A Graph Theoretic Algorithm for Forced Harmonic Input Localization in Large Power Networks

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Abstract—In this paper we consider the problem of localizing inputs in swing dynamic models of large power system networks in the form of forced oscillations with unknown amplitude and frequency. Such harmonic oscillations commonly result from internal failures of control actuators in synchronous machines, and are particularly dangerous because their frequency often lies in the range of the inter-area oscillation modes of the system, resulting in unwanted sub-synchronous resonance phenomena. We first develop the concept of discrete nodal domains for second-order network dynamic systems, and relate these nodal domains to the residues of the system transfer function. Thereafter, we develop a graph-theoretic algorithm based on the magnitude as well as the sign of these residues that detects the location of the forced oscillation input. This algorithm extends our recent work in [1] for first-order consensus networks to second-order oscillatory networks, and shows that if the system has a clustered topology resulting in two-time-scale behavior in the phase oscillations then one can accurately localize the cluster in which the resonance is initiated using the residues of the inter-area modes. Our discussion includes both undamped and weakly damped swing models. We simulate a 60-generator 4-area power system to illustrate the different steps of our algorithm.

Index Terms—Disturbance localization, identification, algebraic graph theory.

I. INTRODUCTION

The US electric power transmission grid is a critical aspect of our national infrastructure, and is constantly under stress from various types of disturbances. The three basic types of disturbances that may occur in a power system can be classified as ambient, transient, and forced [2]. Ambient disturbances are small-signal inputs that are commonly attributed to intermittent loads and random variations in the system parameters due to transmission line overheating, frictional losses in the mechanical components of the generators, capacitor switching, etc. Transient or impulsive disturbances, on the other hand, are large-signal events that are caused due to faults and switchings such as line-to-ground faults followed by tripping of transmission lines, loss of generators and loads, sudden changes in controller setpoints, etc. The most deceptive and yet increasingly frequent type of failure among these three types of disturbances is forced disturbances. Forced disturbances, or more commonly referred to as forced oscillation disturbances, are limit cycle oscillation inputs that are typically caused by malfunctioning of control equipments such as power system stabilizers (PSS), voltage regulators, governor controls, and most commonly by the pole-blocking phenomenon in power electronic converters, especially in high-voltage DC (HVDC) transmission lines. A common example, for instance, is a malfunctioning valve at a power plant cycling full on and full off [2], and thereby exciting the electro-mechanical swing dynamics of the generators by a limit cycle, pulse train or saw tooth waves, all of which can be highly detrimental to regular system operations. Detection and localization of such forced oscillation disturbances are, in fact, gradually becoming a major threat for power system operators, not only because of increasing transmission expansion and power flow congestions, but also due to frequent cyber intrusions in Energy Management System (EMS) control centers, several examples of which have recently been reported in [3], [4]. For example, on November 30, 2005, a hydro-governor on a generator in the western North American Power System (wNAPS) went into a limit cycle due to its internal controller failure. The frequency of the forced oscillation just happened to align with a major electromechanical mode in the system thereby causing a resonance effect due to which the limit cycle input was amplified throughout the system resulting in very large oscillations in the power flows [5].

Motivated by this critical problem of current interest, in this paper we propose a measurement-based input localization algorithm by which forced oscillation inputs in large power system networks can be successfully localized by exploiting the algebraic graph-theoretic properties of it underlying network structure. In our recent paper [1] we developed a related localization method for first-order consensus networks using the concept of nodal domains of clustered graphs that result in certain sign patterns in the residues of the network transfer function. Compared to the first-order models studied in [1], the network models in this paper are, however, coupled second-order oscillators resulting in complex eigenvalues and eigenvectors, and therefore, complex residues. When the swing dynamics of the system exhibit a coherency structure leading to a two time-scale behavior in the phase angle oscillations, the algorithm can be naturally extended to localize forced harmonic inputs entering the swing dynamics. Our basic approach to achieve this is as follows. Assuming the power system to consist of p coherent areas, we first use Synchronized measurements of both phase angle and frequency of selected generators to estimate the p − 1 inter-area modes and their corresponding modal residues in the transfer function of the system model.
We then derive several system-theoretic conditions by which the contribution of the forced input can be detected from the magnitude and signs of these residues. Thereafter, we develop a graph-theoretic algorithm based on the relationship between the residues and the weak nodal domains of these fast slow modes that detects the location of this input. Our results include derivations of the nodal domains of both the undamped and weakly damped swing models. Our approach is more theoretically concrete compared to the simulation-based triangulation method for input detection proposed in [6]. It also captures the graph-theoretic interpretation of the system dynamics compared to the heuristic ellipsoid algorithm proposed in [6]. Furthermore, in contrast to the results presented in [7] where the authors nicely formalize conditions for which an attack (or disturbance) in a network can be detected and then identified via decentralized monitors with knowledge of the underlying network, we consider outputs from a single measurement point, hence no communication is required. Most importantly, our algorithm is measurement-based, and therefore, needs minimal information about the actual operating model of the system.

The remainder of the paper is organized as follows. Section II introduces definitions and notation used throughout the paper. Section III formulates the measurement-based localization problem using a second-order swing model of the power system. Section IV briefly reviews the basic concept localization using nodal domains. Section V presents theoretical results on the nodal domains of the eigenvectors of second-order synchronization networks. Section VI develops the input localization algorithm for forced inputs, and then discusses methods for distinguishing the exogenous input frequency from the natural inter-area modes. Section VII illustrates the steps of the algorithm with a numerical example. Section VIII concludes the paper.

II. Preliminaries

In this section, we review the graph-theoretic definitions that will be used throughout the paper. A graph $G = (V, E)$ is defined by a vertex (or node) set $V = \{1, \ldots, n\}$ and an edge set $E \subseteq V \times V$. The number of vertices in $G$ is $|V|$. If there is an edge $kl \in E$, then vertices $k, l \in V$ are adjacent, which is denoted $k \sim l$. The set of nodes adjacent to $l \in V$ is $N_l = \{k | k \sim l\}$. The graph $G' = (V', E')$ is a subgraph of $G$ if $V' \subseteq V$ and $E' \subseteq E$. If $G'$ contains every edge $kl \in E$ such that $k, l \in V'$, then $G'$ is the subgraph induced by the vertex set $V'$. The unit indicator vector of a node $i$ is $\delta_i = [\delta_{i1}, \ldots, \delta_{in}] \in \mathbb{R}^n$. The number of vertices in $G$ is $|V|$ is undirected, i.e., $l \sim k \Rightarrow k \sim l$, and that there are no loops or multiple edges between nodes. We assume that every node $k \in V$ has a real-valued node-weight $m_k > 0$. We also assume that every edge $kl \in E$ has a real-valued edge-weight $a_{kl} > 0$. The (weighted) degree of node $k \in V$ is $d_k(G) = \sum_{l \in N_k} a_{kl}$. The adjacency matrix $A(G)$ is defined as $[A(G)]_{kl} = a_{kl}$ if $k \sim l$, and $[A(G)]_{kl} = 0$ otherwise. The adjacency matrix is positive definite, denoted $A(G) > 0$. The degree matrix $D(G)$ is a diagonal matrix defined by $[D(G)]_{ii} = d_i(G)$. The (edge-weighted) graph Laplacian matrix defined as $L(G) = L(G)^T = D(G) - A(G) \geq 0$. We assume the eigenvalues of $L(G)$ are ordered as $0 = \lambda_1 \leq \ldots \leq \lambda_n$. The number of zero eigenvalues of $L(G)$ is equal to the number of connected components of $G$. Unless otherwise noted, we will assume that $G$ is connected, i.e., $0 = \lambda_1 < \lambda_2$. Throughout this paper we will use the following definition of (weak) discrete nodal domains, abbreviated simply as nodal domains for a connected graph $G$.

Definition 2.1 (Discrete Nodal Domain [9]): A positive (negative) weak nodal domain $D$ of a real-valued vector $x \in \mathbb{R}^n$ is a maximal connected induced subgraph of $G$ on nodes $k \in V$ such that $|x|_k \geq 0$ ($|x|_k \leq 0$), where $|V| = n$ and $|x|_k$ denotes the $k$th component of $x$.

In general, if $v$ is an eigenvector of $L(G)$ corresponding to eigenvalue $\lambda_i$ then we will refer to the nodal domains of $v$ as the nodal domains of $G$ or $L(G)$. Specifically, we will refer to the weak nodal domains of $v_i$ as the $\lambda_i$ nodal domains.

We denote the unit indicator vector $e_i \in \mathbb{R}^n$, where $[e_i]_l = 1$ if $l = i$ and $[e_i]_l = 0$ otherwise. We denote the identity matrix as $I_n \in \mathbb{R}^{n \times n}$, the vector of ones as $1_n \in \mathbb{R}^n$, and reserve the imaginary unit $j = \sqrt{-1}$.

III. Problem Formulation

Consider a $n$-bus power system model defined over a graph $G = (V, E)$, where $|V| = n$. Each of the $n$ nodes represents a synchronous generator, and the edges represent tie-lines, which for simplicity, we assume are directly connected to the generator buses. The internal voltage phasor of the $i^{th}$ machine is denoted as

$$\hat{E}_i = E_i \angle \delta_i, \quad i = 1, 2, \ldots, n$$

where, from synchronous machine theory [10], $E_i$ is constant, $\delta_i$ is the angular position of the generator rotor, and $E_i \angle \delta_i$ denotes the polar representation $E_i \exp(j \delta_i)$. The tie-line connecting the $i^{th}$ and the $k^{th}$ machines, $ik \in E$, is assumed to have an impedance $\tilde{z}_{ik} = r_{ik} + j x_{ik}$, where '$r$' denotes the resistance and 'x' denotes the reactance. If two generators do not share a connection then the impedance corresponding to that non-existing edge is infinite (i.e., open circuit), or equivalently, the admittance $\tilde{y}_{ik} = 1/\tilde{z}_{ik} = 0 \forall k \notin N_i$. The mechanical inertia of the $i^{th}$ machine is denoted as $H_i$. The dynamic electro-mechanical model of the $i^{th}$ generator can be written as [10]

$$\dot{\delta}_i = \omega_i - \omega_s,$$

$$2H_i \omega_i = P_{m,i} - d_i \omega_i - \sum_{k \in N_i} \left( E_i^2 r_{ik} - E_i E_k p_{ik} \cos(\delta_{ik} + \alpha_{ik}) \right) \overline{p_{ik}},$$

where $\delta_{ik} = \delta_i - \delta_k$, $\omega_s = 120\pi$ is the synchronous speed for a 60 Hz system, $\omega_i$ is the rotor angular velocity, $0 \leq d_i < 1$ is the damping factor, $P_{m,i}$ is the mechanical power, $p_{ik} = \sqrt{r_{ik}^2 + x_{ik}^2}$ and $\alpha_{ik} = \tan^{-1}(x_{ik}/r_{ik}) \approx \frac{x_{ik}}{r_{ik}}$ when $x_{ik} \gg r_{ik}$ as in typical tie-lines. All quantities are in per unit except for the phase angles which are in radians. We linearize (2)-(3) about an initial equilibrium $(\delta_{i0}, \omega_s)$ where $0 < \delta_{i0} < \ldots < \delta_{in}$.
In order to formalize the localization algorithm we make the following assumptions about the network.

**Assumption 1:** The network can be partitioned into \( p \) coherent areas [11]. Each coherent area can be described by a vertex-disjoint induced subgraph \( G[k] \), where \( V = \bigcup_{k=1}^{p} V[k] \) and \( V[k] \cap V[l] = \emptyset \) if \( k \neq l \).

This assumption has a very natural interpretation for the power system network (6) due to coherency of generators [11]. Because of these \( p \) clusters, \( L_m \) will have \( p-1 \) slow eigenvalues. The next assumption is required for accurate identification of these slow eigenvalues from \( y(t) \).

**Assumption 2 (Measurement-based Transfer Function):**

The transfer function containing the \( p-1 \) slow poles of (9)–(10) has a minimum realization. Using these assumptions we can relate the location of the input node relative to the output node through the sign patterns of the residues of the transfer function of (9)–(10). Let \( R^k_{ij} \) denote the residue associated with \( \lambda_k \) for the \((k, l)\) input-output pair, then the impulse response can be written as

\[
g_{kl}(s) = \sum_{i=1}^{n} \frac{R^k_{i}}{(s + \lambda_i)},
\]

with

\[
R^k_{ij} = [Cv_k w_k^T B_m]_{ij} = (e_j^T v_k) (w_k^T M^{-1} e_i),
\]

where \( v_k \) and \( w_k \) are the left and right eigenvectors, respectively, corresponding to \( \lambda_k \) of \( L_m \). The first result relating the residues to nodal domains is that the nodal domains are identical for left and right eigenvectors—despite the asymmetry in \( L_m \) [1].

**Lemma 4.1:** The nodal domains of the left and right eigenvectors of \( L_m \) are identical.

**Corollary 4.1:** The residues defined in (12) are symmetric, i.e., \( R^T_{ij} = R^k_{ji} \), \( i, j \in \mathcal{V} \), \( k = 1, \ldots, n \).

Following these two results, the next lemma describes the sign of \( R^k_{1l} \) [1].

**Lemma 4.2:** Let \( v_i \) be an eigenvector of \( L_m \) corresponding to \( \lambda_i \), and let \( D_1 \) and \( D_2 \) be adjacent nodal domains of \( v_i \), then

\[
\text{sign}(R^k_{1l}) = \begin{cases} + & \text{if } k, l \in D_1 \setminus D_2, \\ - & \text{if } k \in D_1 \setminus D_2 \text{ and } l \in D_2 \setminus D_1, \\ 0 & \text{if } k \in D_1 \cap D_2 \text{ or } l \in D_1 \cap D_2. \end{cases}
\]

It should now be clear that if an input enters any node inside of a given area \( k \) and an output measurement is taken at any node in a given area \( l \), then there is a specific and predictable sign pattern of the residues from (12) which only depends on the relative location of the two areas with respect to the nodal domains. Furthermore, this sign pattern will remain invariant as long as the relative locations remain the same, despite possible changes to the system parameters. Formally, we define a \( p \times p \) array of localization keys \( K \in [K_{kl}] \), where

\[
K_{kl} = \{\text{sign}(R^k_{ij}), \ldots, \text{sign}(R^k_{lj})\}.
\]

Each key in \( K \) provides the aforementioned residue sign pattern between two areas (hence the need for a \( p \times p \) table)
based on nominal network parameters. These keys, therefore, can be used for area-to-area disturbance localization. In particular, after (11) has been estimated via system identification an estimated key $K_{ij}$ can be matched to the entries in $K$ defined in (13). A matching indicates an area in which the disturbance may have entered.

In this paper we build upon these results in two ways. First, we generalize the concept of nodal domains to complex-valued eigenvectors, and show that the nodal domains corresponding to the state matrix in (6) are identical to those of $L_m$, despite the influence of damping, as long as the system remains underdamped. Second, using these results we detail an algorithm to detect and localize inputs of the form $u(t) = f_0 \sin(\omega_f t)$ in (6) where $f_0$ and $\omega_f$ may not be known.

V. NODAL DOMAINS OF SECOND-ORDER NDS

In this section we study the relationship between the dynamics of second-order systems of the form (6) and the generalized nodal domains of the corresponding quadratic eigenvalue problem (QEP). These results will be used to derive the input localization algorithm presented in Section VI. For simplicity of notation, let $\Delta_0(t) = x_1(t) \in \mathbb{R}^n$ and $\Delta(t) = x_1(t) = x_2(t) \in \mathbb{R}^n$, then (6) may be written as

$$M \ddot{x}_1(t) + D \dot{x}_1(t) + Lx_1(t) = c_i u(t),$$

where $L = L(G)$, and $M$ and $D$ are defined as before. The modes (eigenvalues) and corresponding eigenvectors of (14) follow from corresponding QEP [12]. Specifically, they are scalars (eigenvalues) $\mu \in \mathbb{C}$ and vectors (right and left eigenvectors) $\psi, \phi \in \mathbb{C}^n$, $\phi \in \mathbb{C}^n$ that satisfy

$$(\mu^2 + \mu D + L)\psi = 0, \quad \phi^* (\mu^2 M + \mu D + L) = 0,$$

where $*$ denotes the conjugate transpose, and where $M$, $L$, and $D$ are $n \times n$ complex matrices. Solutions to (15) are largely dependent on the structure and properties of $M$, $L$, and $D$ [12], which we will note throughout this section. However, before preceding we must first define nodal domains for these eigenvectors since $\psi, \phi \in \mathbb{C}^n$, the $n$-dimensional complex space.

Definition 5.1: $D$ is a nodal domain of $\psi \in \mathbb{C}^n$ if $D$ is a nodal domain of $Re(\psi)$.

This is a generalization of Definition 2.1, and later we will see how it allows us to relate the dynamic response of the networked system (14) to the underlying graph $G$. Analogous to Lemma 4.1 for the first-order system, we have that the nodal domains of left and right eigenvectors are identical.

Lemma 5.1: Let $\psi \in \mathbb{C}^n$ and $\phi \in \mathbb{C}^n$ be right and left eigenvectors of (14) corresponding to eigenvalue $\mu \in \mathbb{C}$, then the nodal domains of $\psi$ and $\phi$ are identical.

Proof: Because $M, D,$ and $L$ in (14) are real symmetric matrices, the left and right eigenvectors coincide [12]. The nodal domain property follows by definition.

We will often restrict our analysis, without loss of generality, to right eigenvectors, denoted by $\psi$. Lastly, we can also formulate the state-variable representation of (6) as

$$\begin{bmatrix} \dot{x}_1(t) \\ M\dot{x}_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ e_i \end{bmatrix} u(t),$$

where

$$A = \begin{bmatrix} 0 & I_n \\ -L & D \end{bmatrix}$$

This state-variable representation will be used frequently in the next two sections—first to analyze solutions to (15) via the linearized QEP, and second to generalize residue sign pattern results in Lemma 4.2 for each output states in (8).

A. Undamped Systems

We first study the case where damping in (14) is ignored, i.e., assume $\lambda_i = 0, \forall i = 1, \ldots, n$. The system is said to be undamped, and (14) reduces to

$$M \ddot{x}_1(t) + Lx_1(t) = c_i u(t).$$

The solution to the QEP of (18) is well known. The nodal domains of $\psi$ are identical to those of $L(G)$ and there is a natural generalization of Lemma 4.2. These results are stated in the following theorem and corollaries.

Theorem 5.1: The nodal domains of $\psi$ in (15) with $D = 0$ are identical to the nodal domains of $L_m.$

Proof: The nodal domains of $L_m$ are equivalently the nodal domains of the eigenvectors $v \in \mathbb{R}^n$ which satisfy

$$(L_m - \lambda I_n) v = 0.$$  

Letting $D = 0$ in (15), we have

$$(\mu^2 M + L)v = 0.$$  

Taking advantage of the invertibility of $M$, it follows from direct substitution that $-\lambda = \mu^2$ and $\psi = v \in \mathbb{R}^n$. Since the eigenvectors are equivalent, so are the nodal domains.

The proof to Theorem 5.1 also reveals the well known form of the eigenvalues of (20), which are the mode pairs of (14) when $D = 0$. Consequently, we can naturally relate the sign patterns given in Lemma 4.2 to the second-order system (18) corresponding to output states $x_2$ and $x_1$. This is summarized in the next two corollaries, respectively.

Corollary 5.1: Assuming the input enters at node $k$ and output is $[x_2(t)]$, the matrix of every complex residue pair associated with eigenvalues $\pm \mu$ is $P = \lambda vw^T$, where $v$ and $w$ are the right and left eigenvectors of $L_m$ corresponding to $\lambda$.

Proof: Consider an equivalent representation of (18) as the set of $2n$ first-order differential equations

$$\begin{bmatrix} -I_n & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} I_n & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ e_i \end{bmatrix} u(t).$$

Solution to this linearized QEP are

$$\begin{bmatrix} -I_n & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \psi \psi \\ \mu \psi \end{bmatrix} = 0.$$  

where $\psi = v$ and $\mu = \pm j\sqrt{\lambda}$. The matrix of transfer function residues is $P = (\pm \mu v)(\pm \mu^2 \psi)^*$ $= \lambda vw^T$. This is precisely
the matrix of residue pairs for the first-order system scaled by a positive constant $\lambda \in \mathbb{R}$.

This result was stated in [1] without proof. The following corollary extends Corollary 5.1 if the output of (18) is $x_1(t)$.

**Corollary 5.2:** Assuming the input enters at node $k$ and output is $[x_1(t)]_k$, the matrix of every complex residue pair associated with eigenvalues $\pm \mu$ is $Q = \pm \mu uv^T$.

**Proof:** The proof is analogous to Corollary 5.1. ■

The two corollaries imply that $\text{sign}(\pm \mu Q) = -\text{sign}(P)$ for every element of $P$ and $Q$. This result enables the use of both phase angle and frequency measurements in (8) to localize $u(t)$ using the algorithm presented in Section VI. We next turn our attention to a more realistic scenario where the damped matrix $D$ is non-zero.

**B. Weakly Damped Systems**

Damping in synchronous generator models of a power system is typically small and is often assumed negligible for analysis, then added back to match the model’s response to actual measured data [5]. We use a similar approach to extend the analysis of the previous section to the weakly damped system (14) with $0 < d_i < \sqrt{m_i}/2, \forall i$. The solution to the corresponding QEP (15) has only complex eigenvalues, except the two corresponding to the DC modes, and these eigenvalues come in complex-conjugate pairs [12].

For the analysis in this section it will be useful to denote the complex conjugate pairs of eigenvalues $\mu = \alpha \pm j\beta$, where $\beta = \sqrt{\lambda + \delta \lambda}$. From (15), we also see that

$$\mu = \frac{-(D\psi, \psi) \pm \sqrt{(D\psi, \psi)^2 - 4(M\psi, \psi)(L\psi, \psi)}}{2(M\psi, \psi)}, \tag{23}$$

hence $\text{Re}(\mu) = \alpha < 0$. It is also clear from (23) that $\alpha \to 0$ and $\delta \lambda \to 0$ as $D \to 0 \in \mathbb{R}^{n \times n}$. We also denote the corresponding eigenvectors as $\psi = \xi + j\zeta$, where $\xi = v + \delta v$, and notice that $\zeta, \delta v \to 0$ as $D \to 0 \in \mathbb{R}^{n \times n}$. The lemma and theorem that follow show that the nodal domains of $\psi$ from (15) are invariant under a small real perturbation $D$, corresponding to the lightly damped model (14).

**Lemma 5.2:** Denote solution to the (15) of the weakly damped system as $\mu = \alpha \pm j\beta$ with $\beta = \sqrt{\lambda + \delta \lambda}$, and $\psi = \xi + j\zeta$, with $\xi = v + \delta v$, where $(-\lambda M + L)v = 0$ corresponds to the undamped solution to (15). Then the real eigenvector perturbation $\delta v$ is component-wise smaller than $v$, i.e. $|\delta v|_i < |v|_i, \forall i$.

The result of Lemma 5.2 is the proof to the following theorem.

**Theorem 5.2:** For the weakly damped system (14), the nodal domains of $\psi$ from (15) are identical to those of $L_m$.

**Proof:** Since $\text{Re}(\psi) = v + \delta v$, Lemma 5.2 guarantees that the element-wise sign pattern of $v$ is invariant. ■

Theorem 5.2 is an intuitive result from a system-theoretic perspective, where adding damping does not change the direction of the real component of the mode shape (the sign pattern of eigenvectors), but may change the slope of these mode shapes (corresponding to changing the magnitude of the real and imaginary eigenvector components) as long as the system remains underdamped. These results now allow us to apply Corollary 5.1 and Corollary 5.2 to weakly damped systems. In other words, although impulse response residues of (14) are neither purely real nor purely imaginary, we can build the estimated localization key using the sign pattern of either the real part of the residue (when the output is $[x_2(t)]_k$) or the imaginary part of the residue compared to the sign of the imaginary part of the pole (when the output is $[x_1(t)]_k$). This point will be illustrated in Section VII where we present our simulation results.

**VI. LOCALIZING THE RESONANT INPUT**

The remainder of the paper is devoted to the specific application of localizing forced harmonic inputs of the form $u(t) = f_0 \sin \omega t$. Our basic approach is as follows. Recalling Assumption 1, the system (14) consists of $p$ coherent areas, hence the inter-area dynamics are governed by $p - 1$ inter-area modes. Using the phase angles and frequency measurements available from sensors such as Phasor Measurement Units (PMU) at a selected number of generator buses, we first run a system identification routine to estimate the transfer function of the entire power system model (6), isolate the residues of the $p - 1$ slow poles, and finally construct a localization key and compare it with the table of keys $K$ described previously in Section IV.

**A. Transfer Function Derivations**

We begin by analyzing the input-output transfer function to see how the effective transfer-function residues are related to the impulse-response residues used in the localization algorithm. For the sake of clarity in the following derivations, we consider a single input and a single output in the undamped system (18), where the measured state is $x_1$ (phase-angle). Let the input enter node $k$ and let the output be measured at node $l$. We first assume that the incoming disturbance is impulsive, and run a subspace identification algorithm (e.g. the Eigensystem Realization Algorithm [13]). Since the network has $p$ areas the identified transfer function can be written in a truncated pole-residue form as

$$\frac{Y_i(s)}{U_k(s)} = \sum_{i=2}^{p} \frac{R_i}{(s^2 + \lambda_i)} + \tilde{g}(s), \tag{24}$$

where $\tilde{g}(s)$ contains the zero modes and the fast modes, $R_i \in \mathbb{R}$ is the residue corresponding to $\lambda_i = -\mu_i^2$, and $Y_i(s)$ and $U_k(s) = 1$ are the Laplace transforms of the output and input, respectively. The actual input to the system, however, is $u(t) = f_0 \sin \omega t$, the Laplace transform of which is $U(s) = f_0 \omega / (s^2 + \omega^2)$. Then the actual expression for $Y_i(s)$, assuming $f_0 = 1$ and neglecting the contribution from
\[ \hat{g}(s) \text{ without loss of generality, is} \]

\[ Y_1(s) = \left( \frac{\omega_f}{s^2 + \omega_f^2} \right) \sum_{i=2}^{p} \frac{R_i}{s^2 + \lambda_i} \]

\[ = \frac{1}{4} \left( \frac{j \omega_f}{s + j \omega_f} - \frac{-j \omega_f}{s - j \omega_f} \right) \times \sum_{i=2}^{p} \left( P_i^* + j Q_i \right) \left( s + \mu_i \right) + P_i^* - j Q_i \left( s - \mu_i \right) \right) \]

where \( \pm \mu_i = \pm j \sqrt{\lambda_i} \) and \( P_i^* \in \mathbb{R} \) and \( Q_i \in \mathbb{R} \) are the real and complex parts of the residues corresponding to modes \( \pm \mu_i \). Accordingly from Corollary 5.2, it is easy to verify \( P_i = 0 \) and \( j Q_i = \mu_i R_i \). Then, (26) can be written as

\[ Y_1(s) = \frac{1}{4} \sum_{i=1}^{p} \left[ \left( j \omega_f Q_i \right) \left( s + j \omega_f (s + \mu_i) \right) + \omega_f Q_i \left( s - j \omega_f (s + \mu_i) \right) \right] \]

\[ + \omega_f Q_i \left( s + j \omega_f (s - \mu_i) \right) - \omega_f Q_i \left( s - j \omega_f (s - \mu_i) \right) \]

\[ = \frac{1}{4} \sum_{i=1}^{p} \left[ \left( \frac{-j \omega_f Q_i}{s + j \omega_f (s + \mu_i)} + \frac{\omega_f Q_i}{s - j \omega_f (s + \mu_i)} \right) \right] \]

\[ + \left( \frac{-j \omega_f Q_i}{s + j \omega_f (s - \mu_i)} + \frac{\omega_f Q_i}{s - j \omega_f (s - \mu_i)} \right) . \]

(28)

It can be shown that the output decomposes into a component corresponding to the input frequency \( \omega_f \) and components corresponding to the system’s slow poles as

\[ Y_1(s) = \frac{1}{2} \sum_{i=1}^{p} \left[ \left( \frac{-j \omega_f Q_i}{s + j \omega_f (s + \mu_i)} + \frac{\omega_f Q_i}{s - j \omega_f (s + \mu_i)} \right) \right] \]

\[ + \left( \frac{-j \omega_f Q_i}{s + j \omega_f (s - \mu_i)} + \frac{\omega_f Q_i}{s - j \omega_f (s - \mu_i)} \right) . \]

(29)

From (29), we see that the effective residues associated with the system slow poles are

\[ \hat{R}_i = \frac{j \omega_f Q_i}{(s + \mu_i)} = \frac{\omega_f^2}{\omega_f^2 - \lambda_i} \mu_i R_i, \]

(30)

for \( i = 2, \ldots, p \). From (30) we see that not only do these effective residues grow in magnitude compared to their nominal values as the input frequency approaches the resonance condition \( \omega_f^2 = \lambda_i \), but they also change signs when \( \omega_f^2 < \lambda_i \). This will be accounted for in the following input localization algorithm.

**B. Input Localization Algorithm**

This section details an algorithm to localize the source of a forced harmonic input in a large-scale power system. We will also discuss the ability to quickly detect the input from a measurement-based perspective; however, the focus of the algorithm is localization.

As mentioned in Section IV, the algorithm for the first-order network requires minimal a-priori information in the form of localization keys, and the requirement is no different here. To recollect, a central operator has a nominal knowledge of the clustering structure of the network graph \( G \). Denote the nominal Laplacian matrix by \( L_m \). Using \( L_m \), construct a \( p \times p \) array of input localization keys \( K = [K_{ij}] \). Construction of this key is a one-time task as long as the clustering remains the same as \( L_m \). In practice, this means \( K \) can be used despite changes in real-time operating conditions, such as loss of generators and tie-lines, so long as the clustering structure does not change. Once this table of keys has been computed, the \( l^{th} \) row of the table is associated with Area \( l \) (i.e. given to the local operator of Area \( l \)), which encompasses the node set \( \mathcal{V}^l \). The local operator taking measurements at \( q_i \in \mathcal{V}^l \) runs a system identification procedure to estimate the transfer function \( Y_i(s) = \hat{g}_{\text{slow}}(s) \).

If the number of slow modes detected exceeds the expected number, i.e., the transfer function contains \( \hat{p} > p \) modes in the slow spectrum, then we say that the input pole has been detected. Once the pole has been detected, the operator attempts to distinguish the input from the system’s natural response, after which the residue signs can be calculated according to equation (30). The input localization steps are summarized in Algorithm 1. It should be noted that the complexity of the first step in the algorithm—the system identification subroutine—is by far the most complex and time consuming step.

**C. Distinguishing the Exogenous Mode**

The unwanted input is easily detected in Algorithm 1 if the system identification subroutine successfully identifies \( \omega_f \) in addition to the slow poles of (14). However, distinguishing the unwanted mode \( \omega_f \) from \( \{ \lambda_i \}, i = 1, \ldots, p \) requires additional information and additional computation. In this section we consider the two cases where \( \omega_f^2 \neq \lambda_i \) for all \( i = 1, \ldots, p \), and...
where \( \omega_f^2 \rightarrow \lambda_i \) for some \( i, i = 2, \ldots, p \).

First, we consider these cases in the weakly damped system (14). The presence of any damping in the natural frequencies is enough information to distinguish \( \omega_f \) which is assumed to have zero damping. The distinguishing step, line 7, in Algorithm 1 can simply include an all-to-all comparison of \( \text{Re}(\mu_i) \). In the first scenario when \( \omega^2 \neq \lambda_i, \forall i = 1, \ldots, p \), this procedure results in an additional \( \binom{p}{2} \) operations. In the second case, when \( \omega_f \) approaches the resonance condition, this procedure can be reduced to a single additional operation as the candidate modes have been previously identified in the algorithm (line 2).

In a second scenario, we consider the response of the undamped system (18). Here damping cannot be used as a distinguishing factor. However, if one has a-priori information about the magnitude of the nominal residues \( \bar{R}^k \) and \( \omega_f^2 \neq \lambda_i, \forall i = 2, \ldots, p \), then one could formulate a non-linear least-squares problem using this information with the effective residues described by (30) to determine \( \omega_f \) from a set of candidates. Namely,

\[
\omega_f = \arg \min_{\omega} \| \bar{R}^k - f_k(\omega; \lambda_2, \ldots, \lambda_p) \| \tag{31}
\]

where

\[
f_k(\omega; \lambda_2, \ldots, \lambda_p) = \frac{\omega^2}{(\omega - \lambda_i)} \mu_i \bar{R}^k, \tag{32}
\]

and \( \bar{R}^k \) is assumed known. However, as \( \omega^2 \rightarrow \lambda_i \), the effective residue of the input also approaches the magnitude of its \( \lambda_i \) counterpart, making it impossible to distinguish \( \omega_f \) from \( \lambda_i \). The ‘distinguish’ step in Algorithm 1, therefore, works best when the system is weakly damped.

VII. Simulation Results

We demonstrate the application of Algorithm 1 to large-scale power networks by simulating a 60-generator power system network. The simulated network is spatially and dynamically clustered into four areas as illustrated in Fig. 1. The network was constructed by taking the Cartesian product\(^1\) of two graphs \( G_1 \) which resembles the wide-area (aggregate) network and \( G_2 \) which is a random graph that composes each of the local clusters. The specific parameters including node weights, edge weights, and damping are arbitrary, but again we emphasize that the network should retain the clustered structure. To perform the a-priori analysis, a central operator constructs the \( 4 \times 4 \) array of input localization keys \( K \). This can be done efficiently by selecting a single test node from each area, running an eigenvalue decomposition on the nominal Laplacian \( \bar{L}_m \), then doing an all-to-all sign comparison of the corresponding test node elements of the slow-mode eigenvectors. The array of localization keys for this simulation is given in Table I.

We next simulate a disturbance input \( u(t) = \sin \omega_f t \) with \( \| \omega_f^2 - \lambda_3 \| \leq 0.001 \). The forcing function enters at node 32 which is in Area 3 (highlighted in red). Output responses from a node in each area are shown in Fig. 2a to illustrate the effects of the resonance condition as measured by local system operators. The specific nodes (highlighted in blue in Fig. 1) are chosen so that Assumption 2 has been met.

Finally, we illustrate how a particular local operator, in this case in Area 1, proceeds through Algorithm 1. A system identification is run using both state measurements from node 4, resulting in a matrix of two transfer functions. The recovered phase angle and frequency are compared against the measurements in Fig. 2. The close matching indicates that the system identification from the measurements is accurate (Assumption 2 has been met). Next, the system poles are distinguished from the forcing frequency \( \omega_f \). The poles and corrected residues are shown in Table II and Tables III. Next, two estimated localization keys are built by applying Corollaries 5.1 and 5.2 to the respective table. In both cases the estimated key is \( \{-, +, -\} \). Comparing the estimated key to the first row of Table I (corresponding to outputs from

\(^1\)See, e.g. [14], for details on network construction using the Cartesian product for graphs.

---

**TABLE I**

<table>
<thead>
<tr>
<th>Area</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{+, +, +}</td>
<td>{-, -, -}</td>
<td>{-, +, -}</td>
<td>{-, +, +}</td>
</tr>
<tr>
<td>2</td>
<td>{+, +, +}</td>
<td>{+, +, -}</td>
<td>{+, +, +}</td>
<td>{-, +, -}</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Fig. 1. Large-scale network used for simulation examples. The clustering is apparent. The area assignments are ordinal: \( V^1 = \{1 : 15\} \), \( V^2 = \{16 : 30\} \), \( V^3 = \{31 : 45\} \), and \( V^4 = \{46 : 60\} \). The graph was constructed by taking the Cartesian product of two smaller graphs: \( G_1 \) is a four node graph the resembles the aggregate structure, and \( G_2 \) is a random graph on 15 nodes. Spacing between vertices is approximately (effective resistance)\(^{1/3}\). Measurement nodes are highlighted in blue, while the disturbance location is highlighted in yellow.
Area 1), the operator finds a match in column 3 and only in column 3. This indicates that the input must have occurred in Area 3—which is indeed the case.

VIII. CONCLUSION

In this paper we presented a measurement-based graph-theoretic algorithm to localize potentially dangerous sub-synchronous resonant inputs in the swing dynamics of large interconnected power systems. Such harmonic inputs can be triggered in synchronous machines due to internal failures or interconnections. Modeling the network with weakly damped second-order swing dynamics, we established an extension of our previous work reported for consensus networks in [1]. We established the relationship between the nodal domains of the eigenvectors of such coupled oscillation dynamics and the corresponding modal residues. Assuming the network to be divided into $p$ distinct clusters, we finally developed an algorithm that uses the residues of a system transfer function, obtained from system identification using Synchrophasor measurements, to localize the cluster in which the resonant input may have occurred. We demonstrated the execution of this algorithm with a numerical example. Future work in this direction will include generalizing the detection and correction procedures in the algorithm to more generic classes of network faults with unknown functional forms.

REFERENCES