Distance Characterization and Input Localization in Networked Dynamic Systems

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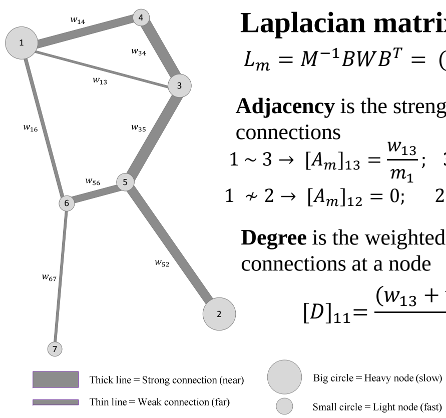
Introduction

"How far?" has different answers depending on the setting and context. For example in Manhattan the answer may be 5 blocks, or 1 mile (heading north-south), and 5 blocks, or ¼ of a mile (heading east-west).

In networked dynamic systems (NDS) this question and its answers have applications in input localization. To understand how distance plays a role in the system dynamics we write the LTI dynamic equations in terms of a node- and edge-weighted graph Laplacian matrix[1], or equivalently the asymmetric combinatorial Laplacian. We then formulate the definitions of distance in terms of this Laplacian and relate these distances to the system outputs.

This poster introduces the node- and edge-weighted graph and the associated graph Laplacian, defines distances in terms of this Laplacian, and presents input localization results for simple topologies.

Networked Dynamic System



Laplacian matrix

$$L_m = M^{-1}BWB^T = (A_m - D_m)$$

Adjacency is the strength of the neighboring connections

$$1 \sim 3 \rightarrow [A_m]_{13} = \frac{w_{13}}{m_1}; \quad 3 \sim 1 \rightarrow [A_m]_{31} = \frac{w_{13}}{m_3}$$

$$1 \nsim 2 \rightarrow [A_m]_{12} = 0; \quad 2 \nsim 1 \rightarrow [A_m]_{21} = 0$$

Degree is the weighted sum of all connections at a node

$$[D]_{11} = \frac{(w_{13} + w_{14} + w_{16})}{m_1}$$

State-space model

$$\dot{x}(t) = -L_m x(t) + M^{-1} b_i u(t)$$
$$y(t) = c_j^T x(t)$$

b_i and c_j are unit vectors

Transfer function

$$g(s) = c_j^T (sI + L_m)^{-1} M^{-1} b_i$$
$$= \sum_{k=1}^{n} \frac{R_{ij}^k}{s - \lambda_k}$$

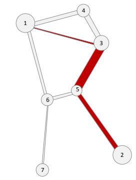
Residue is the influence of the k^{th} eigenvalue on the evolution of the input-output pair *(i,j)*

$$R_{ij}^k = (c_j^T v_k)(w_k^T M^{-1} b_i)$$

where v_k and w_k are the associated right and left eigenvectors, respectively. *Residues in the asymmetric system are symmetric*

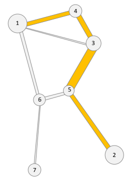
$$R_{ij}^k = R_{ji}^k$$

Distances



Hop distance is the minimum number of edges between two nodes. The hop distance is related the relative degree of the system

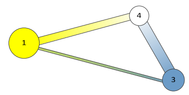
$$h(i,j) = r - 1 = \min a \ni L^a_{m_{ij}} \neq 0$$



Geodesic distance is the shortest weighted path between two nodes. This distance is asymmetric

$$d(i,j) = \min \sum_{k,l \in path(i \rightarrow j)} L_{m_{kl}}$$

Effective distance is analogous to the effective resistance in an electrical network

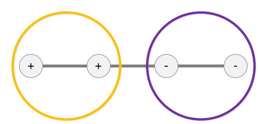


$$\Omega(i,j) = L^+_{m_{ii}} + L^+_{m_{jj}} - L^+_{m_{ij}} - L^+_{m_{ji}}$$

"+" indicates the Moore-Penrose psuedoinverse

Input Localization

Weak Nodal Domain Theorem guarantees there are exactly two domains, one positive and one negative, in a connected graph corresponding to the second smallest eigenvalue λ_2 [2]. The sign of node j corresponds to the sign of the j^{th} element in v_2 (or w_2)



$$R_{ij}^2 = c_j^T v_2 w_2^T M^{-1} b_i$$

$$sign(v_2) = sign(w_2) = \begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix}$$

$$sign(R^2) = \begin{bmatrix} + & + & - & - \\ + & + & - & - \\ - & - & + & + \\ - & - & + & + \end{bmatrix}$$

This means the sign of the residue is *positive* for *(i,j)* pairs within a domain, and *negative* for pairs between adjacent domains

References

[1] S.Y. Shafi, M. Arcak, and L. El Ghaoui. Designing node and edge weights of a graph to meet laplacian eigenvalue constraints. In Communication, Control, and Computing (Allerton), 2010 48th Annual Allerton Conference on, pp. 1016—1023, 29 October 2010.

[2] Biyikoglu Turker, Josef Leydold, and Peter F. Stadler. *Laplacian Egienvectors of Graphs*. New York:Springer, 2007, pp.26—27.

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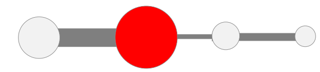
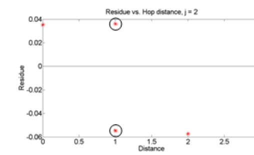
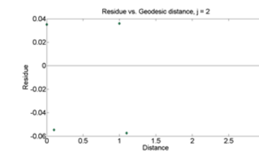
Examples

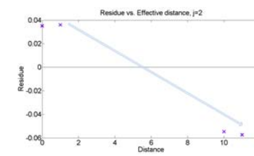
These examples demonstrate input localization in a first order LTI system for simple two-area topologies where the nodal domains also correspond to area partitioning.. The residues and relative degree of the system can be estimated with measurements taken at a fixed output node (red).

Line graph with output taken at the largest node(j =2)





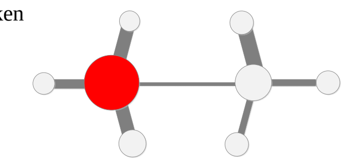


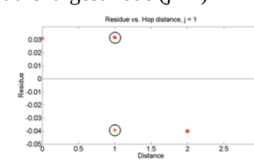


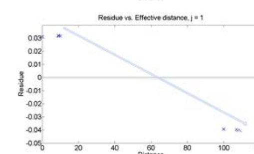
There is not correlation between the geodesic distance and residue

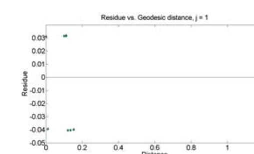
Notice the monotonic decrease in residue vs effective distance, and the negative residues for inter-area i/o pairs (highlighted in the Hop distance plot)

Star areas with the output taken at the largest node (j =1)









The star topology follows the monotonic decrease in residue for distances > 0, and the sign change for inter-area i/o pairs

Future work

We would like to know how to guarantee monotonicity in the residues with respect to distance, and how to guarantee that the nodal domains align with the optimal area partitioning

We are also working on using higher-order nodal domains for high resolution input localization in large networked dynamic systems