A/D and D/A conversion

Many real-world signals are analog. In order to enjoy the advantages of digital processing, we must convert from analog to digital, perform the DSP, and then convert back to analog.

Application: Speech signals are often sampled at 8 K samples/sec. The processing is performed digitally in real time, and we convert back to analog.

A/D conversion can be considered within the following schematic:

\[ X_a(t) \rightarrow \text{sampler} \rightarrow X(n) \rightarrow \text{Quantizer} \rightarrow X_q(n) \rightarrow \text{Coder} \rightarrow \text{bits} \]

- **Sampling**: \[ X(n) = X_a(nT) \], where \( T \) is the sampling interval.
- **Quantization**: truncate/round \( X(n) \), which is continuous-valued, to discrete set of values.
- **Coding**: translate the discrete-valued \( X_q(n) \) into actual bits.
1) Non-uniform samples [e.g., instead of sampling interval $T$, it varies] can be used — and in fact have some advantages — but require more math & algorithms.

2) Non-uniform quantization is also advantageous; I’ve performed research in this area, but in practice the simplest and more commonly used quantizers round to the nearest of a set of uniformly-spaced reproduction levels.

3) Data compression can be performed by allocating fewer bits to quantization levels that are common, and more bits to infrequent ones.

D/A conversion — the simplest approach is a zero-order hold, which holds each digital level stable at the output for $T$ time. This gives a staircase approximation.

A first-order hold is linear interpolation; it connects the dots:
Sampling

we perform uniform sampling,

\[ x(n) = x_a(nT), \]

where \( T \) is the sampling interval and \( F_s = \frac{1}{T} \) is the sampling rate (samples per unit time).

Note that \( t = nT = \frac{n}{F_s} \) are times when we sample.

We sketch a sampling operation as follows:

\[
\begin{array}{c}
\text{analog } x_a(t) \\
\text{input} \\
\text{Fs} = \frac{1}{T} \\
\end{array}
\rightarrow
\begin{array}{c}
x(n) = x_a(nT) \\
\text{discrete time signal} \\
\end{array}
\]

Now consider \( x_a(t) = A \cos (2\pi ft + \theta) \)

Let's sample it:

\[ x(n) = x_a(nT) = A \cos \left( \frac{2\pi nf}{F_s} + \theta \right) \]

Recall the form for a discrete sinusoid:

\[ x(n) = A \cdot \cos (2\pi fn + \theta) \]

we see that \( f = \frac{f}{F_s} \), seeing that \( \omega = 2\pi f \), \( W = 2\pi f \), and \( T = \frac{1}{F_s} \); we also have \( W = \omega T \).

Now, recall that \( -\infty < f, \omega < +\infty \).

But \( -\frac{1}{2} < f < \frac{1}{2} \) and \( -\pi < \omega < +\pi \).

In order for the (discrete-time) frequency to fall into the correct range, we need \( \frac{f}{F_s} \in (-\frac{1}{2}, \frac{1}{2}) \).
keeping in mind that $\frac{1}{F_s} = T$.

$F \in \left(-\frac{F_s}{2}, \frac{F_s}{2}\right)$ or

$-\frac{F_s}{2} = -\frac{1}{2T} \leq F \leq \frac{1}{2T} = +\frac{F_s}{2}$

IF the (analog frequency) $F$ is small enough, then all spectral components of the signal will be mapped into (discrete time) frequencies between $-\frac{1}{2}$ and $\frac{1}{2}$, and we will be able to discern (separate) between different components in the discrete domain.

BUT if there are analog frequencies beyond $(-\frac{F_s}{2} + \frac{F_s}{2})$, then we have a mess!

To see why, suppose that

$x_1(t) = \cos(2\pi \cdot 10 \cdot t)$,

$x_2(t) = \cos(2\pi \cdot 50 \cdot t)$,

and $F_s = 40$ Hz. Let's look at what happens in discrete time,

$x_1(n) = \cos\left(2\pi \cdot \frac{10}{40} \cdot n\right) = \cos\left(\frac{\pi}{2} \cdot n\right)$,

$x_2(n) = \cos\left(2\pi \cdot \frac{50}{40} \cdot n\right) = \cos\left(\frac{5\pi}{2} \cdot n\right)$,

but $\cos\left(\frac{5\pi}{2} \cdot n\right)$ is identical to $\cos\left(\frac{\pi}{2} \cdot n\right)$. That is, these two analog inputs get sampled to the same output. This is called **aliasing**.
Returning to the general case, consider
\[ X_a(t) = A \cos(2\pi f_0 t + \theta), \]
and sample it with sampling rate \( F_s = \frac{1}{T} \):
\[ x(n) = A \cos(2\pi f_0 n + \theta), \]
where \( f_0 = \frac{F_0}{F_s} \).

Case #1: \( -\frac{F_s}{2} \leq F_0 \leq \frac{F_s}{2} \), and then \( -\frac{1}{2} \leq f_0 \leq \frac{1}{2} \), and we have a one-to-one relationship between \( X_a(t) \) and \( x(n) \).

Case #2: \( |F_0| > \frac{F_s}{2} \), and it is impossible to determine the analog frequency unless more information is available.

**Example 1.4.2**

\[ X_a(t) = 3 \cos(100\pi t). \]

(a) What is the minimum sampling rate required to avoid aliasing?

Answer: the analog frequency is \( F = 50 \text{ Hz} \), and the minimum sampling rate is double that, \( F_s = 100 \text{ Hz} \).

(b) Suppose \( F_s = 200 \text{ Hz} \), what is the discrete time signal?

Answer: at sample \( n \) we process time \( \frac{n}{200} \) and
\[ x(n) = X_a(n) = 3 \cos \left( \frac{100\pi n}{200} \right) = 3 \cos \left( \frac{5\pi n}{2} \right). \]

(c) \( F_s = 75 \text{ Hz} \), what is the discrete time signal?

Answer: \( x(n) = 3 \cos \left( \frac{100\pi n}{75} \right) = 3 \cos \left( \frac{4\pi n}{3} \right) \)
\[ = 3 \cos \left( (2\pi - \frac{2\pi}{3}) n \right) = 3 \cos \left( \frac{2\pi}{3} n \right), \]
it can be shown that \( y_a(t) = 3 \cos (50\pi t) \) yields identical samples.
sampling theorem

Suppose we know something about the frequency content of an analog signal, in particular that it is band-limited. For example, speech signals are dominated by spectral components below 3000 Hz.

By sampling the speech signal fast enough, we can avoid aliasing. Given $F_{\text{max}}$, we choose a sampling rate $F_s = \frac{1}{T}$ such that all analog frequencies are below $\frac{F_s}{2}$.

$F_s > 2F_{\text{max}}$.

Suppose now that the analog signal is

$$X_a(t) = \sum_{i=1}^{N} A_i \cos(2\pi F_i t + \theta_i),$$

where $|F_i| < F_{\text{max}}$, this is a superposition of bandlimited components. By sampling at $F_s$, $F_i$ will be mapped to a discrete-time sinusoid with frequency

$$f_i = \frac{F_i}{F_s},$$

and we have $f_i \in [-\frac{1}{2}, \frac{1}{2}]$.

Theorem

If the highest frequency in $X_a(t)$ is $F_{\text{max}} = B$ and we sample at rate $F_s > 2F_{\text{max}} = 2B$, then $X_a(t)$ can be recovered perfectly:

$$X_a(t) = \sum_{n=-\infty}^{\infty} X_a(\frac{n}{F_s}) g(t - \frac{n}{F_s}),$$

where $g(t) = \frac{\sin(2\pi B t)}{2\pi B t}$.
This sampling rate is known as the Nyquist rate, and the sampling theorem is due to several people including Nyquist and Shannon, among others.

This means that if we sample an analog signal fast enough, there is no aliasing, and we can later reconstruct the original analog input perfectly. However, the sinc function used in ideal reconstruction is not causal, and in practice there is always some distortion, because causal processing is required in real-world implementations.

Example 1.4.4

\[ X_a(t) = 3 \cos(2000 \pi t) + 5 \sin(6000 \pi t) + 10 \cos(12000 \pi t) \]

(a) Active learning component

what is the Nyquist rate?

(b) We use \( F_S = 5000 \) samples/sec, what is the discrete time signal?

\[ x(n) = x_a(nT) = x_a\left(\frac{n}{T}\right) = 3 \cos\left(\frac{2\pi}{5} n\right) + 5 \sin\left(2\pi \cdot \frac{3}{5} n\right) + 10 \cos\left(2\pi \cdot \frac{3}{5} n\right) \]

\[ = 3 \cos\left(2\pi \cdot \frac{3}{5} n\right) + 5 \sin\left(2\pi \cdot \frac{6}{5} n\right) + 10 \cos\left(2\pi \cdot \frac{6}{5} n\right) \]

and \( \cos\left(2\pi \cdot \frac{6}{5} n\right) \) is aliased to \( \cos\left(2\pi \cdot \frac{1}{5} n\right) \).
(c) What analog signal would be obtained with ideal interpolation?

Note that $X(n)$ can be simplified as follows:

$X(n) = 13 \cos\left(2\pi \left(\frac{1}{5}\right) n\right) + 5 \sin\left(2\pi \left(\frac{2}{5}\right) n\right)$.

The first component corresponds to analog frequency $1000 \, \text{Hz}$, and the second to frequency $-2000 \, \text{Hz}$. Therefore,

$y(t) = 13 \cos\left(2000 \frac{\pi}{5} t\right) - 5 \sin\left(4000 \frac{\pi}{5} t\right)$.

\[1000 \, \text{Hz}\] \[\uparrow\]
\[2000 \, \text{Hz}\] \[\uparrow\]

Owing to negative frequency