domain Theorem** [15]: Let \(M\) be a generalized Laplacian of a connected graph with \(n\) vertices. Then any eigenvector \(x\) corresponding to the eigenvalue \(\lambda_k\) with multiplicity \(r\) has at most \(k + r - 1\) strong nodal domains and at most \(k\) weak nodal domains.

Figure 1 illustrates the weak nodal domains corresponding to \(\lambda_2\) in simple (unweighted) graph. Weak nodal domains of opposite sign are indicated with ovals of different color. The domains overlap at a zero vertex, called the characteristic vertex for nodal domains corresponding to \(\lambda_2\).

![Fig. 1. Weak nodal domains corresponding to \(\lambda_2\)](image-url)

III. POWER SYSTEM MODEL AND SYSTEM IDENTIFICATION

In order to motivate the input localization problem of Section IV, we first develop the power system models of interest, and describe how the transfer function for this model can be identified using PMU data.

A. Swing Models

Consider a network of \(n\) synchronous generators (nodes) connected to each other through \(m\) tie-lines (edges) with \(m \leq n(n-1)/2\), forming a connected graph \(G(n,m) = (V,E)\), such that no more than one edge exists between any two nodes. Let the internal voltage phasor of the \(i^{th}\) machine be denoted as
\[
\bar{E}_i = E_i/\delta_i, \quad i = 1,2,\ldots,n \tag{5}
\]
where, following synchronous machine theory [18], \(E_i\) is constant, \(\delta_i\) is the angular position of the generator rotor, and \(E_i/\delta_i\) denotes the polar representation \(E_i \exp(j\delta_i)\) (\(j = \sqrt{-1}\)). The transmission line connecting the \(i^{th}\) and the \(k^{th}\) machines is assumed to have an impedance \(Z_{ik} = r_{ik} + jx_{ik}\),
where ‘r’ denotes the resistive part and ‘x’ denotes the reactive part. For $i,k \in V$ and $k \in \mathcal{N}_i$, the total number of tuples formed by pairing $i$ and $k$ is $m$. We denote the edge connecting the $i^{th}$ and the $k^{th}$ nodes by $ik \in E$. If two nodes do not share a connection then the impedance corresponding to that non-existing edge is infinite (i.e., open circuit), or equivalently, the admittance $\tilde{y}_ik = 0 \forall k \not\in \mathcal{N}_i$. The mechanical inertia of the $i^{th}$ machine is denoted as $H_i$. The dynamic electro-mechanical model of the $i^{th}$ generator, neglecting damping, can be written as [18]

$$\dot{\delta}_i = \omega_\ell - \omega_s,$$

(6)

$$2H_i \omega_i = P_{mi} - \sum_{k \in \mathcal{N}_i} \left( \frac{E_i^2 r_{ik} - E_i E_k p_{ik} \cos(\delta_{ik} + \alpha_k)}{p_{ik}^2} \right),$$

(7)

where $\delta_{ik} = \delta_i - \delta_k$, $\omega_s = 120\pi$ is the synchronous speed for a 60 Hz system, $\omega_\ell$ is the rotor angular velocity, $P_{mi}$ is the mechanical power input, $p_{ik} = \sqrt{r_{ik}^2 + x_{ik}^2}$ and $\alpha_k = \tan^{-1}(x_{ik}/r_{ik})$. All quantities are in per unit except for the phase angles which are in radians. We assume that the network structure is known, i.e., the set $\mathcal{N}_i \forall i \in V$ in (6)-(7) is known.

We linearize (6)-(7) about an initial equilibrium $(\delta_0, 0)$ where $0 < \delta_0 < 90^\circ \forall i \in V$, and denote the perturbed state variables as

$$\Delta \delta = [\Delta \delta_1 \ldots \Delta \delta_n]^T,$$

$$\Delta \omega = [\Delta \omega_1 \ldots \Delta \omega_n]^T.$$

(8)

(9)

We assume that the input $u$ enters the system through the $j^{th}$ node, $j \in V$. The aggregate LTI model is

$$\frac{[\Delta \dot{\delta}]}{\Delta \omega} = - \mathcal{L}_m \begin{bmatrix} I_n \cr 0 \cr 0 \cr M^{-1}e_j \end{bmatrix} u,$$

(10)

where $I_n$ is the $n$-dimensional identity matrix, $e_j$ is the $j^{th}$ unit vector with all elements zero except the $j^{th}$ element which is $1$, $M = \text{diag}[m_1, m_2, \ldots, m_n]$, $m_i = 2H_i$ is the inertia of the $i^{th}$ generator, and $\mathcal{L}_m = M^{-1}L$ is the $n \times n$ asymmetric Laplacian matrix with elements

$$[\mathcal{L}_m]_{ik} = \begin{cases} \sum_{j \in \mathcal{N}_i} w_{ij}/m_i & \text{if } k = i, \\
-w_{ik}/m_i & \text{if } k \in \mathcal{N}_i, \\
0 & \text{otherwise}, \end{cases}$$

(11)

where $w_{ik} = \frac{E_i E_k}{p_{ik}} \sin(\delta_{ik} - \delta_{0k} + \alpha_k)$. It follows that if $m_i = m_j, \forall i, j \in V$, then $A = A^T$. However, in general each machine will have distinct inertia as a result of which the symmetry property does not hold. The eigenvalues of the system are $\lambda(A) = \pm \sqrt{-\lambda_k}$, $k = 1, \ldots, n$, where $\lambda_k$ is the $k^{th}$ eigenvalue of $\mathcal{L}_m$. By choosing the output as frequency measurements $\omega_i$ from the PMU located at node $i$ we can define a new variable $x = \Delta \delta$ and define a $n^{th}$-order system analogous to (10) as

$$\begin{bmatrix} \dot{x} \\
y \end{bmatrix} = \begin{bmatrix} -\mathcal{L}_m x + b_j u, \\
e_j^T x \end{bmatrix},$$

(12)

where $b_j = (m_j)^{-1}e_j$. The transfer function of the new system (12) is

$$g(s) = e_j^T(sI + \mathcal{L}_m)^{-1}b_j,$$

$$= \frac{\beta_0 - \rho s^{n-p} + \beta_{n-p-1}s^{n-p-1} + \cdots + \beta_1 s}{\alpha_n s^n + \cdots + \alpha_1 s},$$

(13)

where $R^k_{ij}$ is the $(i,j)$-residue associated with the $k^{th}$ eigenvalue. The residue $R^k_{ij}$ represents the influence of a unit impulse at vertex $j$ on mode $\lambda_k$ as measured at vertex $i$, and is the product of the mode controllability factor and mode observability factor [19], defined as

$$R^k_{ij} = \left( e_j^T r_{ik} \right) \left( e_i^T b_j \right).$$

(15)

where $\ell_k, r_k$ are the left and right eigenvectors corresponding to $\lambda_k$, normalized such that $\ell_i^T r_k = 1$. It is important to note that with this choice of output the transfer function of the original model (10) is identical to (13) or (14), except that $s$ must be replaced by $s^2$. Next we establish that the residues $R^k_{ij}$ of the asymmetric system (12) are symmetric. This fact will be used for developing the input localization method in Section IV.

**Lemma 1:** The matrix $[R^k_{ij}]$ is symmetric.

**Proof:** Define $[R^k_{ij}]$ by taking every $b_i$ and $e_j^T$ pair,

$$[R^k_{ij}] = \begin{bmatrix} e^T_1 & e^T_2 & \cdots & e^T_n \end{bmatrix} r_k \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix},$$

(16)

Recall $\mathcal{L}_m = M^{-1}BW BT$ and $\mathcal{L}_m^T = BW^T M^{-1}$. Since $\ell_k$ is in the nullspace of $(\mathcal{L}_m^T - \lambda_k I)$ and $r_k$ is in the nullspace of $(\mathcal{L}_m - \lambda_k I)$ we have

$$(BW^T - \lambda_k M)M^{-1}r_k = 0,$$

$$M^{-1}(BW^T - \lambda_k M)r_k = 0.$$

Since the node-weight matrix is non-singular, $M^{-1}r_k$ is in the nullspace of both $(BW^T - \lambda_k M)$ and $M^{-1}(BW^T - \lambda_k M)$. However, the nullity of $M^{-1}(BW^T - \lambda_k M)$ is unity, which implies

$$r_k = \gamma_k M^{-1}r_k,$$

(17)

where $\gamma_k$ is a positive, real-valued constant. Substituting this result into (16) yields

$$[R^k_{ij}] = \frac{1}{\gamma_k} r_k \ell^T_k = [R^k_{ij}]^T.$$
B. System Identification

We next describe how PMU measurements can be used to identify the transfer function of the system and generate the residues, which will be used for input localization in Section IV. Subspace identification methods, such as the Eigenvalue Realization Algorithm (ERA) [20], can be used for this purpose. The idea is briefly described as follows. Consider a discrete-time LTI system with \( d \) inputs and \( q \) outputs

\[
x(k+1) = Ax(k) + Bu(k); \quad y(k) = Cx(k),
\]

(19)

where \( x(k) \in \mathbb{R}^n \) (\( n \) is a known quantity), \( k = 0, \ldots, N \). The impulse response of the system is

\[
y(k) = CA^{k-1}B.
\]

(20)

Given measurement \( y(k) \), we construct two \( \kappa \times q \) Hankel matrices \( H_0 \) and \( H_1 \) as

\[
H_0 = \begin{bmatrix} y_0^1 & y_0^2 & \cdots & y_0^q \\ y_1^0 & y_1^1 & \cdots & y_1^q \\
\end{bmatrix}, \quad H_1 = \begin{bmatrix} y_0^1 & y_1^1 & \cdots & y_1^q \\ y_0^2 & y_1^2 & \cdots & y_1^q \\
\end{bmatrix},
\]

(21a)

(21b)

where

\[
y_i^0 = \begin{bmatrix} y(i) & y(i+1) & \cdots & y(i+k-1) \\ \end{bmatrix}^T, \quad y_i^1 = \begin{bmatrix} y(i+1) & y(i+2) & \cdots & y(i+k) \\ \end{bmatrix}^T.
\]

(22a)

(22b)

It can be easily shown that \( H_0 = OC \) and \( H_1 = OC \), where \( O \) and \( C \) are observability and controllability matrices of \( (19) \), respectively. Next, consider the truncated SVD of \( H_0 \)

\[
\tilde{H}_0 = \tilde{O}\tilde{\Sigma}\tilde{\nu}^T.
\]

(23)

Estimates for the triplet \((\tilde{A}, \tilde{B}, \tilde{C})\) are calculated as

\[
\tilde{A} = \tilde{\Sigma}^{-1/2}\tilde{O}^TH_1\tilde{\Sigma}^{-1/2}, \quad \tilde{B} = \tilde{\Sigma}^{1/2}\tilde{\nu}E_d, \quad \tilde{C} = E_q^T\tilde{\Sigma}^{1/2}.
\]

(24a)

(24b)

where \( E_d = [I_q \ 0 \ \cdots \ 0]^T \) and \( E_q = [I_q \ 0 \ \cdots \ 0]^T \). From \((\tilde{A}, \tilde{B}, \tilde{C})\), we can obtain for a given input-output pair \( i, j \)

\[
y_i(t) = R_{ij}^1 + R_{ij}^2 \exp(\pm t\sqrt{-\lambda_2}) + \cdots + R_{ij}^p \exp(\pm t\sqrt{-\lambda_p})
\]

\[
+ R_{ij}^{p+1} \exp(\pm t\sqrt{-\lambda_{p+1}}) + \cdots + R_{ij}^{q} \exp(\pm t\sqrt{-\lambda_q}),
\]

(25)

where, by assumption, there are \( p \) areas, or \( p-1 \) slow modes. Next, we use the signs of the estimated \( R_{ij}^k \), \( k = 2, 3, \ldots, p \), to develop our main result.

IV. INPUT LOCALIZATION

In this section we develop our PMU-based input localization algorithm assuming that the system is defined over a certain class of graphs with specific nodal domain properties. Before defining this class and proving the main results, we illustrate the relationship between nodal domains and clustering in asymmetric systems with a simple example.

Example 1: Consider a family of 4-node graphs \( S_1 \) and \( S_2 \) with identical \( V = \{1, 2, 3, 4\} \), \( E = \{12, 23, 24, 34\} \). Let \( S_1 \) be a simple graph and \( S_2 \) be a node- and edge- weighted graph. The graph \( S_1 \) was shown previously (Fig. 1), and \( S_2 \) is shown in Figure 2(a) with its node and edge weights indicated.

First, consider the nodal domains corresponding to \( \lambda_2 \) and \( \lambda_3 \) in the simple graph \( S_1 \). The \( \lambda_2 \) nodal domains contain a characteristic vertex (i.e., a zero vertex) where two weak nodal domains overlap, and we would find that there are only two weak nodal domains corresponding to \( \lambda_3 \) with multiple zero vertices. Contrast this with the weak nodal domains of \( S_2 \), which contains the maximum number of weak (and strong) nodal domains for each of \( \lambda_2 \) and \( \lambda_3 \) (shown in Figures 2(b) and 2(c), respectively). The node- and edge- weights play an important role in the distribution and existence of nodal domains.

Now consider the class of \( p \)-area complete node- and edge-weighted graphs of order \( n \) with the following properties:

- the graph can be partitioned into \( p \leq n \) disjoint areas \( A_1, \ldots, A_p \);
- the eigenvector \( r_i \) corresponding to \( \lambda_k \), \( k = 2, 3, \ldots, p \), contains no vanishing coordinates;
- there are exactly \( k \) nodal domains (weak or strong) corresponding to \( \lambda_k \), \( k = 2, 3, \ldots, p \), composed of the \( p \) areas—i.e., for a given \( \lambda_k \), all areas belong to exactly one nodal domain.

It follows from the above conditions that the first \( p \) eigenvalues of a \( p \)-area complete graph must be simple. If this were not the case then \( r_k \) would at least one vanishing coordinate [17]. This \( p \)-area completeness will be a necessary condition for our main result in Theorem 1. While the construction of a \( p \)-area complete graph may initially seem restrictive, we note that the graph \( S_2 \) from Example 1 is indeed \( 3 \)-area complete. Furthermore we can also construct a \( p \)-area complete graph of arbitrary size using the following lemma.

Lemma 2: Consider two graphs \( G_1 \) and \( G_2 \), where \( G_1 \) is \( p \)-area complete with eigenvalues \( 0 = \lambda_1 < \lambda_2 < \cdots < \lambda_p \leq \cdots \leq \lambda_n \), and the Fiedler value of \( G_2 \) is \( \mu_2 > \lambda_p \). Then \( H = G_1 \cup G_2 \) is \( p \)-area complete.

Proof: The first \( p \) eigenvalues and eigenvectors of \( G_1 \cup G_2 \) are

\[
\lambda'_i = \lambda_i, \quad r'_i = r_i \cup u_1, \quad i = 1, \ldots, p,
\]

where the eigenvector \( u_1 \) corresponding to eigenvalue \( \mu_1 = 0 \) has only positive entries [17], due to which \( r'_i \) has the same nodal domain partitioning as \( r_i \) for \( i = 1, \ldots, p \). That is, the first \( p \) dominant clusters of \( H \) are identical to those of \( G_1 \) with each node in \( G_1 \) replaced by a copy of the graph \( G_2 \). Therefore, \( H \) is \( p \)-area complete.

To this end, we are ready to state the following two lemmas that will motivate Theorem 1. Both lemmas follow from Lemma 1 and the properties of weak discrete nodal domains.
Nodal domains corresponding to $\lambda_1$, $\lambda_2$, and $\lambda_3$.

**Proposition 1:** The sign of the residues $R_{ij}^k$, $k = 2, \ldots, p$ can be used to localize an input using a table of localization keys, each of length $p - 1$ that contains entries $\{\text{sign}(R_{ij}^1), \ldots, \text{sign}(R_{ij}^p)\}$.

Consider $S_2$ from Example 1. There are three areas $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3, 4\}$, and two slow modes $\lambda_2$ and $\lambda_3$. For $j = 1 \in A_1$, the key $\{+, +\}$ implies the location of $i$ is $A_1$, the key $\{+, -\}$ localizes $i \in A_2$, and the key $\{-, +\}$ localizes $i \in A_3$. The table of localization keys is shown in Table I.

The input localization table is symmetric. In other words, the key corresponding to $j \in A_1$ and $i \in A_2$ is identical to the key corresponding to $j \in A_2$ and $i \in A_1$. This result follows directly from Lemma 1. Also note that the diagonal elements of the table are all positive—a trivial corollary to Lemma 4.

**Theorem 1:** In a $p$-area complete system, given an output node, the localization key is unique for any input node $j \in A_k$.

**Proof:** Proceed by contradiction. For $i \in A_1$, assume there are two identical keys corresponding to $j \in A_q$ or $j \in A_s$, $A_q \neq A_s$. Now continue through the key, verifying the nodal domain location of $A_q$, $A_s$. There are exactly two weak nodal domains corresponding to $\lambda_2$. Thus, $A_q$, $A_s$ must be in identical domains (zero number of domains apart). There are exactly three weak nodal domains corresponding to $\lambda_3$. Thus, at this point they may either lie in the same nodal domain or separate domains that share an adjacent nodal domain (zero nodal domains apart or an even number of nodal domains apart). Proceed in this manner for $k \leq p$, recalling that there are exactly $k$ disjoint nodal domains corresponding to $\lambda_k$, at which point every area $A_1, \ldots, A_p$ is adjacent to an area with opposite sign. Thus $A_q = A_s$, a contradiction.

**V. SIMULATION RESULTS**

We illustrate our results using a five-area 33-node power system $\mathcal{A}_5$ (Fig. 3). Each area of the system is generated as a random graph of order $2 \leq n_i \leq n_{\text{max}}$, with uniformly distributed node and edge weights. The areas are then connected in a path via arbitrarily chosen boundary nodes. The inter-area edge weights are on average five to ten times smaller than the local connections. This method of generating random graphs obviously does not guarantee a $p$-area complete system. If the graph is initially not $p$-area complete we use the heuristics developed in Example 1 to increase the appropriate inter-area sparsity and intra-area density. If this fails we simply reject the graph and start afresh. The system has four slow eigenvalues $\lambda_2 = 0.0025$, $\lambda_3 = 0.0085$, $\lambda_4 = 0.0157$, $\lambda_5 = 0.0582$. The next mode $\lambda_6 = 1.4253$ is much larger than the slow modes.

After generating the system, we identify the location of the areas $A_1, A_2, A_3, A_4, A_5$ with respect to each of the slow-mode nodal domains (Fig. 5). Using this information and Lemma 4 we construct the table of input localization keys (Table II). Recall that the keys are symmetric. For simplicity, we only present the upper triangle.

Now assume that we have a PMU at each boundary node. A disturbance is simulated as an impulsive input occurring...
at an unknown location (node 29). Figure 4 shows the input response as seen from the boundary nodes in areas $A_1 - A_4$. Operators in each area may independently run a system ID (ERA) with their available measurements to obtain a minimum-order realization of the system model, and from this obtain the transfer function residues and localization keys. The pole-residue form of the recovered transfer function is

$$g(s) = \sum_{k=1}^{5} \frac{R^k_{ij}}{(s^2 + \bar{\lambda}_k)},$$

where $\bar{\lambda}_k$ is the mode recovered by each operator. The first five modes are $\bar{\lambda}_1 = 0$, $\bar{\lambda}_2 = 0.0032$, $\bar{\lambda}_3 = 0.0106$, $\bar{\lambda}_4 = 0.0196$, and $\bar{\lambda}_5 = 0.0728$, with corresponding residues from each area:

<table>
<thead>
<tr>
<th>Area</th>
<th>$R^1_{ij}$</th>
<th>$R^2_{ij}$</th>
<th>$R^3_{ij}$</th>
<th>$R^4_{ij}$</th>
<th>$R^5_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.0001</td>
<td>-0.0003</td>
<td>-0.0011</td>
<td>0.0007</td>
<td>0.0007</td>
</tr>
<tr>
<td>$A_2$</td>
<td>-0.0005</td>
<td>0.0023</td>
<td>-0.0040</td>
<td>-0.0003</td>
<td>-0.0003</td>
</tr>
<tr>
<td>$A_3$</td>
<td>-0.0011</td>
<td>-0.0034</td>
<td>0.0019</td>
<td>0.0003</td>
<td>0.0003</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.0005</td>
<td>0.0024</td>
<td>0.0025</td>
<td>-0.0080</td>
<td>-0.0080</td>
</tr>
</tbody>
</table>

The localization keys are $\mathbf{K}_1 = \{1, -1, -1, 1\}$, $\mathbf{K}_2 = \{-1, 1, -1, -1\}$, $\mathbf{K}_3 = \{-1, -1, 1, 1\}$, $\mathbf{K}_4 = \{1, 1, 1, -1\}$, for areas $A_1 - A_4$, respectively. Each operator compares their recovered key to their stored table (Tab. II), and each operator concludes that the input occurred in area $A_5$. We confirm the correctness of this localization by inspecting Fig. 3 for the input location (node 29).

**VI. CONCLUSION**

In this paper we developed a novel measurement-based disturbance localization method for a large power grid modeled as an asymmetric networked dynamic system. This localization method exploits the system’s globally interconnected topology and its wide-area dynamics, and hence can be particularly effective when the precise local topology is not known. We proved that measurements from any vertex in a $p$-area complete system can be used to uniquely localize the area in which the disturbance occurred. We illustrated our results by simulating a large power grid where each local topology is arbitrarily connected and weighted. Future work will include relaxing the assumption on $p$-area completeness, implementing the algorithm in real-time, and improving the robustness of the localization, especially when the PMU measurements are noisy. Our ultimate goal is to incorporate this valuable disturbance localization information into distributed wide-area control designs.

**REFERENCES**


