1.27. 2013

This module:
1) Analyzes LTI systems in the $z$-domain.
2) Discusses the one-sided $z$-transform.

Suggested reading:
sections 3.5 and 3.6.

Motivation
We've seen already that rational $z$-transforms can be useful in characterizing difference equations. This module will expand on that, and analyze linear time invariant (LTI) systems in the $z$-domain. We will also consider the one sided transform with which it becomes quite convenient to solve difference equations.
LTI systems in $z$ domain \textit{(section 3.5)}

Recall the system function (or transfer function), $H(z) = \frac{B(z)}{A(z)}$, where $B(z)$ contains zeros and $A(z)$ contains poles. Suppose now that the input $X$ takes the form

$$X(z) = \frac{N(z)}{Q(z)}.$$ 

We now have $Y(z) = X(z)H(z) = \frac{B(z)N(z)}{A(z)Q(z)}$.

Some notation: $H$ contains poles $p_1, \ldots, p_m$; $X$ contains poles $q_1, \ldots, q_N$. We further assume that zeros of $B(z)$ and $N(z)$ don't coincide with any poles. It turns out that we can now write:

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{\prod_{k=1}^{m} (p_k z - 1)} + \sum_{k=1}^{N} \frac{Q_k}{\prod_{k=1}^{N} (1 - q_k z^{-1})}$$

where $A_k$ and $Q_k$ are constants. This form is pretty cool... but requires another assumption: the difference equation system is initially at rest.
With this restriction in place, the output is causal, and we can invert its \( z \)-transform:

\[
\gamma(n) = \sum_{k=1}^{\infty} A_k (p_k)^n u(n) + \sum_{k=1}^{\infty} Q_k (q_k)^n u(n)
\]

\( \text{response due to \( H \)'s poles} \quad \text{poles from \( X \)} \)

\( \text{natural response} \quad \text{forced response} \)

The natural response depends on \( H \), and hopefully (location of poles matters) it is transient in nature—that is, it vanishes to zero. The forced response depends on the input. We know that exponents are eigen-functions of LTI systems, and all that happens is amplification by some constant.

In order for the natural response to be transient, we need \(|p_k| < 1\). The rate of decay to zero depends on the poles. Seeing that there might be non-idealities (e.g., calculations on a computer are done with finite precision), it's best to keep the poles of \( H(z) \) away from the unit circle.
Example 3.5.1

\[ y(n) = 0.5 y(n-1) + x(n) \] initially at rest.
\[ x(n) = e^{j \pi n/4} u(n) \]

Note "initially at rest" means that \( y(-1) = 0 \) in this case.

\[ Y(z) = 0.5 z^{-1} Y(z) + X(z) \]
\[ \Rightarrow H(z) = \frac{1}{1-0.5z^{-1}}. \]

The \( z \)-transform of the input \( x \) is
\[ X(z) = \frac{1}{1-e^{j\pi/4}z^{-1}} = \frac{1}{1-(\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}})}z^{-1} \]

Note that the pole in \( H \) lies inside the unit circle, and after some algebra it can be shown that the natural response decays to zero. However, \( X(z) \) has a pole on the unit circle. Therefore, \( y(n) \) will be a superposition of:

1) some natural response, which vanishes.
2) an exponential (sinusoidal) oscillatory forced response.
Causality and Stability

We have already seen that an LTI system is causal if and only if the ROC of $H(z)$ is outside some circle.

Let's see how this relates to BIBO stability. Recall that we need $\sum_{n} |h(n)| < \infty$, and so on the unit circle $(|z|=1)$ we have

$$|H(z)| = \left| \sum_{n=-\infty}^{\infty} h(n) z^{-n} \right|$$

The triangle inequality implies

$$\leq \sum_{n=-\infty}^{\infty} |h(n) z^{-n}|$$

$$= \sum_{n=-\infty}^{\infty} |h(n)| \cdot |z|^{-n} = \sum_{n=-\infty}^{\infty} |h(n)|$$

and the latter term is finite for BIBO stable LTI systems. Therefore, an LTI system is BIBO stable if and only if the ROC contains the unit circle.

Seeing that causal systems have an ROC outside some circle, it takes the following form:

$$\text{ROC} = \{ z : |z| > r \}$$

where $0 < r < 1$. 
Let's look at example 3.5.2 graphically.

\[ H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}} \]

(a) What's the ROC in order for \( h(n) \) to be stable?

Answer: we have poles at \( z = \frac{1}{2} \) and \( z = 3 \). Stability requires the unit circle to be in the ROC:

ROC = \{ z : \frac{1}{2} < |z| < 3 \}. The pole at \( z = \frac{1}{2} \) is outside it! It corresponds to a causal response \( (\frac{1}{2})^n u(n) \), but the pole at \( z = 3 \) is beyond the ROC, and the response is anti-causal: \( 3^n u(-n-1) \).

(b) The system is causal. Is the system stable?

Answer: The ROC must be outside a circle or, in this case, outside both circles:

ROC = \{ z : |z| > 3 \}
The poles both contribute to causal terms of the form \( \frac{1}{2^n} u(n) \) and \( 3^n u(n) \). However, the unit circle isn't in the ROC. Therefore, the system is not BIBO stable.

To see this, note that \( 3^n \) blows up as \( n \) is increased.

**Pole-zero cancellations**

In principle, if a zero and pole fall on the same location, they cancel each other, and the effect of that pole on the response \( y(n) \) is removed.

In practice, this is somewhat fragile. If the zero is near the pole (but not on it), it reduces the response of that pole. As long as all poles are stable this is fine; but amplifying an unstable pole by an arbitrarily small zero (placed imprecisely) won't really work.

In summary, don't try this at home :-)
Multiple order poles
Recall that a pole of multiple order that is located on the unit circle leads to a response of the form polynomial \cdot exponential, where the order of the polynomial is the multiplicity of the pole, and the exponential is oscillatory.

Therefore, such a response will blow up. This is why a bounded-input (oscillating) processed by an LTI system with pole on the unit circle might have unbounded output. (The system's impulse response is bounded, yet it is not BIBO stable.)

One sided $z$-transform (section 3.6)
The one-sided transform is useful for solving difference equations that are not relaxed. (The initial conditions were not necessarily zero.)

We survey this material briefly.
The one-sided transform is defined:

\[ x^+ (z) = \sum_{n=0}^{\infty} x(n) z^{-n} \]

Note that the summation is not over negative time.

Active learning assignment

1) \( x_1(n) = \{ 1, 2, 5, 3 \} \)

What's the one-sided z-transform?

2) \( x_2(n) = \{ 1, \frac{3}{4}, 5 \} \)
The one-sided transform has the following properties:

**Time delay:** \( x(n) \rightarrow x(n-k) \rightarrow z^{-k} \left[ X^+(z) + \sum_{n=0}^{k-1} x(n)z^{-n} \right] \)

**Time advance:** \( x(n+k) \rightarrow z^k \left[ X^+(z) - \sum_{n=0}^{k-1} x(n)z^{-n} \right] \)

**Example 3.6.2:**
Take \( x(n) = a^n \) and \( x_1(n) = x(n-2) \). What is \( x_1^+(z) \)?

Using the formula:

\[
Z^+ \{x(n-2)\} = z^{-2} \left\{ X^+(z) + x(-1)z + x(-2)z^2 \right\} = z^{-2} X^+(z) + a^{-1} z^{-1} + a^{-2}
\]

Now, let's take the transform of \( x \):

\[
X^+(z) = \sum_{n=0}^{\infty} a^n z^{-n}
\]

Thus, the causal transform of \( x(n) \) is:

\[
Z\{x(n)\} = \frac{1}{1 - a z^{-1}}
\]

Therefore, \( Z^+ \{x_1(n)\} = \frac{z^{-2}}{1 - a z^{-1}} + a^{-1} z^{-1} + a^{-2} \)
The old-fashioned way to compute this (without formula):

\[ x_1(0) = x(-2) = a^{-2} \]
\[ x_1(1) = a^{-1} \]
\[ x_1(2) = a^0 \]

\[ x_1^+(z) = a^{-2}z^2 + a^{-1}z + a^0 = a^{-2}\left\{1 + a^{-1}z + a^{-2}z^2 + \ldots\right\} = \frac{a^{-2}}{1 - az^{-1}} \]

Now, let's compare the expressions:

\[ \frac{z^{-2}}{1-az^{-1}} + a^{-1}z + a^{-2} \]
\[ = \frac{1}{1-az^{-1}} \left\{ z^{-2} + (1-az^{-1})(a^{-1}z^{-1} + a^{-2}) \right\} \]
\[ = \frac{1}{1-az^{-1}} \left\{ z^{-2} + \underbrace{a^{-1}z^{-1}}_{m} + \underbrace{a^{-2}z^{-2}}_{m} - a^{-1}z^{-1} \right\} \]
\[ = \frac{a^{-2}}{1-az^{-1}} \]

The two answers coincide.
Let's now learn by example how to solve a difference equation.

We will discuss example 3.65, which analyzes the Fibonacci sequence

\[ x = \{ 1, 1, 2, 3, 5, 8, \ldots \} \]

We can see that \( y(n) = y(n-1) + y(n-2) \) with initial conditions:

1. \( y(0) = y(-1) + y(-2) \)
2. \( y(1) = y(0) + y(-1) \)

We conclude that \( y(-1) = 0 \) and \( y(-2) = 1 \).

We now compute the one-sided transform:

\[
y^+(z) = \left[ 2^{-1} y^+(z) + y(-1) \right] + \left[ 2^{-2} y^+(z) + y(2) + y(-1) z^{-1} \right]
\]

\[
y(-1) = 0
\]

\[
y(-2) = 1
\]

\[
x = (2^{-1} + 2^{-2}) y^+(z) + 1
\]

\[
y^+(z) = \frac{1}{1 - 2^{-1} z^{-2}} = \frac{z^{-2}}{z^{-2} - z^{-1}}
\]

It can be shown that there are poles at \( p_1 = \frac{1 + \sqrt{5}}{2} \) and \( p_2 = \frac{1 - \sqrt{5}}{2} \).

This leads after some algebra to

\[
y(n) = \text{const}_1 \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^n + \text{const}_2 \cdot \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]