This module:

1) Reviews some concepts and definitions in signals and systems.

2) Discusses analog to digital (A/D) and digital to analog (D/A) conversion.

Suggested reading material:
Sections 1.1 - 1.4 in the textbook.

Signals and systems

Signal - a quantity that changes in time or space. Basically it's a function of one (or more) independent variables.

Examples

\[ S_1(t) = \sin (10 \pi t) + 13 \]
\[ S_2(t) = e^{0.1t} + 7t - 2 \]

Consider also \( S_3(t) = \sin (2t_1 + 13t_2) \), it's a function of 2 variables.
Real-world signals typically can't be described so succinctly. One approach is to consider random processes—see ECSE 54.

Another approach is to focus on a short time duration. Under some technical conditions a signal over a short duration can be approximated as a sum of sinusoids:

\[ x(t) = \sum_{i=1}^{N} A_i(t) \sin[2\pi F_i(t) t + \Theta_i(t)] \]

Such representations are convenient for

* Biological signals (ECG's, EEG's).
* Audio signals (musical instruments generate several tones over a short duration).

System—some device that responds to a stimulus.

Signals are generally inputs and outputs of systems. The systems can be physical (e.g., my vocal tract), computational (an echo canceling algorithm), or even conceptual (an ideal lowpass filter).
Systems are often characterized by how they operate on signals. They can be

* linear / nonlinear

* causal / anti-causal

* random / deterministic (we only consider the latter in this course).

Digital systems are the emphasis of this course; they can be implemented in software (for example in Matlab on a computer) or hardware (logic circuit).

The rules that govern the signal processing system are called an algorithm. Often there are multiple algorithms to implement a system; we will focus on efficient algorithms.
One approach to process signals is analog processing, especially because most real-world signals are analog (e.g., speech, temperature, audio, images...).

\[\text{analog input} \rightarrow \boxed{\text{analog}} \rightarrow \text{analog output}\]

Another approach is to convert the analog system to digital, perform signal processing in the digital domain, and convert back to analog.

\[\text{analog input} \rightarrow \boxed{\text{A/D converter}} \rightarrow \text{digital input} \rightarrow \boxed{\text{DSP}} \rightarrow \text{digital output} \rightarrow \boxed{\text{D/A converter}} \rightarrow \text{analog output}\]

That said, sometimes we actually want a digital output (e.g., in a communication system decoder), and no digital to analog conversion is needed.

As we discussed before, digital processing could have various advantages: flexibility in programming, accuracy, storage of results, and even price. However, sometimes analog systems are required to process wideband signals.
Types of signals

Multidimensional - the signal is a function of more than one input (e.g., an image is 2-dimensional).

Multichannel - a signal with more than one component (a complex-valued signal contains 2 channels).

Continuous time or analog - defined for every value of time (or space) and take on continuous values:

\[
x(t)
\]

Discrete time signal - defined over discrete values of time/space.

Example - \( x(n) = 0.9 \ln n \)

Digital Signal - similar to discrete time signal AND takes on a discrete set of values.

Note that digital computers in the real world process numbers that are specified in bits (0's and 1's), these are digital signals; they take on a discretized set of possible levels.
Active learning assignment
(Problem 1.1 in textbook)
classify the signals below as
one/multi dimensional; single/multi channel;
continuous/discrete time; digital/analog amplitude.

a) Closing prices of stocks.

b) A color movie.

c) Weight/height measurement of child every month.
Frequency

Continuous time sinusoid

\[ X_a(t) = A \cos(\omega t + \theta) \]

- \( A \): amplitude (real valued)
- \( \omega \): frequency (radians per second)
- \( \theta \): phase (in radians), \( \theta \in [-\pi, +\pi] \) typically belongs to set \( \text{inclusive of } -\pi, +\pi \)

Can also express the signal in cycles per second:

\[ X_a(t) = A \cos(2\pi F t + \theta) \]

instead of \( \omega \)

Disclaimer - this is the textbook's notation. Sadly, each course/book uses different notation.

Properties

1) For all \( F > 0 \), \( X_a(t) \) is periodic.
   For all \( X_a(t + T_p) = X_a(t) \), \( T_p = \frac{1}{F} \).

2) Sinusoids with different frequencies are different.

3) Increasing \( F \) makes oscillations faster and reduces the period.
Negative frequencies are especially convenient in evaluating complex exponentials. Recall that \( e^{\pm j \phi} = \cos(\phi) \pm j \sin(\phi) \), we will see quite a few of these later.

**Discrete time sinusoids**

\[
X(n) = A \cos(wn + \Theta)
\]

instead of \( t \) instead of \( \tau \)

Here the frequency \( w \) is in radians per sample.

we can also write

\[
X(n) = A \cos(2\pi f n + \Theta), \quad w = 2\pi f,
\]

where \( f \) is cycles per sample.

\[
\text{discrete time sinusoids have slightly different properties. primarily}
\]

1) A discrete-time sinusoid is periodic only if its frequency \( f \) is a rational number.

Periodic requires \( X(N+n) = X(n) \)

\[
\iff \cos(2\pi f_0 (N+n) + \Theta) = \cos(2\pi f_0 n + \Theta)
\]

\[
\iff 2\pi f_0 N = 2\pi k
\]

meaning that \( f_0 = \frac{k}{N} \), it's rational.
2) sinusoids whose frequencies \( w \) are separated by integer multiple of \( 2\pi \) are indistinguishable. 
Explanation: 
\[
\cos ((w_0 + 2\pi k_n) n + \theta) = \cos (w_0 n + 2\pi k_n + \theta) = \cos (w_0 n + \theta)
\]
If two sinusoids are indeed separated by a multiple of \( 2\pi \), the convention is that \( |w_1| < \pi \) is the "real" sinusoid, whereas \( |w_2| > \pi \) is its alias (looks the same in discrete time).

3) Highest rate of oscillation: \( w = \pm \pi \) \((f = \pm \frac{1}{2})\).

Harmonics

Consider \( s_k(t) = e^{jk2\pi f_0 t} \)
where \( k = 0, \pm 1, \pm 2, \ldots \)
\( s_k(t) \) is periodic with period \( T_p / k \), where \( T_p = \frac{1}{F_0} \).
That is \( s_k(t) = s_k(t + T_p) \).
In particular, set \( k = 0 \), then \( s_k(t) = s_0(t + T_p) \)
\( \Rightarrow \) the signals \( s_k(t) \) all have common period \( T_p \).

We can construct a linear combination of all these harmonics, \( x(t) = \sum_{k=-\infty}^{\infty} c_k s_k(t) \).
This \( x(t) \) is periodic with period \( T_p \) \((= \frac{1}{F_0})\), and this representation is the Fourier series.
Here we define $S_k(n) = e^{j2\pi k f_0 n}$, where $k \in \{0, \pm 1, \pm 2, \ldots \}$, where $f_0 = \frac{1}{N}$.

Note that $S_{k+N}(n) = e^{j2\pi \frac{k}{N} (k+N)} = e^{j(2\pi \frac{k}{N} + 2\pi n)} = e^{j2\pi \frac{kN}{N}} = S_k(n)$, there are only $N$ distinct harmonics.

We typically choose $S_k(n)$ with $k \in \{0, 1, \ldots, N-1\}$. We form linear combinations as before:

$$x(n) = \sum_{k=0}^{N-1} c_k S_k(n),$$

this signal has period $N$.

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**Problem 1.2 from textbook:**

What's the fundamental period (if it exists)?

(a) $\cos(0.01 \pi n)$

(b) $\cos(\pi \frac{30n}{105})$

(c) $\sin(3n)$
Tougher problem

\[ x(n) = 0.1 \cos \left( \frac{65}{40} \pi n \right) + 12 \sin \left( \frac{37}{4} \pi n \right) \]

First, forget about the cosine vs. sine stuff — I just tried to make it annoying :-(

The gist of the matter is to realize that each term is periodic, and then find the least common denominator.

1) \[ \frac{65}{40} \pi n K = 2 \pi n \ell \Rightarrow \frac{65}{40} K = 2 \ell \]
\[ \Rightarrow 65 K = 80 \ell \]
\[ \Rightarrow 13 K = 16 \ell \]

And 16 is the period.

Sanity check - \[ \frac{65}{40} \pi n \cdot 16 = \frac{13 \cdot 5 \cdot 16 \pi n}{8 \cdot 5} = 13 \cdot 2 \pi n \]
[13 cycles]

2) \[ \frac{37}{4} \pi n K = 2 \pi n \ell \Rightarrow 37 K = 8 \ell \]
the period is 8, giving
\[ \frac{37}{4} \cdot 8 \pi = 2 \pi \cdot 37 \Rightarrow 37 \text{ cycles}. \]

3) The periods are 8 and 16; they divide nicely; the least common term is 16.