Question 1

Consider a linear time invariant (LTI) discrete time system that responds to an input
\[ x_1(n) = 0.2(0.3)^{n-1}u(n - 1) \]
with output
\[ y_1(n) = \frac{(n-1)!}{3!(n-4)!} (0.2)^4 (0.3)^{n-4}u(n-4). \]
(a) What is the output, \( y_2(n) \), if the input is \( x_2(n) = 0.2(0.3)^{n+1}u(n+1) \)?

Solution: It can be seen that \( x_2(n) = x_1(n+2) \). Due to time invariance, \( y_2(n) = y_1(n+2) \), and
\[
y_2(n) = y_1(n+2) = \frac{((n+2)-1)!}{3!((n+2)-4)!} (0.2)^4 (0.3)^{(n+2)-4}u((n+2)-4)
= \frac{(n+1)!}{3!(n-2)!} (0.2)^4 (0.3)^{n-2}u(n-2).
\]

(b) We learn that the \( z \) transform of \( y_1(n) \) is given by
\[
Y_1(z) = \left( \frac{0.2 z^{-1}}{1 - 0.3 z^{-1}} \right)^4.
\]
Find \( H(z) \), the transfer function of the LTI system.

Solution: We know that \( Y_1(z) = H(z)X_1(z) \), and solve for \( X_1(z) \). One way to solve for \( X_1(z) \) is to realize that \( x_1(n) \) is a delayed version of \( 0.2(0.3)^n u(n) \), whose transform is \( \frac{0.2}{1-0.3z^{-1}} \). Therefore,
\[
X_1(z) = \frac{0.2 z^{-1}}{1 - 0.3 z^{-1}},
\]
where the \( z^{-1} \) in the numerator is due to the time delay. We can see that
\[
H(z) = \frac{Y_1(z)}{X_1(z)} = \left( \frac{0.2 z^{-1}}{1-0.3 z^{-1}} \right)^4 = \left( \frac{0.2 z^{-1}}{1 - 0.3 z^{-1}} \right)^3.
\]
Question 2

Consider a continuous time signal, \( x_a(t) \), with \( X_a(F) = 0 \) for \( |F| > B \). Determine the Nyquist rate, \( F_s \), for the following signals.

(a) \( y_a(t) = x_a(t/2) \).

**Solution:** The new signal fluctuates two times more slowly. Therefore, instead of sampling at \( 2B \), we can sample at half of that, \( F_s = \frac{B}{2} = B \).

(b) \( y_a(t) = x_a(t) \sin(2\pi F_c t) \).

**Solution:** The product in the time domain is convolution in the frequency domain, where the sine has deltas at \( \pm F_c \). Therefore, the bandwidth now has support from \( -B - F_c \) up to \( B + F_c \), and \( F_s = 2(B + F_c) \).

(c) \( y_a(t) = (x_a(t))^4 \).

**Solution:** Taking \( x_a(t) \) to the power of four means three convolutions in the frequency domain between four \( X(F) \) versions, which quadruples the bandwidth and thus also the sampling rate, \( F_s = 4 \cdot 2B = 8B \).
Let $X(k)$ be a 4-point discrete Fourier transform (DFT) of the signal $x(n)$, where $X(0) = 0, X(1) = 3 - j, X(2) = 2, X(3) = 3 + j$. In all parts below, computing $x(n)$ is not required (you will receive full credit only by solving the question without $x(n)$). Moreover, each part can be solved independently of the others, and the question should only require simple arithmetic.

(a) We are given that $X(1) = ax(0) + bx(1) + cx(2) + dx(3)$. Compute $a, b, c,$ and $d$.

**Solution:** We know that

$$X(k) = \frac{1}{4} \sum_{n=0}^{3} x(n)e^{-j2\pi kn/4}$$

$$= \frac{1}{4} \sum_{n=0}^{3} x(n)e^{-j2\pi n/4}$$

$$= \frac{1}{4} \sum_{n=0}^{3} x(n) (e^{-j\pi/2})^n$$

$$= \frac{1}{4} \sum_{n=0}^{3} x(n) (-j)^n,$$

which implies that

$a = (-j)^0 = 1/4$, $b = (-j)^1 = -j/4$, $c = (-j)^2 = -1/4$, $d = (-j)^3 = +j/4$.

(b) Find $x(2)$.

**Solution:** A similar style to part (a) provides

$$x(2) = \frac{1}{4} (X(0)-X(1)+X(2)-X(3)) = \frac{1}{4} (0-(3-j)+2-(3+j)) = -\frac{1}{4} \cdot 4 = -1.$$

(c) Find $\sum_{n=0}^{3} x(n)x(n)^*$.  

**Solution:** Using the Parseval property,

$$\sum_{n=0}^{3} x(n)x(n)^* = \sum_{n=0}^{3} |x(n)|^2 = \frac{1}{4} \sum_{k=0}^{3} |X(k)|^2 = \frac{1}{4} [(0^2)+(3^2+1^2)+(2^2)+(3^2+1^2)] = 6.$$

(d) Suppose that we define $\tilde{x}(n)$ using re-arranged entries of $x(n)$,

$$\tilde{x}(n) = [x(0) \quad x(3) \quad x(2) \quad x(1)].$$

What is $\tilde{X}(k)$, the DFT of $\tilde{x}(n)$?

**Solution:** It can be seen that $\tilde{x}(n)$ is the cyclically time reversed of $x(n)$, in which case $\tilde{X}(k)$ should also be cyclically reversed, $\tilde{X}(k) = [0 \quad 3 + j \quad 2 \quad 3 - j]$.  

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Question 4
Write Matlab code that shifts the vector $x$ to the right by $m$ samples in a cyclical way, and stores the result in $y$. For example, given $x = [1 \ 7 \ 3 \ 5]$ and $m = 3$, the algorithm should generate $y = [7 \ 3 \ 5 \ 1]$. You can assume that (i) the row vector $x$ of unknown length is already in memory and (ii) $m$ is less than the length of $x$, and is also in memory. Make sure that your code can process all versions of $x$ and $m$ that satisfy these properties.

**Solution:** There are many possible solutions. One particularly short one is

$$y = [x(\text{length}(x)-m+1:\text{length}(x)) \ x(1:\text{length}(x)-m)]$$
Question 5

This question explores **deconvolution**, where an input was convolved by some filter, and we want to undo the effect of the convolution. After working out a simple deconvolution system, the question will explore some of its limitations.

Imagine that a few years after studying ECE 421 you are in charge of designing a communication system, where the transmitter sends a discrete time signal \( x(n) \) into a channel, and the receiver measures \( v(n) \) as follows,

\[
v(n) = x(n) - \frac{1}{3}x(n-1).
\]  

(a) Compute the transfer function, \( H(\omega) = \frac{V(\omega)}{X(\omega)} \).

**Solution:** Because \( V(\omega) = X(\omega) - \frac{1}{3}X(\omega)e^{-j\omega} \), we have \( H(\omega) = 1 - \frac{1}{3}e^{-j\omega} \).

(b) The receiver observes \( v(n) \), the output of the channel. Suppose that the receiver also knows \( H(\omega) \). Your goal is to reconstruct (or deconvolve) \( x(n) \) by passing \( v(n) \) through a second filter,

\[
\hat{x}(n) = g(n) \ast v(n).
\]  

Specify a transfer function for \( g \), \( G(\omega) = \frac{\hat{X}(\omega)}{V(\omega)} \), such that \( \hat{x}(n) = x(n) \).

**Solution:** If we choose \( G(\omega) = \frac{1}{H(\omega)} = \frac{1}{1 - \frac{1}{3}e^{-j\omega}} \), then we obtain \( \hat{x}(n) = x(n) \) for all possible inputs.

(c) If you are unsure about your answer in part (b), you may assume that \( \tilde{G}(\omega) = \frac{1}{1+2e^{-j\omega}+3e^{-2j\omega}} \). Using either \( G(\omega) \) from part (b) or \( \tilde{G}(\omega) \), construct a causal difference equation linking \( v(n) \) and \( \hat{x}(n) \).

Once you have designed this filter (2), your manager should be quite pleased, because it should deconvolve \( x(n) \) perfectly.

**Solution:** We solve for both \( G \) and \( \tilde{G} \). For \( G(\omega) = \frac{1}{1 - \frac{1}{3}e^{-j\omega}} \),

\[
\frac{\hat{X}(\omega)}{V(\omega)} = \frac{1}{1 - \frac{1}{3}e^{-j\omega}},
\]

\[
\hat{X}(\omega) \left[ 1 - \frac{1}{3}e^{-j\omega} \right] = V(\omega),
\]

\[
\hat{x}(n) - \frac{1}{3}\hat{x}(n-1) = v(n),
\]

which leads to the causal difference equation,

\[
\hat{x}(n) = \frac{1}{3}\hat{x}(n-1) + v(n).
\]
(d) In parts (a)-(c), you designed a difference equation that performed ideal
deconvolution. However, there are a variety of reasons why this design
methodology might cause problems.

One potential problem occurs if $H(z)$ (we are moving from the Fourier do-
main to the $z$ transfer domain) contains a pole at some $z_1$, and the input $x(n)$
has energy corresponding precisely to that frequency, e.g., $x(n) = (z_1)^nu(n)$.

- Determine a zero $z_1$ of $H(z)$.
- Design a signal that you expect to be “zeroed out” by $H$.
- Compute $v(n)$ for $n = 0, 1, \ldots, 3$; you can assume that the system is
initially at rest (all initial conditions are zero). (Hint: Due to initial
conditions, $v(n)$ might be nonzero for the values of $n$ considered, but
$H$ should drive $v(n)$ toward zero.)

In words, $H$ wiped out our incoming communication signal, and we must
take care to transmit well-designed inputs $x(n)$ into the channel.

**Solution:** We have $H(z) = 1 - \frac{1}{3}z^{-1} = \frac{z - 1}{z}$ with a zero at $z_1 = \frac{1}{3}$. The input
$x(n)$ is designed with this zero, for example $x(n) = (z_1)^nu(n) = 3^{-n}u(n)$.

Next, we can calculate the outputs of the first stage,

\[
\begin{align*}
v(0) &= x(0) - \frac{1}{3}x(0 - 1) = 1 - \frac{1}{3}0 = 1, \\
v(1) &= x(1) - \frac{1}{3}x(1 - 1) = \frac{1}{3} - \frac{1}{3}1 = 0, \\
v(2) &= x(2) - \frac{1}{3}x(2 - 1) = \frac{1}{9} - \frac{1}{3} \frac{1}{3} = 0, \\
v(3) &= x(3) - \frac{1}{3}x(3 - 1) = \frac{1}{27} - \frac{1}{3} \frac{1}{9} = 0.
\end{align*}
\]

Putting aside the initial conditions, which generated nonzero $v(0)$, the rest
of $v(n)$ is zeroed out.

(e) Your manager is content, because your background in ECE 421 has helped
the team realize that $x(n)$ must be well-designed. This is your opportu-
nity to make an even better impression (and hopefully secure a nice bonus!) by
pointing out more problems that the deconvolution approach may have.
Please provide a brief description of one possible problem.

**Solution:** There are many ways to answer this part. One is to say that noise
in the communication system, $v = x * h + noise$, will later get processed by
$G(\omega)$, and the poles of $G$ could magnify noise components. Another answer
is that in practice $H$ would be estimated by $\hat{H}$, and $\frac{1}{\hat{H}}$ might not cancel the
correct $H$ well.