2.24.2013

This module:
1) Discusses transient and steady state responses of LTI systems.
2) The frequency response of rational systems.

Suggested reading:
complete section 5.1 and cover section 5.2.

Motivation - LTI systems and aperiodic inputs
Today's module naturally splits into two components. We begin by motivating the first part. In practice, the inputs of LTI systems are not sinusoidal.

One (simple) approach is to say that the input is actually a superposition of sinusoids,
\( x(n) = \sum_{i=1}^{L} A_i \cos(\omega_i n + \Phi_i), \)
and the output:
\( y(n) = \sum_{i=1}^{L} A_i |H(\omega_i)| \cos(\omega_i n + \Phi_i + \Theta(\omega_i)). \)
However, real-world signals are not finite superpositions of sinusoids either. (To explain why, it is sufficient to point out that a finite superposition of sinusoids continues forever, it is not of finite time duration, in contrast to many practical signals.)

More realistically, the input can be written as an integration over an infinite number of sinusoidal components:

$$X(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(w) e^{jwn} dw,$$

and you can probably guess what the output of the LTI system is.

**Steady state and transient signals**

Suppose that an exponential input appears at some time — it can be \( n = 0 \) to keep things simple. The output of the LTI system—assuming that it is BIBO stable—eventually converges to the ordinary sinusoidal output.
In practice, the transient output decays gradually to zero. However, the decay is typically fast enough to allow us to neglect the transient response.

**Modified example (5.1.5) + active learning**

An LTI system has response $h(n) = \left(\frac{1}{3}\right)^n u(n)$. The input is $x(n) = \left(\frac{1}{3}\right)^n u(n)$. We want to determine the spectrum.
Active learning component

a) what's the frequency response $H(w)$?

b) Similarly, $X(w) = \frac{1}{1 - \frac{1}{2} e^{-jw}}$. 

c) The spectrum (frequency response) of the output is

$$Y(w) = X(w) \cdot H(w)$$

$$= \frac{1}{1 - \frac{1}{2} e^{-jw}} \cdot \frac{1}{1 - \frac{1}{4} e^{-jw}}.$$
**Rational Systems**

As we have discussed, difference equations have rational transfer functions. Because in practical systems many filters are implemented using difference equations, it is useful to build intuition about the corresponding transfer functions.

Recall \( H(\omega) = \frac{B(\omega)}{A(\omega)} \)

\[
= \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}
\]

\[
= b_0 \frac{\prod_{k=1}^{M} (1 - z_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - p_k e^{-j\omega})}
\]

where \( z_k \) and \( p_k \) are zeros and poles, respectively.

Let's begin by looking at the magnitude, in particular \( |H(\omega)|^2 \). Note that

\[
|H(\omega)|^2 = H(\omega) H^*(\omega)
\]

\[
= H(\omega) H(-\omega)
\]

Let's work this out through an example.
Example (5.2.1)
Consider the difference equation system,
\[ y(n) = -0.1 y(n-1) + 0.2 y(n-2) + x(n) + x(n-1). \]
Let's compute \(|H(w)|^2\)

Solution:
\[ H(z) = \frac{1 + z^{-1}}{1 + 0.1 z^{-1} - 0.2 z^{-2}} \]

\[ |H(w)|^2 = H(z) \cdot H(z^{-1}) \]
\[ = \frac{1 + z^{-1}}{1 + 0.1 z^{-1} - 0.2 z^{-2}} \cdot \frac{1 + z}{1 + 0.1 z - 0.2 z^2} \]
\[ = \frac{z + (z + z^{-1})}{1.05 + 0.08(z^2 + z^{-2})} \]

Notice how \(z + z^{-1}\) appear together, and so do \(z^2 + z^{-2}\). But keep in mind \(z = e^{jw}\), and therefore
\[ z + z^{-1} = e^{jw} + e^{-jw} = 2 \cos(w), \]
\[ z^2 + z^{-2} = e^{2jw} + e^{-2jw} = 2 \cos(2w). \]
These are real valued, which makes sense, because we are evaluating \(|H(w)|^2\), which is real valued.

Returning to the problem,
\[ |H(w)|^2 = \frac{2 + 2 \cos(w)}{1.05 + 0.08 \cos(w) - 0.4 \cos(2w)} \]

This can be further simplified.
Note although \( |H(jw)|^2 \) can be determined from \( H(z) \), the opposite is typically not true. To see why, if zeros and poles of \( H(z) \) are \( \{z_K\} \) and \( \{p_K\} \), respectively, then the zeros and poles of \( H(z^{-1}) \) are \( \{\frac{1}{z_K}\} \) and \( \{\frac{1}{p_K}\} \). Poles appear in pairs \( (p_K, \frac{1}{p_K}) \), and so do zeros. In general, it is impossible to determine which component of the pair is the correct one.

Let us see another way how rational transfer functions can be written:

\[
H(w) = b_0 \frac{\prod_{k=1}^{M} (1 - z_K e^{-jw})}{\prod_{k=1}^{N} (1 - p_K e^{-jw})}
\]

\[
= b_0 e^{jw(N-M)} \frac{\prod_{k=1}^{M} (e^{jw} - z_K)}{\prod_{k=1}^{N} (e^{jw} - p_K)}
\]

analogous to multiplying num/denom by \( z's \).

We now write:

\[
e^{jw} - z_K = V_K(w) e^{j\Theta_K(w)}
\]

\[
e^{jw} - p_K = U_K(w) e^{j\Phi_K(w)}
\]

where \( V_K(\cdot) \), \( U_K(\cdot) \), \( \Phi_K(\cdot) \), \( \Theta_K(\cdot) \) are the magnitude and phase.
we can now write for the magnitude:

$$|H(\omega)| = |b_0| \frac{V_1(\omega) \cdots V_m(\omega)}{U_1(\omega)U_2(\omega) \cdots U_N(\omega)}.$$

Similarly, for the phase we have:

$$\angle H(\omega) = \angle b_0 + \omega(NM) + \sum_{k=1}^{N} \theta_k(\omega) - \sum_{k=1}^{N} \phi_k(\omega).$$

These can be interpreted geometrically. Keep in mind that $$z = e^{j\omega}$$, and so we are traversing the unit circle and looking at vectors from the circle to the zero or pole.

Angles in the figure and lengths of vectors can be related to the terms $$U, V, \phi, \Theta.$$
Example (5.2.2)

\[ H(z) = \frac{1}{1-0.8z^{-1}} = \frac{z}{z-0.8} \]

we evaluate the frequency response.

**Solution** - the system has a zero at \( z=0 \) and pole at \( p=0.8 \). The frequency response is

\[ H(\omega) = \frac{e^{j\omega}}{e^{j\omega}-0.8} \]

the magnitude response

\[ |H(\omega)| = \left| \frac{e^{j\omega}}{e^{j\omega}-0.8} \right| = \frac{1}{|\cos(\omega)+j\sin(\omega)-0.8|} = \frac{1}{\sqrt{(-0.8+\cos(\omega))^2 + (\sin(\omega))^2}} = \frac{1}{\sqrt{[0.64 + \cos^2(\omega) + \sin^2(\omega) - 1.6\cos(\omega)]}} = \frac{1}{\sqrt{[1.64 - 1.6\cos(\omega)]}} \]

The phase is

\[ \Theta(\omega) = \phi(e^{j\omega}) - \phi(e^{j\omega}-0.8) = \omega - \tan^{-1}\left( \frac{\sin(\omega)}{\cos(\omega)-0.8} \right) \]
Example (question 5.10 (a))

\[
\begin{align*}
\mathbf{X}(z) & \xrightarrow{\mathbf{z}^{-1}} \mathbf{Y}(z) \\
& \xrightarrow{\frac{1}{2} + \frac{1}{2}z^{-1}} \mathbf{Y}(z)
\end{align*}
\]

Determine the magnitude and phase response.

Solution:

\[
\begin{align*}
y(n) &= \frac{1}{2} (x(n) + x(n-1)) \\
Y(z) &= X(z) \left( \frac{1}{2} + \frac{1}{2}z^{-1} \right) \\
H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{2} + \frac{1}{2}z^{-1}
\end{align*}
\]

\[
H(w) = \frac{1}{2} + \frac{1}{2}e^{-jw} = \frac{1}{2} (e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}) e^{-j\frac{\omega}{2}}
\]

\[
|H(w)| = \left| \frac{1}{2} \cdot 2 \cos \left( \frac{\omega}{2} \right) e^{-j\frac{\omega}{2}} \right| = \left| \cos \left( \frac{\omega}{2} \right) \right|
\]

\[
\mathbb{E}(\omega) = \sqrt{\cos^2 \left( \frac{\omega}{2} \right) - \frac{\omega}{2}}
\]

0 when \( \cos \left( \frac{\omega}{2} \right) \geq 0 \)

\( \pi \) when \( \cos \left( \frac{\omega}{2} \right) < 0 \)