Question 1
Consider the following signal,

\[ x_a(t) = 0.4 \sin(700\pi t) + 0.7 \cos(1000\pi t). \]

1. What is the Nyquist rate required to sample \( x_a(t) \)?

**Solution:** The frequencies in the signal components are \( F_1 = \frac{700\pi}{2\pi} = 350 \) and \( F = \frac{1000\pi}{2\pi} = 500 \). The Nyquist rate is \( F_s = 2 \cdot \max\{F_1, F\} = 1000 \) samples per unit time.

2. Suppose that \( x_a(t) \) from part 1 of this question is sampled at \( F_s = 800 \) samples per unit time. What is the resulting \( x(n) \)? In particular, provide detail about aliasing, including which old frequency gets mapped to which new frequency.

**Solution:** Let us derive \( x(n) \),

\[
\begin{align*}
  x(n) &= x_a(t = \frac{n}{F_s}) \\
        &= 0.4 \sin(700\pi \frac{n}{800}) + 0.7 \cos(1000\pi \frac{n}{800}) \\
        &= 0.4 \sin(\frac{7}{8} \pi n) + 0.7 \cos(\frac{5}{4} \pi n) \\
        &= 0.4 \sin(\frac{7}{8} \pi n) + 0.7 \cos(-\frac{3}{4} \pi n).
\end{align*}
\]

We can see that \( 0.7 \cos(1000\pi t) \), which should have gotten mapped to \( \cos(\frac{5}{4} \pi n) \), gets aliased to \( 0.7 \cos(600\pi t) \).
Question 2
Consider the following system,

\[ y(n) = 3x(2n - 1). \]

1. Is the system \( y(n) \) time invariant? Justify your answer.

**Solution:** Consider the unit impulse input \( x_1(n) = \delta(n) \). For this input, the output is \( y_1(n) = 0 \). It is always zero, because \( x_1(n) \neq 0 \) only when \( n = 0 \). But \( 2n - 1 \) is never zero. In contrast, consider a new shifted input \( x_2(n) = x_1(n - 1) \). The new input \( x_2(n) \) is 1 when \( n = 1 \). The new output satisfies \( y_2(n = 1) = 3x_2(2 \cdot 1 - 1) = 3x_2(1) = 3 \). Note how this new output is not a shifted version of the original input. Therefore, the system \( y(n) \) is not time invariant. (It is time variant.)

2. Suppose that the input is \( x(n) = \{3, 6, -2, 7, 0, 3\} \).

Using the system from part 1 of this question, derive the output \( y(n) \).

**Solution:** For \( n = -2 \), \( y(-2) = 3x(2(-2) - 1) = 3x(-5) = 0 \). For \( n = -1 \), \( y(-1) = 3x(2(-1) - 1) = 3x(-3) = 3 \cdot 3 = 9 \). For \( n = 0 \), \( y(0) = 3x(2(0) - 1) = 3x(-1) = 3 \cdot (-2) = -6 \). For \( n = 1 \), \( y(1) = 3x(2(1) - 1) = 3x(1) = 3 \cdot (0) = 0 \). For \( n = 2 \), \( y(2) = 3x(2(2) - 1) = 3x(3) = 3 \cdot (0) = 0 \). For \( n < -2 \) and \( n > 2 \), clearly \( y(n) = 0 \).
Question 3
Consider the following difference equation,

\[ y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n-1). \]

1. Compute the transfer function \( H(z) = \frac{Y(z)}{X(z)}. \)

**Solution:** We begin by taking the \( z \)-transform of the difference equation,

\[ Y(z) = \frac{5}{6} z^{-1} Y(z) - \frac{1}{6} z^{-2} Y(z) + z^{-1} X(z). \]

Next, we obtain \( H \) by dividing \( Y \) by \( X \),

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}.
\]

2. Determine the zeros and poles of the system \( H(z) \).

**Solution:** For convenience in evaluating the poles, we multiply the denominator and numerator by \( z^2 \),

\[
H(z) = \frac{z}{z^2 - \frac{5}{6} z + \frac{1}{6}} = \frac{z}{(z - \frac{1}{2})(z - \frac{1}{3})}.
\]

There is one zero at \( z_1 = 0 \), and there are two poles at \( p_1 = \frac{1}{2} \) and \( p_2 = \frac{1}{3} \).
3. Suppose that the system is causal.

(a) Determine the region of convergence (ROC).
Solution: Because the system is causal, the ROC is the intersection of ROC’s corresponding to the two poles. For the first pole, \( p_1 = \frac{1}{2} \) and \( \text{ROC}_1 = \{ z : |z| > \frac{1}{2} \} \). For the second pole, \( p_2 = \frac{1}{3} \) and \( \text{ROC}_2 = \{ z : |z| > \frac{1}{3} \} \). The overall ROC is the intersection of the two ROC’s,

\[
\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2 = \{ z : |z| > \frac{1}{2} \}.
\]

(b) Is the system stable?
Solution: The ROC contains the unit circle, thus the system is stable.

(c) Draw the poles, zeros, and ROC.
Solution: Please see Figure 1.

![Figure 1: zero-pole pattern](image-url)
4. Suppose that the input signal is

\[ x(n) = \left( \frac{1}{4} \right)^n u(n), \]

where \( u(n) \) denotes the unit-step function. Using the transfer function \( H(z) \) from part 1 of this question, express \( X(z) \) and \( Y(z) \), the \( z \)-transforms of the input and output signals, respectively.

**Solution:** We begin with \( X(z) \), the \( z \)-transform of the input,

\[ X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}. \]

Next, we multiply \( X(z) \) by the transfer function \( H(z) \),

\[ Y(z) = H(z)X(z) = \frac{1}{(z - \frac{1}{2})(z - \frac{1}{3})(1 - \frac{1}{4}z^{-1})}. \]
Question 4
Consider an input signal $x(n)$ with the following $z$-transform,

$$X(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})}.$$ 

1. Express $X(z)$ using a partial fraction expansion by writing $X(z)$ as a sum of two first-order systems,

$$X(z) = \frac{a}{1 - \frac{1}{2}z^{-1}} + \frac{b}{1 + \frac{1}{2}z^{-1}}.$$ 

In particular, compute the constants $a$ and $b$.

**Solution:** We proceed as follows,

$$1 = a \left(1 + \frac{1}{2}z^{-1}\right) + b \left(1 - \frac{1}{2}z^{-1}\right) = \frac{a - b}{2}z^{-1} + (a + b)$$

Therefore, $\frac{a - b}{2} = 0$ and $a + b = 1$. Substituting $a = b$ into $a + b = 1$, we obtain $(b) + (b) = 2b = 1$, which implies that $b = \frac{1}{2}$. Similarly, $a = b = \frac{1}{2}$.

Therefore,

$$X(z) = \frac{\frac{1}{2}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{2}}{1 + \frac{1}{2}z^{-1}}.$$ 

2. Suppose that in part 1 of this question you got

$$X(z) = \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{7}{1 + \frac{1}{2}z^{-1}},$$

and suppose further that $x(n)$ is a causal signal ($x(n) = 0$ when $n < 0$). Compute $x(n)$.

**Solution:** There are two first-order system. The first one is $X_1(z) = \frac{\frac{1}{2}}{1 - \frac{1}{2}z^{-1}}$, and it gets mapped to $x_1(n) = 2 \left(\frac{1}{2}\right)^n u(n)$. The second one is $X_2(z) = \frac{7}{1 + \frac{1}{2}z^{-1}}$, and it gets mapped to $x_2(n) = 7 \left(-\frac{1}{2}\right)^n u(n)$. Summing these two components,

$$x(n) = \left[ 2 \left(\frac{1}{2}\right)^n + 7 \left(-\frac{1}{2}\right)^n \right] u(n).$$