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On cloaking for elasticity and physical equations with a transformation invariant form

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Abstract. In this paper, we investigate how the form of the conventional elastodynamic equations changes under curvilinear transformations. The equations get mapped to a more general form in which the density is anisotropic and additional terms appear which couple the stress not only with the strain but also with the velocity, and the momentum gets coupled not only with the velocity but also with the strain. These are a special case of equations which describe the elastodynamic response of composite materials, and which it has been argued should apply to any material which has microstructure below the scale of continuum modelling. If composites could be designed with the required moduli then it could be possible to design elastic cloaking devices where an object is cloaked from elastic waves of a given frequency. To an outside observer it would appear as though the waves were propagating in a homogeneous medium, with the object and surrounding cloaking shell invisible. Other new elastodynamic equations also retain their form under curvilinear transformations. The question is raised as to whether all equations of microstructured continua have a form which is invariant under curvilinear space or space-time coordinate transformations. We show that the non-local bianisotropic electrodynamic equations have this invariance under space-time transformations and that the standard non-local, time-harmonic, electromagnetic equations are invariant under space transformations.

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1. Introduction

Invisibility has a long history. Maxwell [1] was aware that no magnetic field would be produced outside a torus with a constant poloidal current flowing on its surface: that it would be like a magnet bent around in a circle, with its poles in contact. In other words it would be magnetically invisible from the outside. Many other current sources turn out to be electromagnetically invisible. Curiously Afanasiev and Dubovik [2] show that there exist current sources for which the electric and magnetic fields are zero outside the sources, but the electromagnetic potential is not, and could theoretically be detected quantum mechanically using the Aharonov–Bohm effect.

Mansfield [3] discovered that certain reinforced holes, which he called neutral inclusions, could be cut out of a uniformly stressed plate without disturbing the stress surrounding the hole. In other words by appropriately reinforcing the hole, one could make it invisible to a uniform field. Hashin [4] found that certain coated elastic spheres could be inserted into a hydrostatically stressed isotropic material without disturbing the field outside each coated sphere (see also the related paper by MacKenzie [5]). Hashin and Shtrikman [6] extended this to conductivity, finding that inserting certain coated spheres would leave undisturbed a uniform current field in the surrounding material. Kerker [7] found that coated confocal ellipsoids could be invisible to long wavelength planar electromagnetic waves, and Chew and Kerker [8] showed that by adjusting both the electrical permittivity and magnetic permeability of the coating and core of the coated spheres, the corrections could be made proportional to the eighth power of the wavenumber (basically the inverse of the wavelength). Alú and Engheta [9] found that an appropriately coated sphere did not have to be exceedingly small compared to the wavelength to get reasonable invisibility for planar electromagnetic waves: the scattering cross-section could be small for spheres with cores as large as one fifth of the wavelength. For more examples of neutral inclusions see subsection 7.11 of [10] and references therein. Another type of invisibility was discovered by Ramm [11] who found that for scattering of a plane scalar wave from an obstacle, one could apply controls on the field on an arbitrarily small section of the boundary to make the total scattering from the object arbitrarily small. Fedotov *et al* [12] found that a certain fish scale structure of copper strips was invisible to polarized normally incident plane waves of a given frequency.

What is remarkable is that one can get invisibility to arbitrary fields, without having to adapt the coating (or the controls) to the applied field. Kohn and Vogelius [13] (following a suggestion of Luc Tartar) found that one could get invisibility to arbitrary DC electric fields by coating the inclusion to be made invisible by an anisotropic coating. Not all inclusions can be made invisible in this way but in three dimensions (3D), it can be done for any shaped inclusion with constant conductivity. The key feature of the conductivity equations which allows this invisibility is the invariance of their form under coordinate transformations. This invariance (see also subsection 8.5 of [10]) is a corollary of the invariance of Maxwell's equations under coordinate transformations as shown by Post [14]. Specifically, given a potential $V(\mathbf{x})$ satisfying the conductivity equations

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) \nabla V(\mathbf{x}) = f(\mathbf{x}), \tag{1.1}$$

for a chosen conductivity tensor field $\boldsymbol{\sigma}(\mathbf{x})$ and source term $f(\mathbf{x})$, we make the mapping from each point $\mathbf{x}$ in a cartesian space to a point $\mathbf{x}'(\mathbf{x})$ in a different cartesian space, and we let $V'(\mathbf{x}'(\mathbf{x})) = V(\mathbf{x})$ be the new potential in the new space. Then it turns out that $V'(\mathbf{x}')$ satisfies conductivity equations in the new space,

$$\nabla' \cdot \boldsymbol{\sigma}'(\mathbf{x}') \nabla' V'(\mathbf{x}') = f'(\mathbf{x}'), \tag{1.2}$$

with

$$\boldsymbol{\sigma}'(\mathbf{x}') = \mathbf{A}\boldsymbol{\sigma}(\mathbf{x})\mathbf{A}^{\mathrm{T}} / \det \mathbf{A}, \quad f'(\mathbf{x}') = f(\mathbf{x}) / \det \mathbf{A}, \tag{1.3}$$

where $\mathbf{A}$ is the matrix with elements

$$A_{ki} = \frac{\partial x'_k}{\partial x_i}. \tag{1.4}$$

(Often this matrix is denoted by $\mathbf{A}^{\mathrm{T}}$ but the above choice of notation is consistent with that of Milton [10]).

Following Kohn and Vogelius [13], suppose we wanted to make a 3D isotropic body occupying a region Ω' invisible, when the body and the exterior matrix have isotropic conductivities σ'_0 and σ_0 respectively. For a point $\mathbf{x}' \in \Omega'$ let $\mathbf{x} = (\sigma'_0/\sigma_0)\mathbf{x}' + \mathbf{b}$, where the shift $\mathbf{b}$ is to be chosen appropriately. Let Ω be the image of Ω' under this linear map. We choose a region Γ containing both Ω and Ω', and we suppose that Γ is free of sources, i.e. $f(\mathbf{x})$ is zero there. We take $\boldsymbol{\sigma}(\mathbf{x}) = \sigma_0\mathbf{I}$ and set $\mathbf{x}' = \mathbf{x}$ outside Γ. We choose the mapping $\mathbf{x}'(\mathbf{x})$ to be continuous and to be a bijective map from $\Gamma - \Omega$ on to $\Gamma - \Omega'$. Then $\boldsymbol{\sigma}'(\mathbf{x}')$ will be $\sigma_0\mathbf{I}$ outside Γ and $\sigma'_0\mathbf{I}$ inside Ω' and anisotropic in between. The fields outside Γ will be exactly the same as with the inclusion absent, no matter what the sources and no matter what the boundary conditions. The inclusion will be invisible to all fields. Of course this invisibility extends to dielectric materials in the quasistatic limit. Another quite different type of invisibility to arbitrary fields was discovered by Nicorovici *et al* [15]. They found that a coated cylinder with a core of dielectric constant 1, and a coating of dielectric constant $-1 + \mathrm{i}\delta$, where the loss δ is vanishingly small, embedded in a matrix of dielectric constant 1 would be invisible to all quasistatic transverse magnetic (TM) waves.

Besides invisibility there is what we call cloaking where the surrounding material does not have to be carefully adapted to suit the object to be made invisible. The cloaking device may be invisible or visible, although obviously the former is more interesting. Thus the simplest way

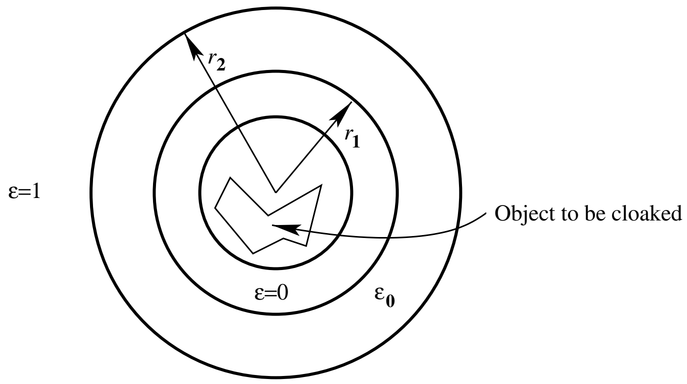

Figure 1. When $\varepsilon_0 = (1 + f_2)/(2f_2)$ where $f_2 = (1 - r_1^3/r_2^3)$, the inclusion will be invisible to long wavelength electromagnetic waves and cloak the dielectric object inside it. It is assumed that the magnetic permeability is close to one everywhere.

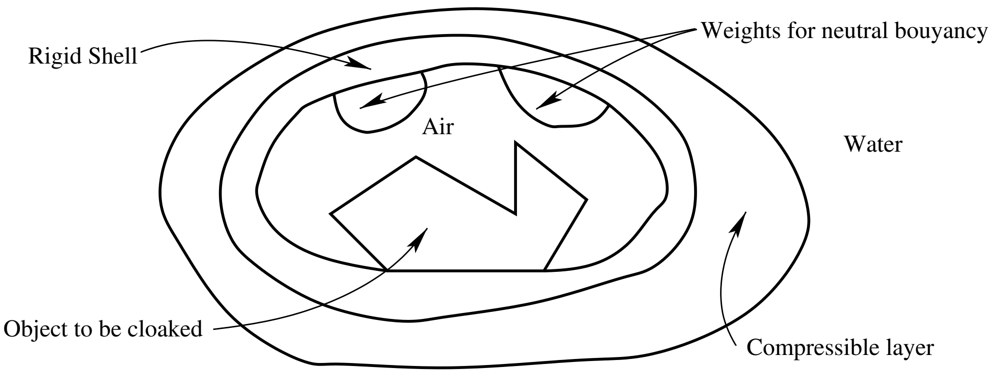

Figure 2. A simple way of cloaking an object to long wavelength acoustic waves.

to cloak a dielectric inclusion from all exterior fields is to surround it by a perfectly conducting spherical shell or, following the ideas of Engheta *et al* [16], a shell having permittivity ε close to zero (the dielectric analogue of a near zero conducting shell). Then following the construction of Hashin and Shtrikman [6], we can make the shell (cloaking device) invisible to uniform electric fields, as illustrated in figure 1. So assuming that μ is close to one everywhere, we get a device which will be invisible and cloak dielectric objects to long wavelength planar electromagnetic waves. Similarly one can build a simple acoustic cloaking device which will work when long wavelength acoustic underwater waves are incident on the object to be cloaked as illustrated in figure 2. First surround the object by a light rigid shell, and attach the object firmly to the inside of the shell. (Here, we assume that the object to be cloaked does not have any viscoelastic or loose components and to a good approximation obeys Newton's law of motion.) Then surround the rigid shell by, for simplicity, a layer of isotropic material that has bulk modulus less than that of water. The thickness of the latter is chosen so that the fractional decrease in volume of the system (enclosed by the outer boundary of the layer) per unit increase in water pressure, exactly matches the bulk modulus of the water. Then rigid weights are added to the inside of the shell so the average density of the system (enclosed by the outer boundary of the layer, and the weight of the object to be cloaked) matches the density of water, so it is neutrally buoyant.

Recently there has been significant progress on cloaking objects to arbitrary fields. This began with the work of Greenleaf *et al* [17, 18]. We now outline their idea, since it is the same basic idea as we are trying to exploit to obtain cloaking for elastodynamics. Suppose the conductivity equations are satisfied in say a homogeneous isotropic medium, with conductivity $\sigma_0\mathbf{I}$. Consider a transformation where outside a radius $|\mathbf{x}| = r_2$ the mapping is the identity mapping $\mathbf{x}' = \mathbf{x}$. We assume there are no sources inside the radius r_2 (i.e. $f(\mathbf{x}) = 0$ for $|\mathbf{x}| < r_2$). Inside a tiny radius r_1 the mapping is $\mathbf{x}' = (r_1'/r_1)\mathbf{x}$, where r_1' is chosen less than r_2, say $r_1' = r_2/2$. For $r_2 > |\mathbf{x}| > r_1$, we choose

$$\mathbf{x}' = \alpha\mathbf{x} + \beta\mathbf{x}/|\mathbf{x}|, \quad \alpha = \frac{r_2 - r_1'}{r_2 - r_1}, \quad \beta = \frac{r_2(r_1' - r_1)}{r_2 - r_1}, \tag{1.5}$$

so the entire mapping is continuous (with $\det \mathbf{A} > 0$). For this mapping

$$\mathbf{A} = \mathbf{I} \text{ for } |\mathbf{x}| > r_2, = (\alpha + \beta/|\mathbf{x}|)\mathbf{I} - \beta\mathbf{x}\mathbf{x}^T/|\mathbf{x}|^3 \text{ for } r_2 > |\mathbf{x}| > r_1, = (r_1'/r_1)\mathbf{I} \text{ for } |\mathbf{x}| < r_1. \tag{1.6}$$

According to (1.3), the new conductivity tensor is therefore

$$\begin{aligned}\boldsymbol{\sigma}' = \sigma_0\mathbf{A}^2/\det\mathbf{A} &= \sigma_0\mathbf{I} \quad \text{for } |\mathbf{x}| > r_2,\\ &= \frac{\sigma_0(\alpha + \beta/|\mathbf{x}|)^2\mathbf{I} - \sigma_0(2\alpha + \beta/|\mathbf{x}|)^2\mathbf{x}\mathbf{x}^T/|\mathbf{x}|^3}{\alpha(\alpha + \beta/|\mathbf{x}|)^2} \quad \text{for } r_2 > |\mathbf{x}| > r_1,\\ &= \sigma_0(r_1/r_1')\mathbf{I} \quad \text{for } |\mathbf{x}| < r_1,\end{aligned} \tag{1.7}$$

which could be re-expressed as functions of $\mathbf{x}'$ although we will not do so. Now the key idea is to modify the conductivity $\boldsymbol{\sigma}'(\mathbf{x}')$ in the region $|\mathbf{x}'| < r_1'$ (i.e. $|\mathbf{x}| < r_1$) by say inserting the conducting object to be cloaked there. This causes a corresponding change in the original conductivity tensor field $\boldsymbol{\sigma}(\mathbf{x})$ but only within the tiny region $|\mathbf{x}| < r_1$. Consequently this causes only a tiny change in the potential $V(\mathbf{x})$ outside the radius r_2 no matter what the conductivity of the object being cloaked, and this tiny change goes to zero in the limit as $r_1 \to 0$. Thus the object, and the surrounding anisotropic conducting shield, become invisible as $r_1 \to 0$.

For quasistatics Milton and Nicorovici [19] discovered that a coated cylinder having a shell dielectric constant close to -1, a matrix and core dielectric constant close to one and core and shell radii of r_c and r_s, would cloak an arbitrary number of polarizable line dipoles aligned with the cylinder axis when they all lie within the specific radius $r_\# = \sqrt{r_s^3/r_c}$ from the centre of coated cylinder. As the loss (the imaginary part of the dielectric constant) goes to zero all these polarizable line dipoles become invisible from outside the radius $r_\#$ to all quasistatic TM waves. Furthermore a finite energy dipole source in the cloaking region becomes cloaked with all its energy being funnelled towards the coated cylinder and vanishingly little escaping beyond the radius $r_\#$. (A related trapping of electromagnetic fields between two opposing sources outside the Veselago lens was discovered by Cui *et al* [20] and Boardman and Marinov [21].) Cloaking still occurs when the core has a dielectric constant $\varepsilon_c \neq 1$ and polarizable line dipoles within the larger radius $r_* = r_s^2/r_c$ get cloaked but the cloaking device (the coated cylinder) becomes visible. Such coated cylinders were studied by Nicorovici *et al* [15] where it was discovered that a line dipole source positioned at a radius r_0 less than r_s^3/r_c^2 would have an image line dipole positioned outside the coated cylinder at the radius r_*^2/r_0. Although not recognized at the time this was the first discovery of superresolution, i.e. resolution of an image finer than the wavelength. It was also discovered that the field had enormous oscillations on the side of the image closest to the coated cylinder. This may have been the first discovery of anomalous localized resonance, where as the loss goes to zero the field blows up to infinity throughout an entire region, not bounded by discontinuities in the moduli, but approaches a smooth field outside this region. A rigorous proof of anomalous localized resonance was given in the paper by Milton *et al* [22] where some small errors in the earlier paper were corrected. Anomalous localized resonance is actually the mechanism behind this cloaking. Cloaking occurs when the resonant field generated by a polarizable line or point dipole acts back on the polarizable line or point dipole and effectively cancels the field acting on it from outside sources, so it has essentially no response to the external field. Numerically it was also found that the polarizable line or point dipole is effectively invisible to the external time harmonic field. Cloaking was also shown to extend to the Veselago lens [23] for the full Maxwell equations: a polarizable line dipole located less than a distance $d/2$ from the lens, where d is lens thickness, is cloaked due to the presence of a resonant field in front of the lens. Also a polarizable point dipole near a slab lens was shown be cloaked in the quasistatic limit. Finally cloaking was shown for the lossless Veselago lens with time evolving fields [24]. The recent discovery by Fang *et al* [25] of ultrasonic metamaterials for which the group and phase velocities are in opposite directions, presents the possibility that this type of cloaking might extend to acoustics.

Leonhardt *et al* [26, 27] and independently Pendry *et al* [28] discovered a mechanism for electromagnetic cloaking which was similar to that of Greenleaf *et al* [17, 18] for conductivity. The central and beautiful idea was to guide electromagnetic radiation around the region to be cloaked. Leonhardt's study was for 2D geometric optics whereas the study of Pendry *et al* [28] was for the full Maxwell's equations at fixed frequency. The key to the study of Pendry *et al* [28] is the result of Ward and Pendry [29], which is a corollary of the result of Post [14] (see also [30]), that Maxwell's equations are form invariant under coordinate transformations. Specifically, Maxwell's equations at fixed frequency

$$\nabla \times \mathbf{E} + \mathrm{i}\omega\boldsymbol{\mu}\mathbf{H} = 0, \quad \nabla \times \mathbf{H} - \mathrm{i}\omega\boldsymbol{\varepsilon}\mathbf{E} = 0, \tag{1.8}$$

with (possibly complex and possibly anisotropic) magnetic permeability $\boldsymbol{\mu}(\mathbf{x})$ and electrical permittivity $\boldsymbol{\varepsilon}(\mathbf{x})$ under the mapping transformation

$$\mathbf{x}' = \mathbf{x}'(\mathbf{x}), \quad \mathbf{E}'(\mathbf{x}') = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{E}(\mathbf{x}), \quad \mathbf{H}'(\mathbf{x}') = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{H}(\mathbf{x}), \tag{1.9}$$

where $\mathbf{A}$ is given by (1.4), retain their form

$$\nabla' \times \mathbf{E}' + \mathrm{i}\omega\boldsymbol{\mu}'\mathbf{H}' = 0, \quad \nabla' \times \mathbf{H}' - \mathrm{i}\omega\boldsymbol{\varepsilon}'\mathbf{E}' = 0, \tag{1.10}$$

with new tensors

$$\boldsymbol{\mu}'(\mathbf{x}') = \mathbf{A}\boldsymbol{\mu}(\mathbf{x})\mathbf{A}^{\mathrm{T}}/\det\mathbf{A}, \quad \boldsymbol{\varepsilon}'(\mathbf{x}') = \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{x})\mathbf{A}^{\mathrm{T}}/\det\mathbf{A}, \tag{1.11}$$

which is the same transformation as for the conductivity tensor in (1.3). For the convenience of the reader, we provide an alternative proof of this result in appendix A. The crux of their analysis is then a similar argument (made independently) as that of Greenleaf *et al* [17, 18]. The technical question remains as to what constraints need to be placed on the object being cloaked to ensure that outside the radius r_2 the fields $\mathbf{E}' = \mathbf{E}$ and $\mathbf{H}' = \mathbf{H}$ are unperturbed by the presence of the cloaking object in the limit $r_1 \to 0$.

2. Transformation of the elasticity equations and cloaking

The electromagnetic cloaking of Leonhardt [26, 27] was for the 2D Helmholtz equation in the limit of geometric optics. As such his analysis immediately applies to the 2D acoustic wave equation in the small wavelength limit. This left open the question of whether one could get acoustic or elastic cloaking under less restrictive conditions.

In this section, we consider the transformation of the elastodynamic wave equation

$$\nabla \cdot \boldsymbol{\sigma} = -\omega^2 \rho \mathbf{u}, \quad \boldsymbol{\sigma} = \mathbf{C}\nabla\mathbf{u}, \tag{2.1}$$

under the mapping $\mathbf{x} \to \mathbf{x}'(\mathbf{x})$ and $\mathbf{u}(\mathbf{x}) \to \mathbf{u}'(\mathbf{x}')$. In this section, we let $\boldsymbol{\sigma}(\mathbf{x})$ denote the stress field, not the conductivity field. We are free to choose any mappings $\mathbf{x} \to \mathbf{x}'$ and $\mathbf{u}(\mathbf{x}) \to \mathbf{u}'(\mathbf{x}')$ but if we wish to preserve the symmetries of the elasticity tensor we need to take

$$\mathbf{u}'(\mathbf{x}') = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{u}(\mathbf{x}), \quad \text{where } A_{ij} = \frac{\partial x'_i}{\partial x_j}. \tag{2.2}$$

Notice that ordinarily if we think of $\mathbf{u} = \mathbf{x}_a - \mathbf{x}_b$ as a (rescaled) vector separating two points $\mathbf{x}_a$ and $\mathbf{x}_b$ spaced an infinitesimal distance apart in $\mathbf{x}$-space, then one would physically expect that $\mathbf{u}'$ should be the (rescaled) vector separating the image points. This would imply that

$$\mathbf{u}'(\mathbf{x}_a) = \mathbf{x}'(\mathbf{x}_a) - \mathbf{x}'(\mathbf{x}_b) = \mathbf{x}'(\mathbf{x}_a) - \mathbf{x}'(\mathbf{x}_a - \mathbf{u}) = \frac{\partial \mathbf{x}'}{\partial x_i} u_i = \mathbf{A}\mathbf{u}, \tag{2.3}$$

which obviously differs from (2.2). However, mathematically the transformation given by (2.2) is not prohibited, since after all it can be viewed as just a change of variables. In appendix B, we show that the elastodynamic wave equation (2.1) actually *changes its form* under the mapping transformation (2.2) and transforms to the equations

$$\nabla' \cdot \boldsymbol{\sigma}' = \mathbf{D}'\nabla'\mathbf{u}' - \omega^2 \boldsymbol{\rho}'\mathbf{u}', \quad \boldsymbol{\sigma}' = \mathbf{C}'\nabla'\mathbf{u}' + \mathbf{S}'\mathbf{u}', \tag{2.4}$$

where the density is now matrix valued and it and the new elasticity tensor have elements

$$\rho'_{pq} = \frac{\rho}{a}\frac{\partial x'_p}{\partial x_i}\frac{\partial x'_q}{\partial x_i} + \frac{1}{a}\frac{\partial^2 x'_p}{\partial x_i \partial x_j} C_{ijk\ell} \frac{\partial^2 x'_q}{\partial x_k \partial x_\ell}, \qquad C'_{pqrs} = \frac{1}{a}\frac{\partial x'_p}{\partial x_i}\frac{\partial x'_q}{\partial x_j} C_{ijk\ell} \frac{\partial x'_r}{\partial x_k}\frac{\partial x'_s}{\partial x_\ell}, \tag{2.5}$$

in which $a = \det(\mathbf{A})$ is the Jacobian and the third-order tensors $\mathbf{D}'(\mathbf{x}')$ and $\mathbf{S}'(\mathbf{x}')$ entering the new terms in the equations have elements

$$S'_{pqr} = \frac{1}{a}\frac{\partial x'_p}{\partial x_i}\frac{\partial x'_q}{\partial x_j} C_{ijk\ell} \frac{\partial^2 x'_r}{\partial x_k \partial x_\ell} = S'_{qpr}, \qquad D'_{pqr} = \frac{1}{a}\frac{\partial^2 x'_p}{\partial x_i \partial x_j} C_{ijk\ell} \frac{\partial x'_q}{\partial x_k}\frac{\partial x'_r}{\partial x_\ell} = S'_{qrp}. \tag{2.6}$$

The analysis given in appendix B is more general, showing that any equation of the form (2.4) preserves its form under the mapping transformation (2.2) and provides expressions for the new moduli in terms of the old moduli.

The concept of a mass density matrix may be foreign. However Willis [31] demonstrated rigorously, for a model composite in which only the density varied, that the effective density operator was indeed a second-order tensor. A simple model with an anisotropic density matrix

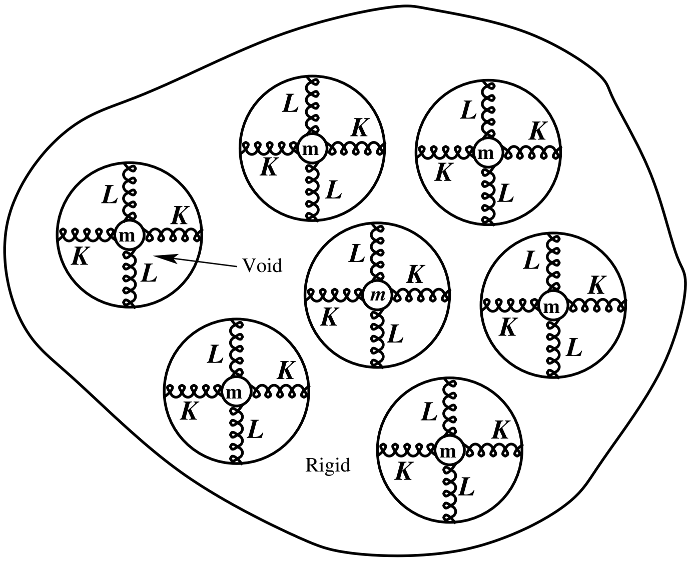

Figure 3. A material with an anisotropic dynamic density. The spring constants K and L in the two directions are different and are attached to a mass m at the centre of each cavity.

is shown in figure 3. This model is based on the models of Sheng *et al* [32] and Liu *et al* [33] where they show the effective mass density at a given frequency is not simply the average mass density and can even be negative. This model and an extension of it are discussed in more detail in Milton and Willis [34].

There is one case where the original form (2.1) of the equations is preserved, aside from the introduction of the density matrix, and that is under harmonic mappings, i.e. mappings satisfying

$$\Delta x'_p(\mathbf{x}) = \frac{\partial^2 x'_p}{\partial x_i \partial x_i} = 0, \quad \text{for all } p, \tag{2.7}$$

for the acoustic wave equations in a perfect fluid having zero shear modulus and zero shear viscosity in the absence of body forces, where the equations take the form

$$-\nabla \cdot \boldsymbol{\sigma} = \omega^2 \rho \mathbf{u}, \quad \boldsymbol{\sigma} = \kappa \mathbf{I}\,\mathrm{Tr}(\nabla \mathbf{u}) \equiv P\mathbf{I}. \tag{2.8}$$

Here $P(\mathbf{x})$ is the hydrostatic stress, $\mathbf{u}(\mathbf{x})$ is the displacement field, $\kappa(\mathbf{x})$ is the bulk modulus, $\rho(\mathbf{x})$ is the density and ω is the frequency. We allow for spatial variations in the density $\rho(\mathbf{x})$ and bulk modulus $\kappa(\mathbf{x})$, as may occur if the fluid is inhomogeneous, but we do not allow these parameters to vary with time. The above equations can be written in the equivalent simpler form

$$-\nabla P = \omega^2 \rho \mathbf{u}, \quad P = \kappa\,\mathrm{Tr}(\nabla \mathbf{u}), \tag{2.9}$$

but the first form of the equations has the advantage of making evident the similarity with elastodynamic wave equations, having a (singular) elasticity tensor field

$$C_{ijk\ell}(\mathbf{x}) = \kappa(\mathbf{x})\delta_{ij}\delta_{k\ell}. \tag{2.10}$$

Under these harmonic mappings, the new equations become

$$\boldsymbol{\sigma}' = \mathbf{C}'\nabla'\mathbf{u}', \quad -\nabla'\cdot\boldsymbol{\sigma}' = \omega^2\boldsymbol{\rho}'\mathbf{u}', \quad \boldsymbol{\sigma}' = \mathbf{C}'\nabla'\mathbf{u}', \tag{2.11}$$

with the new (still singular) elasticity tensor field with elements

$$C'_{pqrs} \equiv \frac{\kappa}{a}\frac{\partial x'_p}{\partial x_i}\frac{\partial x'_q}{\partial x_i}\frac{\partial x'_r}{\partial x_j}\frac{\partial x'_s}{\partial x_j}, \tag{2.12}$$

where $a = \det(\mathbf{A})$, and with density matrix

$$\boldsymbol{\rho}' = \frac{\mathbf{A}\rho\mathbf{A}^{\mathrm{T}}}{\det(\mathbf{A})}. \tag{2.13}$$

The transformed elasticity matrix has the interesting property that the associated stress field has the form

$$\boldsymbol{\sigma}' = \alpha\mathbf{A}\mathbf{A}^{\mathrm{T}}, \quad \text{where } \alpha = \mathrm{Tr}\,[(\kappa/a)\mathbf{A}\mathbf{A}^{\mathrm{T}}\boldsymbol{\varepsilon}'], \quad \varepsilon'_{rs} = \frac{1}{2}\left(\frac{\partial u'_r}{\partial x_s} + \frac{\partial u'_s}{\partial x_r}\right), \tag{2.14}$$

i.e., it only depends on the strain $\boldsymbol{\varepsilon}'$ through the scalar $\alpha(\mathbf{x})$. Locally such materials can only support multiples of a single stress. They are called pentamode materials supporting the stress $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ because at a given $\mathbf{x}$, there is a 5D space of symmetric matrices $\boldsymbol{\varepsilon}'$ such that $\alpha = 0$, i.e. there are five independent easy modes of deformation. As shown by Milton and Cherkaev [35], one can construct periodic pentamode materials which can support any desired average stress. These materials, illustrated in figure 4, are quite unstable however and should be stiffened a little bit, at the cost of a slight reduction in their performance as pentamode materials. Also presumably one could allow the microstructure to vary smoothly with position to obtain a close approximation to the desired elasticity tensor field $C'_{pqrs}(\mathbf{x}')$.

Although this analysis for the acoustic equations is interesting it does not seem helpful for cloaking or making objects invisible, because the assumption of harmonic maps is too restrictive. Indeed if $\mathbf{x}' = \mathbf{x}$ outside a region Γ, and $\Delta x'_i = 0$ everywhere, for $i = 1, 2, 3$ then necessarily $\mathbf{x}' = \mathbf{x}$ inside Γ too.

At first sight one might think there would be no hope of finding materials for which the elastodynamic equations take the form of (2.4). However, the equations (2.4) are a special case of elastodynamic equations for random media which follow directly from a formulation developed by Willis [36]; see also [31, 34, 37]. For a time harmonic disturbance, in the absence of external forces, the equations have the form

$$\nabla\cdot\langle\boldsymbol{\sigma}\rangle = -\mathrm{i}\omega\langle\mathbf{p}\rangle, \qquad \langle\mathbf{e}\rangle = [\nabla\langle\mathbf{u}\rangle + (\nabla\langle\mathbf{u}\rangle)^{\mathrm{T}}], \qquad \langle\boldsymbol{\sigma}\rangle = \mathbf{C}^{\mathrm{eff}}\langle\mathbf{e}\rangle - i\omega\mathbf{S}^{\mathrm{eff}}\langle\mathbf{u}\rangle,$$
$$\langle\mathbf{p}\rangle = \mathbf{S}^{\mathrm{eff}\,\dagger}\langle\mathbf{e}\rangle - i\omega\boldsymbol{\rho}^{\mathrm{eff}}\langle\mathbf{u}\rangle, \tag{2.15}$$

where $\mathbf{e}$ is the strain, $\mathbf{p}$ the momentum density, the angular brackets denote ensemble averages (which, assuming an ergodic hypothesis and fine microstructure, we associate with local volume averages) and $\mathbf{C}^{\mathrm{eff}}$, $\mathbf{S}^{\mathrm{eff}}$ and $\boldsymbol{\rho}^{\mathrm{eff}}$ are non-local operators depending on the frequency ω of oscillation. Here $\mathbf{S}^{\mathrm{eff}\,\dagger}$ is the adjoint of $\mathbf{S}^{\mathrm{eff}}$ by which we mean that the identity

$$\int \mathrm{Tr}\,[\mathbf{V}(\mathbf{S}^{\mathrm{eff}\,\dagger}\mathbf{v})]\,\mathrm{d}\mathbf{x} = \int \mathbf{v}\cdot(\mathbf{S}^{\mathrm{eff}}\mathbf{V})\,\mathrm{d}\mathbf{x}, \tag{2.16}$$

holds for all second order tensor fields $\mathbf{V}$ and vector fields $\mathbf{v}$.

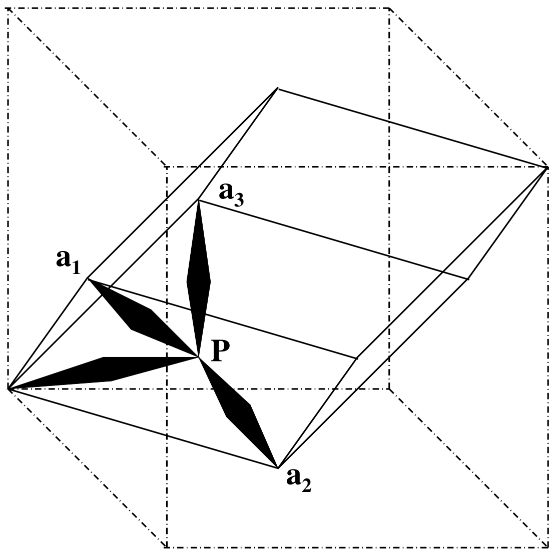

Figure 4. The diamond lattice of linkages is a pentamode material. Shown here are the four linkages inside the primitive unit cell. The cell is outlined by the solid lines and has basis vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$. It sits inside the conventional unit cell of the face centred cubic lattice, outlined by the broken lines. Depending on which side of the unit cell the point **P** lies the material supports a stress with eigenvalues all of the same sign, or with mixed signs. Adapted from [35].

If the microstructure is sufficiently small then these non-local effective operators might be approximated by local operators, in which case we arrive at equations of the desired form (2.4). (This claim is supported by explicit calculations in the 1D case (paper in preparation) where exact expressions for the effective operators can be obtained and the appropriate limits taken.) This suggests that it may be possible to design a cloaking device for elasticity, so that an object inside the device is invisible to arbitrary elastic waves of a given frequency. The shell of the device would be constructed from anisotropic composite materials in such a way that the moduli have the form (2.4) with the transformation being given, say, by (1.5). One would have to check that the cloaking held in a single realization and not just in the ensemble averaged sense. Of course it is far from clear that the desired composites with the required combinations of tensors $\boldsymbol{\rho}'$, $\mathbf{C}'$ and $\mathbf{S}'$ can actually be realized by some microgeometry and, even if they are, it may be a tall order to physically build these materials. The model of figure 3 can be used to achieve any desired tensor $\boldsymbol{\rho}'$, and Milton and Cherkaev [35] show how to achieve any positive definite tensor $\mathbf{C}'$ satisfying the symmetries of elasticity tensors (see also chapter 30 of [10] for an essentially complete proof in the 2D case, and also see Camar-Eddine and Seppecher [38] where possible non-local behaviours are characterized). On the other hand, we presently have little insight into how $\mathbf{S}'$ is related to the microgeometry.

Some difficulties of this sort might occur with the electromagnetic cloaking scheme of Pendry *et al* [28]. It is proposed that the electromagnetic cloaking shall be built from anisotropic composite materials and it is assumed that their behaviour is governed by the effective constitutive laws $\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$ and $\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$. However, if the constituent materials have a large contrast in their properties then it may be a better approximation to use the bianisotropic equations where at fixed frequency the electric displacement $\mathbf{D}$ and magnetic induction field $\mathbf{B}$ are coupled to both the electric field $\mathbf{E}$ and magnetic field intensity $\mathbf{H}$ through the equations

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E} - \mathrm{i}\boldsymbol{\kappa}\mathbf{H}, \quad \mathbf{B} = \mathrm{i}\boldsymbol{\kappa}^{\mathrm{T}}\mathbf{E} + \boldsymbol{\mu}\mathbf{H}, \tag{2.17}$$

where in general $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$ and $\boldsymbol{\mu}$ are non-local. The form of the equations for a composite follows easily from a formulation of Willis [39] which is the analogue for electrodynamics of that of Willis [36] for elastodynamics. Again if the microstructure is sufficiently small then these non-local effective operators might be approximated by local operators. If the medium is anisotropic and non-chiral then the coupling tensor $\boldsymbol{\kappa}$ is antisymmetric but not necessarily zero (see subsubection 2.10.1 of Serdyukov *et al* [40]). For electromagnetic cloaking, it is not yet known whether it is possible to construct media with the required combinations of the tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ among those composites for which $\boldsymbol{\kappa}$ is small.

3. Physical equations with a transformation invariant form

It seems rather amazing that the elastodynamic equations under a curvilinear coordinate transformation map to equations that are a special case of those of Willis. In Milton and Willis [34], it is argued that the Willis equations could be viewed as the proper equations of continuum elastodynamics, since at some level or another all materials have microstructure. If the motion is not necessarily time-harmonic the Willis equations, in the absence of forces, take the form

$$\begin{aligned} &\mathrm{div}\langle\boldsymbol{\sigma}\rangle = \langle\dot{\mathbf{p}}\rangle, \qquad \langle\mathbf{e}\rangle = [\nabla\langle\mathbf{u}\rangle + (\nabla\langle\mathbf{u}\rangle)^{\mathrm{T}}], \\ &\langle\boldsymbol{\sigma}\rangle = \mathbf{C}^{\mathrm{eff}} * \langle\mathbf{e}\rangle + \mathbf{S}^{\mathrm{eff}} * \langle\dot{\mathbf{u}}\rangle, \qquad \langle\mathbf{p}\rangle = \mathbf{S}^{\mathrm{eff}\,\dagger} * \langle\mathbf{e}\rangle + \boldsymbol{\rho}^{\mathrm{eff}} * \langle\dot{\mathbf{u}}\rangle, \end{aligned} \tag{3.1}$$

where the $*$ denotes a time convolution and the dot denotes differentiation with respect to time. In appendix C, it is shown that these equations remarkably also retain their form under curvilinear coordinate transformations. It may be the case that there are voids in the material where the displacement is not defined, or hidden areas where the displacement is not observable. Then it makes sense to look at the equations governing the motion of an ensemble average $\langle\mathbf{u}_w\rangle$ of a weighted displacement field $\mathbf{u}_w(\mathbf{x}, t) = w(\mathbf{x})\mathbf{u}(\mathbf{x}, t)$. Such equations are derived in Milton and Willis [34] and in the absence of forces take the form

$$\mathrm{div}\langle\boldsymbol{\sigma}\rangle = \langle\dot{\mathbf{p}}\rangle, \qquad \langle\mathbf{e}_w\rangle = [\nabla\langle\mathbf{u}_w\rangle + (\nabla\langle\mathbf{u}_w\rangle)^{\mathrm{T}}], \qquad \begin{pmatrix}\langle\boldsymbol{\sigma}\rangle \\ \langle\mathbf{p}\rangle\end{pmatrix} = \begin{pmatrix}\mathbf{C}_w^{\mathrm{eff}} & \mathbf{S}_w^{\mathrm{eff}} \\ \mathbf{D}_w^{\mathrm{eff}} & \boldsymbol{\rho}_w^{\mathrm{eff}}\end{pmatrix} * \begin{pmatrix}\langle\mathbf{e}_w\rangle \\ \langle\dot{\mathbf{u}}_w\rangle\end{pmatrix}, \tag{3.2}$$

where now $\mathbf{D}_w^{\mathrm{eff}}$ is not typically the adjoint of $\mathbf{S}_w^{\mathrm{eff}}$. Also in [34] equations were derived which govern the behaviour of $\langle\mathbf{u}_w\rangle$ when the material contains spinning masses below the chosen scale of modelling. These equations in the absence of forces take the form

$$\mathrm{div}\langle\mathbf{s}\rangle = \langle\dot{\mathbf{p}}\rangle, \qquad \langle\mathbf{d}_w\rangle = \nabla\langle\mathbf{u}_w\rangle, \qquad \begin{pmatrix}\langle\mathbf{s}\rangle \\ \langle\mathbf{p}\rangle\end{pmatrix} = \begin{pmatrix}\mathbf{C}_w^{\mathrm{eff}} & \mathbf{S}_w^{\mathrm{eff}} \\ \mathbf{D}_w^{\mathrm{eff}} & \boldsymbol{\rho}_w^{\mathrm{eff}}\end{pmatrix} * \begin{pmatrix}\langle\mathbf{d}_w\rangle \\ \langle\mathbf{u}_w\rangle\end{pmatrix}, \tag{3.3}$$

where $\mathbf{s}$ is a not necessarily symmetric stress field. One can apply a similar analysis as that in appendix C to show that equations (3.2) and (3.3) retain their form under curvilinear coordinate transformations. These results lead one to ask if all equations of microstructured continua have a form which is invariant under curvilinear coordinate transformations. This is rather speculative but if true could provide a useful principle for discovering new physical equations.

If this invariance were true, one would naturally expect it to extend to space-time coordinate transformations. In fact such an invariance holds for the non-local bianisotropic electromagnetic equations as follows from a simple extension of an argument of Post [14] (see also Leonhardt and Philbin [41] and Schurig *et al* [42]). Let $x_0 = t$ and introduce the electromagnetic field tensors with components

$$(F_{\alpha\beta}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \quad (G^{\alpha\beta}) = \begin{pmatrix} 0 & -D_1 & -D_2 & -D_3 \\ D_1 & 0 & H_3 & -H_2 \\ D_2 & -H_3 & 0 & H_1 \\ D_3 & H_2 & -H_1 & 0 \end{pmatrix}, \tag{3.4}$$

in which we are now distinguishing covariant and contravariant components. Maxwell's differential constraints on the fields take the form

$$F_{\alpha\beta,\mu} + F_{\beta\mu,\alpha} + F_{\mu\alpha,\beta} = 0, \quad G^{\alpha\beta}_{\ ,\alpha} = J^{\beta}, \tag{3.5}$$

where

$$(J^{\beta}) = \begin{pmatrix} \varrho \\ J_1 \\ J_2 \\ J_3 \end{pmatrix} \tag{3.6}$$

is the suitably normalized current source density. The field tensors are related by the bianisotropic non-local constitutive law

$$G^{\alpha\beta}(\mathbf{x}) = \int \mathrm{d}\mathbf{y}\; L^{\alpha\beta\mu\nu}(\mathbf{x},\mathbf{y}) F_{\mu\nu}(\mathbf{y}), \tag{3.7}$$

where the integral is over the space-time variable $\mathbf{y} = (y_0, y_1, y_2, y_3)$ and the $L^{\alpha\beta\mu\nu}(\mathbf{x},\mathbf{y})$ are the coefficients of the non-local operator representing the response of the bianisotropic material.

We introduce a general space-time coordinate transformation through the relation $\mathbf{x}' = \boldsymbol{\phi}(\mathbf{x})$, so that

$$x'^{i} = \phi^{i}(\mathbf{x}), \quad y'^{i} = \phi^{i}(\mathbf{y}). \tag{3.8}$$

As previously we let

$$A^{\alpha'}_{\alpha} = \phi^{\alpha'}_{\ ,\alpha} \tag{3.9}$$

be the transformation matrix where now α and α' range from 0 to 3. For convenience, we define $\mathbf{B} = \mathbf{A}^{-1}$. Under this transformation the electromagnetic field tensors and current source densities satisfy the tensor transformation laws

$$F'_{\mu'\nu'}(\mathbf{y}') = B^{\mu}_{\mu'}(\mathbf{y}) B^{\nu}_{\nu'}(\mathbf{y}) F_{\mu\nu}(\mathbf{y}), \qquad G'^{\alpha'\beta'}(\mathbf{x}') = a^{-1}(\mathbf{x}) A^{\alpha'}_{\alpha}(\mathbf{x}) A^{\beta'}_{\beta}(\mathbf{x}) G^{\alpha\beta}(\mathbf{x}),$$
$$J'^{\beta'}(\mathbf{x}') = a^{-1}(\mathbf{x}) A^{\beta'}_{\beta}(\mathbf{x}) J^{\beta}(\mathbf{x}), \tag{3.10}$$

where, as before, $a = \det \mathbf{A}$ is the Jacobian, which enters because $G^{\alpha\beta}$ and J^{β} are tensor densities of weight one. As is well known these field components satisfy

$$F'_{\alpha'\beta',\mu'} + F'_{\beta'\mu',\alpha'} + F'_{\mu'\alpha',\beta'} = 0, \quad G'^{\alpha'\beta'}_{,\alpha'} = J'^{\beta'}, \tag{3.11}$$

so that the differential constraints retain their Maxwell form. The bianisotropic constitutive law in the transformed coordinates takes the form

$$G'^{\alpha'\beta'}(\mathbf{x}') = \int \mathrm{d}\mathbf{y}'\ L'^{\alpha'\beta'\mu'\nu'}(\mathbf{x}', \mathbf{y}')F'_{\mu'\nu'}(\mathbf{y}'), \tag{3.12}$$

where

$$L'^{\alpha'\beta'\mu'\nu'}(\mathbf{x}', \mathbf{y}') = a^{-1}(\mathbf{x})A^{\alpha'}_{\alpha}(\mathbf{x})A^{\beta'}_{\beta}(\mathbf{x})L^{\alpha\beta\mu\nu}(\mathbf{x}, \mathbf{y})A^{\mu'}_{\mu'}(\mathbf{y})A^{\nu'}_{\nu}(\mathbf{y})a^{-1}(\mathbf{y}), \tag{3.13}$$

in which the extra factor of $a^{-1}(\mathbf{y})$ appears because of the Jacobian which enters when one changes variable in the integration (3.12). One can check (3.12) by substituting (3.13) and (3.10) into the equation and changing the variable of integration from $\mathbf{y}'$ to $\mathbf{y}$, thereby transforming it to the identity (3.7). Reciprocity of the medium is not assumed in this analysis. Thus the form of the electromagnetic equations in non-local bianisotropic media is preserved under general space-time coordinate transformations.

Although the form of the bianisotropic equations is preserved it does not necessarily follow that the bianisotropic response operator given by (3.13) is realizable in any material. Indeed causality, as we understand it, would imply that $L^{\alpha\beta\mu\nu}(\mathbf{x}, \mathbf{y})$ is non-zero only if $\mathbf{y}$ is in the past light cone of $\mathbf{x}$, but this does not imply that $\mathbf{y}'$ is in the past light cone of $\mathbf{x}'$ when $\mathbf{x}'$ and $\mathbf{y}'$ are plotted in Cartesian coordinates. The validity of the Post constraint (see equation (6.18) of Post [14]) that the completely antisymmetric part of L vanishes has been the subject of debate (Lakhtakia [43], Hehl and Obukhov [44]). However, if $L^{\alpha\beta\mu\nu}(\mathbf{x}, \mathbf{y})$ satisfies the Post constraint then so does $L'^{\alpha'\beta'\mu'\nu'}(\mathbf{x}', \mathbf{y}')$.

One can easily apply a similar analysis to show the invariance of the standard non-local, time-harmonic, electromagnetic equations under space (not space time) transformations. Assuming, for simplicity, that there are no free currents or charges, the fields satisfy Maxwell's equations,

$$\nabla \times \mathbf{E} + \mathrm{i}\omega\mathbf{B} = 0, \quad \nabla \times \mathbf{H} - \mathrm{i}\omega\mathbf{D} = 0, \tag{3.14}$$

and are linked by the nonlocal constitutive relations

$$\mathbf{D}(\mathbf{x}) = \int \mathrm{d}\mathbf{y}\ \boldsymbol{\varepsilon}(\mathbf{x}, \mathbf{y})\mathbf{E}(\mathbf{y}), \quad \mathbf{B}(\mathbf{x}) = \int \mathrm{d}\mathbf{y}\ \boldsymbol{\mu}(\mathbf{x}, \mathbf{y})\mathbf{H}(\mathbf{y}), \tag{3.15}$$

where the integration is over the spatial variable $\mathbf{y} = (y_1, y_2, y_3)$. Introducing the space mapping $\boldsymbol{\phi}$, the space variables $\mathbf{x}' = \boldsymbol{\phi}(\mathbf{x})$, $\mathbf{y}' = \boldsymbol{\phi}(\mathbf{y})$ and the usual transformation matrix $A_{jk}(\mathbf{x}) = \partial\phi_j(\mathbf{x})/\partial x_k$, the fields

$$\mathbf{E}'(\mathbf{x}') = [\mathbf{A}^{\mathrm{T}}(\mathbf{x})]^{-1}\mathbf{E}(\mathbf{x}), \quad \mathbf{H}'(\mathbf{x}') = [\mathbf{A}^{\mathrm{T}}(\mathbf{x})]^{-1}\mathbf{H}(\mathbf{x}),$$
$$\mathbf{D}'(\mathbf{x}') = \mathbf{A}(\mathbf{x})\mathbf{D}(\mathbf{x})/a(\mathbf{x}), \quad \mathbf{B}'(\mathbf{x}') = \mathbf{A}(\mathbf{x})\mathbf{B}(\mathbf{x})/a(\mathbf{x}), \tag{3.16}$$

where $a = \det \mathbf{A}$, satisfy Maxwell's equations in the new coordinates, and are related by the constitutive equations

$$\mathbf{D}'(\mathbf{x}') = \int d\mathbf{y}'\ \boldsymbol{\varepsilon}'(\mathbf{x}', \mathbf{y}')\mathbf{E}'(\mathbf{y}'), \quad \mathbf{B}'(\mathbf{x}') = \int d\mathbf{y}'\ \boldsymbol{\mu}(\mathbf{x}', \mathbf{y}')\mathbf{H}(\mathbf{y}'), \tag{3.17}$$

with

$$\boldsymbol{\varepsilon}'(\mathbf{x}', \mathbf{y}') = a^{-1}(\mathbf{x})\mathbf{A}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{x}, \mathbf{y})\mathbf{A}^{\mathrm{T}}(\mathbf{y})a^{-1}(\mathbf{y}),$$
$$\boldsymbol{\mu}'(\mathbf{x}', \mathbf{y}') = a^{-1}(\mathbf{x})\mathbf{A}(\mathbf{x})\boldsymbol{\mu}(\mathbf{x}, \mathbf{y})\mathbf{A}^{\mathrm{T}}(\mathbf{y})a^{-1}(\mathbf{y}). \tag{3.18}$$

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Appendix A. Alternative proof of the invariance of the form of Maxwell's equations under coordinate transformations

We need to show that the equation

$$\nabla \times \mathbf{E} + \mathrm{i}\omega\boldsymbol{\mu}\mathbf{H} = 0, \tag{A.1}$$

transforms to

$$\nabla' \times \mathbf{E}' + \mathrm{i}\omega\boldsymbol{\mu}'\mathbf{H}' = 0, \tag{A.2}$$

under the mapping $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$, where

$$\mathbf{E}'(\mathbf{x}') = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{E}(\mathbf{x}), \quad \mathbf{H}'(\mathbf{x}') = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{H}(\mathbf{x}), \quad \boldsymbol{\mu}'(\mathbf{x}') = \mathbf{A}\boldsymbol{\mu}(\mathbf{x})\mathbf{A}^{\mathrm{T}}/\det \mathbf{A}, \tag{A.3}$$

and $\mathbf{A} = (\nabla\mathbf{x}')^{\mathrm{T}}$, and its inverse $\mathbf{A}^{-1}$ have elements

$$A_{jk} = \frac{\partial x'_j}{\partial x_k}, \quad \{\mathbf{A}^{-1}\}_{k\ell} = \frac{\partial x_k}{\partial x'_\ell}. \tag{A.4}$$

From (A.3) and (A.1) we have

$$-\mathrm{i}\omega\boldsymbol{\mu}'\mathbf{H}' = -\mathrm{i}\omega\mathbf{A}\boldsymbol{\mu}(\mathbf{x})\mathbf{H}/\det \mathbf{A} = \mathbf{A}\nabla \times \mathbf{E}/\det \mathbf{A}. \tag{A.5}$$

In index notation, this reads

$$-\{\mathrm{i}\omega\boldsymbol{\mu}'\mathbf{H}'\}_h = \frac{1}{\det \mathbf{A}}\frac{\partial x'_h}{\partial x_j}e_{jmk}\frac{\partial E_k}{\partial x_m} = \frac{1}{\det \mathbf{A}}\frac{\partial x'_h}{\partial x_j}e_{jmk}\frac{\partial x'_\ell}{\partial x_m}\frac{\partial E_k}{\partial x'_\ell}, \tag{A.6}$$

in which e_{jmk} is the Levi-Civita (permutation) symbol. We need to compare this with

$$\{\nabla' \times \mathbf{E}'\}_h = \{\nabla' \times (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{E}\}_h = e_{h\ell m}\frac{\partial}{\partial x'_\ell}\left[\frac{\partial x_k}{\partial x'_m}E_k\right]$$

$$= e_{h\ell m}\frac{\partial^2 x_k}{\partial x'_\ell \partial x'_m}E_k + e_{h\ell m}\frac{\partial x_k}{\partial x'_m}\frac{\partial E_k}{\partial x'_\ell} = e_{h\ell m}\frac{\partial x_k}{\partial x'_m}\frac{\partial E_k}{\partial x'_\ell}, \tag{A.7}$$

where one can easily see that the first term in the middle equation vanishes by swapping the dummy indices ℓ and m. This will agree with (A.6) if

$$e_{jmk}\frac{\partial x'_h}{\partial x_j}\frac{\partial x'_\ell}{\partial x_m} = (\det \mathbf{A})e_{h\ell m}\frac{\partial x_k}{\partial x'_m}, \tag{A.8}$$

or equivalently (by multiplying both sides by the non-singular matrix $\mathbf{A}$, i.e. by multiplying by $\partial x'_p/\partial x_k$ and summing over k) if

$$e_{jmk}\frac{\partial x'_h}{\partial x_j}\frac{\partial x'_\ell}{\partial x_m}\frac{\partial x'_p}{\partial x_k} = e_{h\ell p}(\det \mathbf{A}). \tag{A.9}$$

The left-hand side of this equation, like the right-hand side, is completely antisymmetric in h, ℓ and p. (For example to see the antisymmetry in h and ℓ swap h and ℓ, and at the same time swap the dummy indices j and m, and use the fact that $e_{mjk} = -e_{jmk}$.) So it suffices to see if (A.9) holds for $h = 1$, $\ell = 2$ and $p = 3$, which is true because it is the well-known formula for the determinant

$$\det \mathbf{A} = e_{jmk}A_{1j}A_{2m}A_{3k}. \tag{A.10}$$

The invariance of the form of the other Maxwell equation under mapping transformations can easily be seen by swapping the roles of $\mathbf{E}$ and $\mathbf{H}$ in the above analysis and replacing $\boldsymbol{\mu}$ with $\boldsymbol{\varepsilon}$.

Appendix B. Closed class of operators under coordinate transformations: the elasticity case

Proposition B.1. *Let Ω, Ω' be two bounded open sets of $\mathbb{R}^d$, $d \geqslant 2$, and let $\mathbf{x} \mapsto \mathbf{x}'(\mathbf{x})$ be a C^2-diffeomorphism of Ω on to Ω'. Then, the class $\mathcal{C}_{\mathbf{C},\mathbf{S},\mathbf{D},\mathbf{B}}$ of the operators defined on the vector-valued functions $\mathbf{u}$ on Ω by*

$$\mathbf{u} \longmapsto -\nabla \cdot (\mathbf{C}(\mathbf{x})\nabla\mathbf{u} + \mathbf{S}(\mathbf{x})\mathbf{u}) + \mathbf{D}(\mathbf{x})\nabla\mathbf{u} + \mathbf{B}(\mathbf{x})\mathbf{u}, \tag{B.1}$$

where $\mathbf{C}$ is a bounded fourth order tensor-valued function (which need not necessarily satisfy the symmetries of elasticity tensor fields), $\mathbf{S}$, $\mathbf{D}$ are bounded third order tensor-valued functions, and $\mathbf{B}$ a bounded matrix-valued function defined on Ω, is closed under the mapping transformation (change of variables)

$$\mathbf{x}' = \mathbf{x}'(\mathbf{x}) \quad \textit{and} \quad \mathbf{u}'(\mathbf{x}') = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{u}(\mathbf{x}) \quad \text{where } A_{ij} := \frac{\partial x'_i}{\partial x_j}. \tag{B.2}$$

Remark B.2. None of the subclasses $\mathcal{C}_{\mathbf{C},\mathbf{0},\mathbf{D},\mathbf{B}}$, $\mathcal{C}_{\mathbf{C},\mathbf{S},\mathbf{0},\mathbf{B}}$, $\mathcal{C}_{\mathbf{C},\mathbf{S},\mathbf{D},\mathbf{0}}$, $\mathcal{C}_{\mathbf{C},\mathbf{0},\mathbf{0},\mathbf{B}}$, $\mathcal{C}_{\mathbf{C},\mathbf{0},\mathbf{D},\mathbf{0}}$, $\mathcal{C}_{\mathbf{C},\mathbf{S},\mathbf{0},\mathbf{0}}$ or $\mathcal{C}_{\mathbf{C},\mathbf{0},\mathbf{0},\mathbf{0}}$ is closed under the mapping transformations. Indeed, the introduction of $\mathbf{A}(\mathbf{x})$ in B.2. induces zero order terms in the derivation of the new function $\mathbf{u}'$ with respect to the new variables $\mathbf{x}'$. Therefore, there is a mixture between the terms $\mathbf{C}$, $\mathbf{S}$, $\mathbf{D}$, $\mathbf{B}$ after the change of variables. The details of this mixture are given in the following proof:

Proof. We denote

$$\{\nabla\mathbf{u}\}_{ij} := \frac{\partial u_j}{\partial x_i}, \quad \{\nabla'\mathbf{u}'\}_{ij} := \frac{\partial u'_j}{\partial x'_i},$$
$$\{\nabla\cdot\boldsymbol{\xi}\}_j := \frac{\partial \xi_{ij}}{\partial x'_i}, \quad \{\nabla'\cdot\boldsymbol{\xi}'\}_j := \frac{\partial \xi'_{ij}}{\partial x'_i}, \quad \{\nabla\nabla\theta\}_{ij} := \frac{\partial^2\theta}{\partial x_i \partial x_j}, \tag{B.3}$$

where i and j range between 1 and d, $\nabla\nabla$ is the double gradient (not to be confused with the Laplacian ∇^2) and

$$\begin{aligned}
&\mathbf{C}(\mathbf{x})\mathbf{M} : \mathbf{N} := C_{ijkl}(\mathbf{x})M_{kl}N_{ij} && \text{for all matrices } \mathbf{M}, \mathbf{N} \in \mathbb{R}^{d\times d},\\
&\mathbf{S}(\mathbf{x})\boldsymbol{\lambda} : \mathbf{M} := D_{ijk}(\mathbf{x})\lambda_k M_{ij} && \text{for all vectors } \boldsymbol{\lambda} \in \mathbb{R}^d \text{and for all matrices } \mathbf{M} \in \mathbb{R}^{d\times d},\\
&\mathbf{D}(\mathbf{x})\mathbf{M}\cdot\boldsymbol{\lambda} := D_{ijk}(\mathbf{x})M_{jk}\lambda_i && \text{for all vectors } \boldsymbol{\lambda} \in \mathbb{R}^d \text{and for all matrices } \mathbf{M} \in \mathbb{R}^{d\times d},\\
&\mathbf{B}(\mathbf{x})\boldsymbol{\lambda}\cdot\boldsymbol{\mu} := B_{ij}(\mathbf{x})\lambda_j\mu_i && \text{for all vectors } \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^d,\\
&\{\boldsymbol{\lambda}\otimes\boldsymbol{\mu}\}_{ij} := \lambda_i\mu_j && \text{for all vectors } \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^d.
\end{aligned} \tag{B.4}$$

Testing the operator (B.1) against a smooth vector-valued function $\mathbf{v}$ with compact support in Ω and applying the change of variables (B.2). with $\mathbf{u}(\mathbf{x}) = \mathbf{A}(\mathbf{x})^{\mathrm{T}}\,\mathbf{u}'(\mathbf{x}')$ and $\mathbf{v}(\mathbf{x}) = \mathbf{A}(\mathbf{x})^{\mathrm{T}}\,\mathbf{v}'(\mathbf{x}')$ we obtain

$$\begin{aligned}
&\langle[-\nabla\cdot(\mathbf{C}(\mathbf{x})\nabla\mathbf{u} + \mathbf{S}(\mathbf{x})\mathbf{u}) + \mathbf{D}(\mathbf{x})\nabla\mathbf{u} + \mathbf{B}(\mathbf{x})\mathbf{u}], \mathbf{v}\rangle\\
&\quad = \int_\Omega [\mathbf{C}(\mathbf{x})\nabla\mathbf{u} : \nabla\mathbf{v} + \mathbf{S}(\mathbf{x})\mathbf{u} : \nabla\mathbf{v} + (\mathbf{D}(\mathbf{x})\nabla\mathbf{u})\cdot\mathbf{v} + \mathbf{B}(\mathbf{x})\mathbf{u}\cdot\mathbf{v}]\,\mathrm{d}\mathbf{x}\\
&\quad = \int_{\Omega'} [\mathbf{C}'(\mathbf{x}')\nabla'\mathbf{u}' : \nabla'\mathbf{v}' + \mathbf{S}'(\mathbf{x}')\mathbf{u}' : \nabla'\mathbf{v}' + (\mathbf{D}'(\mathbf{x}')\nabla'\mathbf{u}')\cdot\mathbf{v}' + \mathbf{B}'(\mathbf{x}')\mathbf{u}'\cdot\mathbf{v}']\,\mathrm{d}\mathbf{x}'\\
&\quad = \langle[-\nabla\cdot(\mathbf{C}'(\mathbf{x}')\nabla'\mathbf{u}' + \mathbf{S}'(\mathbf{x}')\mathbf{u}') + \mathbf{D}'(\mathbf{x}')\nabla'\mathbf{u}' + \mathbf{B}'(\mathbf{x}')\mathbf{u}'], \mathbf{v}'\rangle,
\end{aligned} \tag{B.5}$$

where the new coefficients are given by

$$C'_{pqrs}(\mathbf{x}') := a^{-1}\mathbf{C}(\mathbf{x})(\nabla x'_r \otimes \nabla x'_s) : (\nabla x'_p \otimes \nabla x'_q), \tag{B.6}$$

$$S'_{pqr}(\mathbf{x}') := a^{-1}\mathbf{C}(\mathbf{x})\nabla\nabla x'_r : (\nabla x'_p \otimes \nabla x'_q) + a^{-1}\mathbf{S}(\mathbf{x})\nabla x'_r : (\nabla x'_p \otimes \nabla x'_q), \tag{B.7}$$

$$D'_{pqr}(\mathbf{x}') := a^{-1}\mathbf{C}(\mathbf{x})(\nabla x'_q \otimes \nabla x'_r) : \nabla\nabla x'_p + a^{-1}\mathbf{D}(\mathbf{x})(\nabla x'_q \otimes \nabla x'_r)\cdot\nabla x'_p, \tag{B.8}$$

$$\begin{aligned}
&B'_{pq}(\mathbf{x}') := a^{-1}\mathbf{C}(\mathbf{x})\nabla\nabla x'_q : \nabla\nabla x'_p + a^{-1}\mathbf{S}(\mathbf{x})\nabla x'_q : \nabla\nabla x'_p\\
&+a^{-1}(\mathbf{D}(\mathbf{x})\nabla\nabla x'_q)\cdot\nabla x'_p + a^{-1}\mathbf{B}(\mathbf{x})\nabla x'_q\cdot\nabla x'_p.
\end{aligned} \tag{B.9}$$

where $a = \det\mathbf{A}(\mathbf{x})$. □

Remark B.3. Let θ be a fixed real number. If $\mathbf{C}$ is a bounded fourth order tensor-valued function satisfying the symmetries of elasticity tensors such that the imaginary part of $e^{i\theta}\mathbf{C}$ is coercive on the space of symmetric second order tensors for some angle θ so is $\mathbf{C}'$. It is enough to prove this when $\mathbf{C}$ is symmetric and coercive:

On the one hand, the symmetry properties of $\mathbf{C}'$ are a straightforward consequence of the ones of $\mathbf{C}$. On the other hand, set $\tilde{\mathbf{A}} := |\det \mathbf{A}|^{-1/4}\,\mathbf{A}$ and let $(e_1, \ldots, e_d)$ be the canonic basis of $\mathbb{R}^d$. Then, for any symmetric matrix $\mathbf{M} \in \mathbb{R}^{d\times d}$, we have

$$\mathbf{C}'\mathbf{M} : \mathbf{M} = C'_{pqrs} M_{rs} M_{pq} = C_{ijkl}(M_{rs}\tilde{A}_{rk}\tilde{A}_{sl})(M_{pq}\tilde{A}_{pi}\tilde{A}_{qj}) = C_{ijkl}(\tilde{\mathbf{A}}^T\mathbf{M}\tilde{\mathbf{A}}e_l \cdot e_k)(\tilde{\mathbf{A}}^T\mathbf{M}\tilde{\mathbf{A}}e_j \cdot e_i)$$
$$= \mathbf{C}(\tilde{\mathbf{A}}^T\mathbf{M}\tilde{\mathbf{A}}) : (\tilde{\mathbf{A}}^T\mathbf{M}\tilde{\mathbf{A}}) \geqslant c|\tilde{\mathbf{A}}^T\mathbf{M}\tilde{\mathbf{A}}|^2 \geqslant c'\,|\tilde{\mathbf{A}}^{-1}|^4|\mathbf{M}|^2, \qquad \text{(B.10)}$$

hence the coerciveness.

Remark B.4. If $\mathbf{C}$ is an arbitrary symmetric fourth order tensor-valued function, satisfying the symmetries of elasticity tensors then the subclass $\mathcal{C}_{\mathbf{C},\mathbf{S},\mathbf{D},\mathbf{B}}$ where $S_{ijk} = D_{kji}$ for any $1 \leqslant i, j, k \leqslant d$, is closed under the mapping transformations (B.2), and is the smallest closed subclass containing $\mathcal{C}_{\mathbf{C},\mathbf{0},\mathbf{0},\mathbf{0}}$.

Remark B.5. If $\mathbf{C}$ satisfies the symmetries

$$C_{ijk\ell} = -C_{jik\ell} = C_{k\ell ij}, \qquad \text{(B.11)}$$

then the subclass $\mathcal{C}_{\mathbf{C},\mathbf{0},\mathbf{0},\mathbf{B}}$ is closed under the mapping transformations (B.2). This implies the closure of the time harmonic Maxwell equations under mapping transformations since they can be expressed in the form

$$\{\omega^2\boldsymbol{\varepsilon}\mathbf{E}\}_j = -\{\nabla\times\boldsymbol{\mu}^{-1}\nabla\times\mathbf{E}\}_j = -e_{jim}\frac{\partial}{\partial x_i}\left[(\boldsymbol{\mu}^{-1})_{mn}e_{nk\ell}\frac{\partial E_\ell}{\partial x_k}\right] = \frac{\partial}{\partial x_i}\left(C_{ijk\ell}\frac{\partial E_\ell}{\partial x_k}\right), \qquad \text{(B.12)}$$

with

$$C_{ijk\ell} = e_{ijm}e_{k\ell n}\{\boldsymbol{\mu}^{-1}\}_{mn}, \qquad \text{(B.13)}$$

which clearly satisfies the symmetries (B.11). Furthermore using (B.6) and (A.8), the transformed tensor takes the required Maxwell form

$$C'_{ijk\ell} = e_{ijm}e_{k\ell n}\{(\boldsymbol{\mu}')^{-1}\}_{mn}, \quad \text{with } \boldsymbol{\mu}' = a^{-1}\mathbf{A}\boldsymbol{\mu}\mathbf{A}^{\mathrm{T}}. \qquad \text{(B.14)}$$

Appendix C. Invariance of the Willis equations under coordinate transformations

This appendix will demonstrate that equations (3.1) are closed with respect to transformations of the coordinates. The notation is reduced by omitting the angled brackets (for ensemble means) and the superscripts 'eff'. The equations of motion, together with the constitutive relations (3.1), imply the weak formulation

$$\int\!\!\int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\{v_{i|j}(\mathbf{x}) * C^{ijkl}(\mathbf{x},\mathbf{y}) * u_{k|l}(\mathbf{y}) + v_{i|j}(\mathbf{x}) * \dot{S}^{ijk}(\mathbf{x},\mathbf{y}) * u_k(\mathbf{y})$$
$$+u_{i|j}(\mathbf{y}) * \dot{S}^{ijk}(\mathbf{y},\mathbf{x}) * v_k(\mathbf{x}) + v_i(\mathbf{x}) * \ddot{\boldsymbol{\rho}}^{ik}(\mathbf{x},\mathbf{y}) * u_k(\mathbf{y})\} = 0, \qquad \text{(C.1)}$$

for all test fields $\mathbf{v}$. The spatial arguments have been displayed because they are to be subject to a transformation; time-dependence is left implicit except for the presence of the symbol $*$ to denote convolution. Although, so far, the weak formulation (C.1) applies when the coordinates are Cartesian, for future convenience covariant and contravariant components are indicated. So is covariant differentiation, though this reduces down to partial differentiation for Cartesian coordinates.

Now introduce a set of curvilinear coordinates through the relation $\mathbf{x}' = \boldsymbol{\phi}(\mathbf{x})$, so that

$$x'^i = \phi^i(\mathbf{x}), \quad y'^i = \phi^i(\mathbf{y}). \tag{C.2}$$

As earlier in this study,

$$A^i_j = \phi^i{}_{,j}. \tag{C.3}$$

It is also convenient to define $\mathbf{B} = \mathbf{A}^{-1}$. Employing these coordinates, the weak formulation becomes

$$\begin{aligned}\iint \mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{y}' a^{-1}(\mathbf{x})\{&v'_{i|j}(\mathbf{x}') * C'^{ijkl}(\mathbf{x}',\mathbf{y}') * u'_{k|l}(\mathbf{y}') + v_{i|j}(\mathbf{x}') * \dot{S}'^{ijk}(\mathbf{x}',\mathbf{y}') * u'_k(\mathbf{y}')\\ &+u'_{i|j}(\mathbf{y}') * \dot{S}'^{ijk}(\mathbf{y}',\mathbf{x}') * v'_k(\mathbf{x}') + v'_i(\mathbf{x}') * \ddot{\boldsymbol{\rho}}'^{ik}(\mathbf{x}',\mathbf{y}') * u'_k(\mathbf{y}')\}a^{-1}(\mathbf{y}) = 0,\end{aligned} \tag{C.4}$$

where a denotes the Jacobian $|\partial\boldsymbol{\phi}/\partial\mathbf{x}| = \det\mathbf{A}$. The primed tensor components follow the natural pattern, so that (for instance)

$$v'_i(\mathbf{y}) = B_i{}^p(\mathbf{y})v_p(\mathbf{y}), \quad C'^{ijkl}(\mathbf{x}',\mathbf{y}') = A^i{}_p(\mathbf{x})A^j{}_q(\mathbf{x})C^{pqrs}(\mathbf{x},\mathbf{y})A^k{}_r(\mathbf{y})A^l{}_s(\mathbf{y}). \tag{C.5}$$

The only requirement now is to evaluate the covariant derivatives: for instance,

$$v'_{i|j}(\mathbf{x}') = v'_{i,j}(\mathbf{x}') - \Gamma^r_{ij}v'_r, \tag{C.6}$$

where

$$\Gamma^r_{ij} = \tfrac{1}{2}g^{rs}(g_{si,j} + g_{sj,i} - g_{ij,s}), \tag{C.7}$$

with the contravariant and covariant metric tensors given by $g^{ij} = \delta^{rs}A^i_rA^j_s$, $g_{ij} = \delta_{rs}B^r_iB^s_j$. The connection coefficients Γ always appear in the same combination, which can be simplified by elementary manipulations to the following form:

$$\Gamma^k_{mn}A^m_pA^n_q = -\frac{\partial^2 x'^k}{\partial x^p\partial x^q} \equiv -\phi^k_{,pq}(\mathbf{x}). \tag{C.8}$$

It follows that the integrand of (C.4), with the covariant derivatives explicitly expanded, has exactly the same form as in the original (C.1), in which the covariant derivatives are already partial derivatives, if the following replacements are made:

$$\begin{aligned}&C^{ijkl}(\mathbf{x},\mathbf{y}) \to a^{-1}(\mathbf{x})C'^{ijkl}(\mathbf{x}',\mathbf{y}')a^{-1}(\mathbf{y}),\\ &\dot{S}^{ijk}(\mathbf{x},\mathbf{y}) \to a^{-1}(\mathbf{x})[\dot{S}'^{ijk}(\mathbf{x}',\mathbf{y}') + A^i_p(\mathbf{x})A^j_q(\mathbf{x})C^{pqrs}(\mathbf{x},\mathbf{y})\phi^k_{,rs}(\mathbf{y})]a^{-1}(\mathbf{y}),\\ &\ddot{\rho}^{ik}(\mathbf{x},\mathbf{y}) \to a^{-1}(\mathbf{x})[\ddot{\rho}'^{ik}(\mathbf{x}',\mathbf{y}') + \phi^i_{,pq}(\mathbf{y})\dot{S}^{pqr}(\mathbf{y},\mathbf{x})A^k_r(\mathbf{x}) + \phi^k_{,rs}(\mathbf{x})\dot{S}^{rsp}(\mathbf{x},\mathbf{y})A^i_p(\mathbf{y})\\ &+\phi^i_{,pq}(\mathbf{x})C^{pqrs}(\mathbf{x},\mathbf{y})\phi^k_{,rs}(\mathbf{y})]a^{-1}(\mathbf{y}).\end{aligned} \tag{C.9}$$

These results reduce immediately to those in appendix B in the case that the material response is local.

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