Please remember to justify your answers carefully.

There are 10 pages in total on this test.

Name: ___________________ Student ID: ___________________
Consider a causal notch filter with the system function

$$H(z = e^{j\omega}) = G \left( \frac{1 - e^{-j\frac{\pi}{3}}z^{-1}}{1 - \frac{4}{5}e^{-j\frac{\pi}{3}}z^{-1}} \right) \left( \frac{1 - e^{+j\frac{\pi}{3}}z^{-1}}{1 - \frac{4}{5}e^{+j\frac{\pi}{3}}z^{-1}} \right),$$

where $G$ is the gain of the filter, and it is a positive real-valued constant.

(a) Where are the zeros and poles of the filter located?

**Solution:** The zeros are located at $z_1 = e^{-j\frac{\pi}{3}}$ and $z_2 = e^{+j\frac{\pi}{3}}$. The poles are located at $p_1 = \frac{4}{5}e^{-j\frac{\pi}{3}}$ and $p_2 = \frac{4}{5}e^{+j\frac{\pi}{3}}$.

(b) Sketch the pole-zero plot.

**Solution:** The solution should include the real and imaginary axes (horizontal and vertical, respectively) of the $z$ plane; the unit circle; zeros on the unit circle at angles $\pm\frac{\pi}{3}$ (the angles should be marked on the sketch); and poles near the zeros but at reduced radii of $\frac{4}{5}$.

(c) Find the gain $G$ such that the $|H(\omega = 0)| = 1$.

**Solution:** Let us derive the magnitude of the filter,

$$|H(\omega)| = \left| G \left( \frac{1 - e^{-j\frac{\pi}{3}}z^{-1}}{1 - \frac{4}{5}e^{-j\frac{\pi}{3}}z^{-1}} \right) \left( \frac{1 - e^{+j\frac{\pi}{3}}z^{-1}}{1 - \frac{4}{5}e^{+j\frac{\pi}{3}}z^{-1}} \right) \right|. $$

We now substitute $z = e^{j\omega} = e^0 = 1$,

$$|H(0)| = G \left| \frac{1 - e^{-j\frac{\pi}{3}} - e^{+j\frac{\pi}{3}} + 1}{1 - \frac{4}{5}e^{-j\frac{\pi}{3}} - \frac{4}{5}e^{+j\frac{\pi}{3}} + \frac{41}{25}} \right| = G \left| \frac{2 - 2\cos\left(\frac{\pi}{3}\right)}{\frac{11}{25} - \frac{8}{5}\cos\left(\frac{\pi}{3}\right)} \right| = G \left| \frac{2 - 2\frac{1}{2}}{\frac{11}{25} - \frac{8}{5}\frac{1}{2}} \right|. $$

We now have that $|H(0)| = G\frac{1}{25} = G\frac{25}{21}$, which is 1 based on the condition provided. Therefore, $G = \frac{21}{25}$.

(d) Sketch the magnitude response of the filter, $|H(\omega)|$, as a function of $\omega$.

**Solution:** Due to zeros on the unit circle, the notch filter’s response is zero at frequencies $\omega = \pm\frac{\pi}{3}$. Because the poles are near the zeros, the response rebounds quickly as we move away from this frequency. (We have a sharp notch at that frequency.)

(e) Which frequency $\omega_0$ has the largest magnitude for the filter? (That is, find $\omega_0$ such that $|H(\omega_0)| \geq |H(\omega)|$ for all $\omega$.)

**Solution:** From the plot, it is easy to see that the maximum is at either $\omega = 0$ or $\omega = \pi$. We know that $|H(0)| = 1$, and let us evaluate the other
frequency response,

\[
|H(\omega = \pi)| = G \left| \frac{1 - e^{-j\frac{\pi}{3}}(-1) - e^{+j\frac{\pi}{3}}(-1) + 1}{1 - \frac{4}{5} e^{-j\frac{\pi}{3}}(-1) - \frac{4}{5} e^{+j\frac{\pi}{3}}(-1) + \frac{42}{25}} \right|
\]

\[
= \frac{21}{25} \left| \frac{2 + 2 \cos\left(\frac{\pi}{3}\right)}{\frac{11}{25} + \frac{8}{5} \cos\left(\frac{\pi}{3}\right)} \right|
\]

\[
= \frac{21}{25} \left| \frac{2 + 1}{\frac{11}{25} + \frac{4}{5}} \right|
\]

\[
= \frac{21}{25} \left| \frac{3}{\frac{61}{25}} \right|
\]

\[
= \frac{63}{61}.
\]

This value is greater than one, and so \( \omega_0 = \pi \).
Consider a continuous time signal, \( x_a(t) \), with \( X_a(F) = 0 \) for \(|F| > B\).

Determine the Nyquist rate, \( F_s \), for the following signals.

(a) \( y_a(t) = \frac{dx_a(t)}{dt} \).

**Solution:** The derivative merely multiplies the Fourier response by a term that is linear in the frequency, but the overall bandwidth is unchanged. Therefore, the support of \( Y(F) \) is \([-B, +B]\), and \( F_s = 2B \).

(b) \( y_a(t) = x_a(t/3) \).

**Solution:** The new signal fluctuates three times more slowly. Therefore, instead of sampling at \( 2B \), we can sample at a third of that, \( F_s = \frac{2}{3}B \).

(c) \( y_a(t) = x_a(t) \cos(2\pi F_ct) \).

**Solution:** The product in the time domain is convolution in the frequency domain, where the cosine has deltas at \( \pm F_c \). Therefore, the bandwidth now has support from \(-B - F_c\) up to \( B + F_c\), and \( F_s = 2(B + F_c) \).

(d) \( y_a(t) = (x_a(t))^3 \).

**Solution:** Taking the cube of \( x_a(t) \) is a triple convolution in the frequency domain, which triples the bandwidth and thus also the sampling rate, \( F_s = 6B \).
Question 3

Let \( x(n) = \{3, 4, 1, 2\} \) be the input sequence to a finite impulse response (FIR) filter \( h(n) = \{2, 3, 1\} \) (the underline corresponds to time zero). In this question, we will compare the computational cost of computing \( y(n) \) in the time and frequency domains.

(a) Compute the linear convolution, \( y(n) \), using the time domain formula, i.e., \( y(n) = \sum_m x(m)h(n-m) \).

**Solution:** It is easy to see that

\[
\begin{align*}
y &= 2 \cdot \{3, 4, 1, 2\} + 3 \cdot \{0, 3, 4, 1, 2\} + 1 \cdot \{0, 0, 3, 4, 1, 2\} \\
&= \{6, 8, 2, 4\} + \{0, 9, 12, 3, 6\} + \{0, 0, 3, 4, 1, 2\} \\
&= \{6, 17, 17, 11, 7, 2\}.
\end{align*}
\]

(b) How many zeros need to be padded to \( x(n) \) and \( h(n) \), respectively, in order to avoid aliasing in the output, \( y = x \ast h \), if we want to use the discrete Fourier transform (DFT) and inverse DFT (IDFT) to calculate \( y(n) \)? (Make sure to provide two numbers, one for \( x \) the other for \( h \).)

**Solution:** The length of \( x \) is \( L_x = 4 \), the length of \( h \) is \( L_h = 3 \), and so the length of \( y \) is \( L_y = L_x + L_h - 1 = 4 + 3 - 1 = 6 \). We need to pad \( L_y - L_x = 6 - 4 = 2 \) zeros to \( x \) and \( L_y - L_h = 6 - 3 = 3 \) zeros to \( h \).

(c) Suppose that \( x \) and \( h \) have been defined in Matlab as row vectors. The following Matlab code is supposed to calculate \( y(n) \). However, we have made quite a few mistakes. Please identify 3 mistakes. For each mistake, note the line number that it is on, explain the mistake, and suggest how to correct it.

1. \texttt{Lx=length(x);}  
2. \texttt{Lh=length(H);}  
3. \texttt{Ly=Lx+Lh+1;}  
4. \texttt{xpad=[x zeros(1,Ly-Lx)];}  
5. \texttt{hpad=[h zeros(1,Ly-Lx)];}  
6. \texttt{xf=fft(x);}  
7. \texttt{hf=fft(hpad);}  
8. \texttt{yf=xf*hf;}  
9. \texttt{y=ifft(xf);}  

**Solution:** Line 2 is taking the length of \( H \) instead of \( h \); Line 3 should be subtracting instead of adding 1; Line 5 should be padding with \( Ly - Lh \), not \( Ly - Lx \), zeros; Line 6 should take the FFT of \( xpad \), not \( x \); Line 8 should use a dot multiply; and Line 9 should compute the IDFT of \( yf \), not \( xf \).

(d) We now compare the computational cost of computing \( y \) in the time and frequency domains. Roughly how many operations are needed to compute \( y \) in the time domain? And roughly how many in the frequency domain? (You may assume that a length-\( N \) DFT or IDFT can be computed using \( N \log_2(N) \) operations.) Please express your two answers (for time and frequency) in terms of \( L_x \) and \( L_h \).

**Solution:** In the time domain, the number of operations is proportional to \( L_x \cdot L_h \). In the frequency domain, we must compute two DFTs of length \( L_x \) and \( L_h \), and then multiply the results. The total number of operations is roughly \( 2 \cdot L_x \cdot L_h \).
$L_y$ and one IDFT of the same length. Because $L_y = L_x + L_h - 1$, this is $3(L_x + L_h - 1) \log_2(L_x + L_h - 1)$.

(e) Suppose further that $L_x = L_h = 10$. Using the results of part (d), is it faster to compute $y$ in the time domain or frequency domain? (If you don’t have a calculator, you can assume that the logarithm takes on some simple integer value such as 3 or 7.)

**Solution:** The time domain approach requires $L_x \cdot L_h = 10^2 = 100$ operations. The frequency domain approach requires $3(10 + 10 - 1) \log_2(10 + 10 - 1) = 242$ operations. We can see that it is faster to compute $y$ in the time domain.
Question 4

Compute the 10-point DFT, \( X(k) \), for the following sequences.

(a) \( x(n) = u(n - 3) - u(n - 5) \).

Solution: The input \( x(n) \) takes on the value 1 only for \( n = 3 \) and \( n = 4 \). Therefore, \( X(k) = \sum_{n=0}^{9} x(n)e^{-j\frac{2\pi nk}{10}} = e^{-j\frac{2\pi 3k}{10}} + e^{-j\frac{2\pi 4k}{10}} \).

(b) \( x(n) = \left(\frac{1}{3}\right)^n, 0 \leq n \leq 9 \).

Solution:

\[
X(k) = \sum_{n=0}^{9} x(n)e^{-j\frac{2\pi kn}{10}} = \sum_{n=0}^{9} \left(\frac{1}{3}e^{-j\frac{2\pi k}{10}}\right)^n.
\]

Now define \( a = \frac{1}{3}e^{-j\frac{2\pi k}{10}} = \frac{1}{3}e^{-j\frac{\pi k}{5}} \), and we can see that \( X(k) = \frac{1-a^{10}}{1-a} \), where we note in passing that \( a^{10} = 3^{-10} \).
Question 5

Recall from Matlab Project 1 that aliasing can be used to our advantage when demodulating amplitude modulated (AM) signals using band-pass sampling. This question will explore this property.

(a) A major reason why band-pass sampling works is because multiple frequencies in continuous time have the same discrete time representation for a given sampling frequency, and only one of those multiple frequencies are present in the signal. Consider the discrete sequence $x(n) = \cos(\frac{\pi}{5}n)$. For a sampling frequency $F_s = 2$ kHz, find two different continuous time signals that produce the same discrete time signal $x(n)$.

Solution: For a continuous time signal $x_a(t)$ that satisfies these conditions, we have that $x(n) = x_a(t = \frac{n}{2000}) = \cos(\frac{\pi}{5}n)$. If $x_a(t) = \cos(2\pi F t)$, then $x(n) = \cos(\frac{2\pi F n}{2000})$. In addition to the “standard” $F = 200$, yielding $x_1(t) = \cos(400\pi t)$, signals with $F = 200 + 2000k$ alias to the same result. Possible such signals include $x_2(t) = \cos(4400\pi t)$ for $k = 1$ and $x_3(t) = \cos(8400\pi t)$ for $k = 2$.

(b) Consider an AM system that maps a baseband $[0, 15]$ kHz (you may view this as $[-15, +15]$ kHz if more convenient to do so, because the Fourier transform of a real valued signal is conjugate symmetric) to a double side band $[F_L, F_H] = [F_c - 15, F_c + 15]$ kHz, where $F_c$ is the carrier frequency. Suppose that the carrier frequency is 140 kHz, and so the modulated frequency band is $[125, 155]$ kHz. Find the minimum sampling frequency $F_s$ that brings the modulated frequency band to baseband.

Solution: The objective is to find the lowest sampling frequency, $F_s$, such that the entire modulated frequency band is shifted into the baseband $[-15, 15]$ kHz. The sampling frequency for the base band signal is $F_{sb} = 30$ kHz. Any lower sampling frequency will cause folding of the base band. Thus $F_s \geq F_{sb}$. The carrier frequency, $F_c = 140$ kHz should become 0 Hz in base band for correct mapping by downsampling. This implies that $F_c = h \times F_s$, where $h$ is a positive integer. Let $h$ be the largest integer multiple such that $h \cdot F_{sb} \leq F_c$. Then $h = \lceil 140/30 \rceil = \lceil 4\frac{2}{3} \rceil = 4$, where $\lceil \cdot \rceil$ is the floor operator. The minimum sampling frequency is $F_s = F_c/h$, which is $140/4 = 35$ kHz.