This module will:

1) Bring together - mathematically - concepts that we've discussed informally:
sinusoids/exponentials are eigen functions of LTI systems.

2) Establish the steady state response of a system to a superposition of sinusoids at the input.

Suggested reading:
Section 5.1.

Introduction

We have already discussed that sinusoids (and/or complex exponentials) are eigen functions of LTI systems. That is, a sinusoid will be processed by such a system by (merely) being multiplied by a constant:

\[ \sin(\omega_n) \rightarrow \boxed{H} \rightarrow H(\omega) \cdot \sin(\omega_n) \]

Note - more correctly, exponentials are the inputs, but I want to emphasize intuition.
seeing that LTI systems are linear, 
the response to a bunch of sinusoids 
at the input is a bunch of sinusoids— 
with the same frequencies—at the output. 
The only effect that the LTI system has 
is to modify the amplitudes:

\[
\sum_{k} c_{k} e^{j\omega_{k}n} \rightarrow \mathcal{L} \rightarrow \sum_{k} c_{k} H(\omega) e^{j\omega_{k}n}
\]

For this reason, the frequency domain is 
a very natural way to think about LTI 
systems. They are filters—they let some 
frequencies pass through (perhaps even 
magnifying them), while blocking others.

The details...

Recall that an LTI system in discrete time 
can be written as convolution,

\[
y(n) = \sum_{k=-\infty}^{+\infty} h(k) x(n-k).
\]

Consider the following input,

\[
x(n) = A e^{j\omega n}.
\]
What is the output?

\[ y(n) = \sum_{k=-\infty}^{\infty} h(k) \left[ A e^{j\omega(n-k)} \right] \]

\[ x(n-k) = A e^{j\omega(n-k)} \]

\[ = A \left[ \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \right] e^{j\omega n} \]

The emphasized component is nothing but the Fourier response,

\[ H(\omega) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k}. \]

Example 5.1

Consider a system with impulse response

\[ h(n) = \left( \frac{1}{2} \right)^n u(n), \]

and an input

\[ x(n) = A e^{j\pi n/2}. \]

What is the output?

We begin by computing the system's response:

\[ H(\omega) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \]

\[ = \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k e^{-j\omega k} \]

\[ = \frac{1}{1-\frac{1}{2} e^{-j\omega}}. \]
The input \( x(n) = A e^{j \pi n} = A e^{j (\frac{\pi}{2}) n} \) corresponds to frequency \( w = \frac{\pi}{2} \), the response \( H(w) \) at that frequency is:
\[
H(w = \frac{\pi}{2}) = \frac{1}{1 - \frac{i}{2} e^{-j \frac{\pi}{4}}} = \frac{1}{1 - \frac{i}{2} (\cos \frac{\pi}{4} - j \sin \frac{\pi}{4})}
\]
\[
= \frac{1}{1 + \frac{i}{2} j}
\]
and the output is:
\[
y(n) = A \cdot H(w) \cdot e^{j \omega n} = A \frac{1}{1 + \frac{i}{2} j} e^{j \frac{\pi}{2} n}.
\]

At other frequencies, the system might respond differently. For example, at \( w = 0 \) the response is:
\[
H(w = 0) = \frac{1}{1 - \frac{i}{2} e^{-j 0}} = \frac{1}{1 - \frac{i}{2}} = 2.
\]

Therefore, a constant input \( x(n) = A \), which corresponds to frequency \( w = 0 \), also called DC, is magnified by a factor of 2.
Response to sinusoids

In practice, most of our LTI systems will be real valued, and they process real valued input. Therefore, the outputs are also real.

Recall that the Fourier response is Hermitian symmetric, and so

\[ |H(-\omega)| = |H(\omega)|, \]
\[ \theta(-\omega) = -\theta(\omega). \]

Now take \( x_1(n) = e^{j\omega n} \), \( x_2(n) = e^{-j\omega n} \).

The corresponding outputs are

\[ y_1(n) = |H(\omega)| e^{j\theta(\omega)} e^{j\omega n}, \]
\[ y_2(n) = |H(-\omega)| e^{j\theta(-\omega)} e^{-j\omega n}, \]
\[ = |H(\omega)| e^{-j\theta(\omega)} e^{-j\omega n}. \]

It can be seen that \( y_1(x(n)) = y_2(n) \).

We can create sines/cosines using a superposition of \( x_1 \) and \( x_2 \),

\[ x_3(n) = \frac{1}{2} x_1(n) + \frac{1}{2} x_2(n) = \frac{1}{2} (e^{j\omega n} + e^{-j\omega n}) = \cos(\omega n), \]
\[ x_4(n) = \frac{1}{2j} x_1(n) - \frac{1}{2j} x_2(n) \]
\[ = \frac{1}{2j} (e^{j\omega n} - e^{-j\omega n}) = \sin(\omega n). \]
The outputs are also superpositions, owing to linearity of the LTI system,
\[ y_3(n) = \frac{1}{2} y_1(n) + \frac{1}{2} y_2(n) \]
\[ = \frac{1}{2} |H(w)| e^{j\Theta(w)} e^{jwn} \]
\[ + \frac{1}{2} |H(w)| e^{-j\Theta(w)} e^{-jwn} \]
\[ = |H(w)| \left( e^{j(wn+\Theta(w))} + e^{-j(wn+\Theta(w))} \right) \]
\[ = |H(w)| \cos (wn + \Theta(w)). \]

Similarly,
\[ y_4(n) = |H(w)| \sin (wn + \Theta(w)). \]

**Example 5.1.2**

Consider the moving average filter,
\[ y(n) = \frac{1}{3} (x(n+1) + x(n) + x(n-1)), \]
which corresponds to the convolution system,
\[ h(n) = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\}. \]

Let's compute \( |H(w)| \) and \( \Theta(w) \).

a) (Active learning): what's \( H(w) \)?
(b) compute $|H(w)|$.

(c) Recall that $\Theta(w)$ is either 0 or $\pi$, because $H(w) \in \mathbb{R}$. What is $\Theta(w)$?

(d) We will plot $|H(w)|$ in Matlab.

```matlab
w = -pi : 0.001 : pi; j
H = ... 010 some function of w
plot(w, abs(H));
```
Remarks

In general, moving average filters are lowpass filters. Intuitively this makes sense, because we smooth the average of adjacent samples. Let's construct in Matlab some nice low pass signal (a sinusoid) and add noise to it. The noise has all sorts of frequency components, and if we apply an average filter with lots of taps it will make the noise small while keeping the sinusoid intact.

More formally, last time we showed that the signal \( h(n) = \begin{cases} \frac{1}{2M+1} & -M \leq n \leq +M \\ 0 & \text{else} \end{cases} \)

has Fourier transform \( H(\omega) = \frac{1}{2M+1} \frac{\sin((M+\frac{1}{2})\omega)}{\sin(\frac{\omega}{2})} \).

The filter will be lowpass, it will peak at \( \omega = 0 \), it has \( M \) lobes, and the width of the main one is thus proportional to \( \frac{1}{M} \).
Example 5.1.4

\[ y(n) = ay(n-1) + bx(n), \quad 0 < a < 1. \]

(a) Compute the magnitude \( |H(w)| \) and phase \( \Theta(w) \) of the frequency response.

Solution:

\[ y(t) = ay(t)z^{-1} + bx(t) \]
\[ y(t) (1-az^{-1}) = bx(t) \]
\[ H(z) = \frac{y}{x} = \frac{b}{1-az^{-1}}. \]

Plugging in \( z = e^{jw} \),
\[ H(w) = \frac{b}{1-a e^{-jw}} \]
\[ = \frac{b}{(1-a \cdot \cos(w)) + j(a \cdot \sin(w))} \]

Therefore,
\[ |H(w)| = \frac{b}{|1-a e^{-jw}|} \]
\[ = \frac{b}{|1-a \cos(w) + j(a \sin(w))|} \]
\[ = \frac{b}{\sqrt{(1-a \cos(w))^2 + (a \sin(w))^2}} \]
\[ = \frac{b}{\sqrt{1+a^2-2a \cos(w)}} \]

The phase can also be derived with some deliberation.
(b) Choose $b$ such that $\max_w |H(w)| = 1$.

**Solution:** Note that the denominator is minimal for $w = 0$, and then

$$|H(w=0)| = \frac{b}{\sqrt{1 + a^2 - 2a}} = \frac{b}{1-a}.$$

Choosing $b = 1-a$ sets $|H(w=0)| = 1$. 
