3.12. 2013

This module covers:
1) Sampling.
2) Aliasing.
3) Reconstruction.

Recommended reading material:
Section 6.1.

We have already discussed the main themes:

1) Sampling - \(x^{(m)} = x_a(nT)\).
2) Aliasing - if we don't sample above the Nyquist rate, higher frequencies will get mixed in erroneously.
3) We reconstruct \(x_a(t)\) as follows,

\[
x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin(\frac{\pi}{T}(t-nT))}{\frac{\pi}{T}(t-nT)}
\]

But we only skimmed the surface. Today we will explain these topics in detail.
Let me first survey the material using the approach that was easiest for me when I was a student!

What is sampling? Instead of moving from continuous to discrete time, consider instead what happens when we multiply $x_A(t)$ by an impulse train, also known as a Dirac comb:

$$\Delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

What is this? A delta component of the form $\delta(t - nT)$ is a delta at location (time) $nT$. Because we have a summation over all $n$'s, we have infinitely many deltas with spacing $T$ between them.
Interestingly, the Fourier transform of an impulse train is also an impulse train. More specifically,

$$\mathcal{F}\{\delta(t)\} = \sum_{k=-\infty}^{\infty} \delta(t-kT).$$

we will derive this later, for now let's continue the narrative.

The impulse train is useful in understanding sampling, because if we multiply \( x_a(t) \) by \( \delta_T(t) \), we will get deltas at locations \( nT \) whose height (area) is \( x_a(nT) \). That is,

$$x_a(t) \delta_T(t) = x_a(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT) \delta(t-nT).$$

Why? The \( \delta(t-nT) \) "picked off" \( x_a \) at time \( t=nT \).
Clearly the "information" in this product $x_a(t)\Delta T(t)$ is identical to that of the discrete time signal

$X(n) = x_a(nT)$.

To make this precise, let's look at their Fourier transforms:

$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$

and in continuous time

$F\{x_a(t)\Delta T(t)\} = \int_{-\infty}^{\infty} x_a(t)\Delta T(t) e^{-j2\pi Ft} dt$

$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{+\infty} x_a(nT) \delta(t-nT) e^{-j2\pi Ft} dt$

$= \sum_{n=-\infty}^{+\infty} x_a(nT) e^{-j2\pi FTn}$
we see that the Fourier transforms resemble each other, where in discrete time we had \( w \), whereas in continuous time \( 2\pi FT \).

Now that we agree that \( X(n) \) and \( X_a(t) \Delta T(t) \) contain the same information, let's understand intuitively what the Fourier transform is.

\[
F \{ X_a(t) \Delta T(t) \} = F\{X_a\} \ast F\{\Delta T\},
\]

let's look at this graphically.

where here the impulse train is the Fourier transform of \( \Delta T(t) \).

Recall that \( X(t) \ast \delta(t) = X(t) \), and similarly

\[
x(t) \ast \delta(t-nT) = x(t-nT).
\]
Therefore, \( X_a(F) * \left\{ \frac{\delta(F+k)}{k} \right\}_{k=-\infty}^{\infty} = X_a(F-k) \).

This is sampling.

Aliasing - graphically, we make copies of \( X_a(F) \) in the Fourier domain after multiplying them by \( \frac{1}{T} \) and shifting by \( \frac{k}{T} \), for all \( k \in \mathbb{Z} \).

![Graph showing sampling and aliasing](image)

**Active learning**

![Graph showing active learning](image)

1. \( T = \frac{1}{2} \). Please plot \( F \{ X_a(t) \Delta_T(t) \} \).
Part (1) involved sampling at the Nyquist rate. Let's now sample above and below.

(2) $T = \frac{1}{5}$. Please repeat the plot.

(3) $T = 1$. Ditto.
we now understand aliasing better than before; shifts of the Fourier transform will be copied around, and we don’t want those shifts to interfere with each other.

Real-world problem
Recall our AM system with bandwidth 10 kHz and carrier frequency 900 kHz. And noise:

```
\[ -910 \quad -890 \quad 890 \quad 910 \]
```

By sampling at 2 M samples/second, we have this copied with shifts of 2 MHz:

```
\[ -1M \quad -900K \quad 900K \quad 1M \]
```

Sadly, the squiggly noise also gets copied!
To prevent this, we put the AM signal through an **anti-aliasing filter** before sampling. This filter blocks frequencies above 1 MHz, and the noise is confined to the range \((-1 MHz, +1 MHz)\).
Reconstruction

Again, \( x(n) \) and \( x_a(t) \Delta \tau(t) \) are basically analogous from the point of view of what "information" they contain.

\( F \{ x_a(t) \Delta \tau(t) \} \) contains the Fourier response of the original signal — if sampled above the Nyquist rate — between frequencies \(-\frac{1}{\tau}\) and \(\frac{1}{\tau}\). The frequencies at higher frequencies are the result of the deltas having an infinite bandwidth.

To get back the signal of interest in the red box, we apply a low pass filter, which corresponds to convolution with a sinc function,

\[
x_a(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \text{sinc}\left(\frac{\pi}{\tau}(t-nT)\right).
\]
Big picture

* Sampling — Multiply $X_a(t)$ by impulse train.

* Fourier response is convolution between $X_a(F)$ and $\mathcal{F}[\Delta(t)]$, which is itself an impulse train, resulting in copies of $X_a(t)$ being shifted around.

* Aliasing — you want to make sure that the shifts don’t overlap.

* Reconstruction — $X_a(F)$ appears in the lower frequencies of the Fourier response of the sampled signal $X_a(t)\Delta(t)$, just need to pick them off with a low pass filter.

And now, some details ...
Fourier transform of impulse train

The impulse train $\Delta_T(t)$ is periodic with period $T$. It can be written as:

$$\Delta_T(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi k \frac{t}{T}}$$

Let us compute the coefficients $C_k$:

$$C_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \Delta_T(t) e^{-j2\pi k \frac{t}{T}} dt$$

In the time range $(-\frac{T}{2}, \frac{T}{2})$ we have a single delta at time $t=0$, which is $\delta(t)$; this can be seen graphically.

$$C_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-j2\pi k \frac{t}{T}} dt$$

$$= \frac{1}{T} \cdot e^{-j2\pi k \frac{0}{T}}$$

$$= \frac{1}{T}$$

where here the explanation is that a delta $\delta(t)$ picks off the value at $t=0$. More generally, $$\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) x(t) dt = x(0),$$

where here $x(t) = e^{-j2\pi k \frac{t}{T}}$. 
Recall that a Fourier series can also be represented as deltas. That is, \( c_k \) will be a delta at frequency \( \frac{k}{T} \) with area \( c_k \), meaning \( \delta(F - \frac{k}{T}) \cdot c_k \).

In our example,
\[
F \{ \Delta \tau(t) \} = \sum_{k=-\infty}^{\infty} \frac{1}{k} \delta(F - \frac{k}{T}).
\]
Example 6.1.1

\[ X_{\text{a}(t)} = \cos(2\pi F_0 t) = \frac{1}{2}(e^{j2\pi F_0 t} + e^{-j2\pi F_0 t}) \]

Now let's sample this signal at rate \( F_s < 2F_0 \).

To keep it simple, \( F_s = 1\frac{1}{2}F_0 \).

If we reconstructed (attempted to) \( X_{\text{a}(t)} \), the lowpass filter would yield the range \((-\frac{3}{4}F_0, +\frac{3}{4}F_0)\), because \( \frac{3}{4}F_0 = \frac{1}{2}F_s \). But within this range there are erroneous deltas, owing to aliasing.
Example 6.1.2

\[ X_a(t) = e^{-A|t|} \]
\[ X_a(F) = \frac{2A}{A^2 + (2\pi F)^2}. \]

This signal is *not* bandlimited.

![Graph showing the signal in the time domain and the frequency domain.]

**Why?** Because the peak of the exponent, which we emphasize in red, contains a discontinuity, and discontinuities (in the derivative in this case) have infinite bandwidth.

Therefore, no matter how fast we sample there will always be some aliasing. But in this example \( X_a(F) \) decays quickly enough at high frequencies to make the aliasing manageable—i.e., we sample reasonably fast.