ECE 421 Introduction to Signal Processing

Midterm 1 – Spring 2019

February 13, 2019

Please remember to justify your answers carefully.

There are 9 pages in total on this test.

Name: _________________________ Student ID: _________________________
Question 1 (sampling)
This question considers continuous time analog signals being sampled at rate $F_s = 2000$ samples per second.

(a) What is the Nyquist frequency $F_N$, which limits the highest analog frequency of signals that can be sampled and later perfectly reconstructed with this sampling rate $F_s$?

(b) Find three different analog signals, $x_1(t)$, $x_2(t)$, and $x_3(t)$, that produce the discrete time signal

$$x(n) = \cos(0.2\pi n)$$

when sampled at $F_s$. 
(c) We are given the following continuous time analog signal,

\[ x_a(t) = \sin(2\pi F_1 t) + \cos(2\pi F_2 t), \]

where \( F_1 = 400 \) Hz and \( F_2 = 1700 \) Hz. We sample \( x_a(t) \) at a sampling rate \( F_s = 2000 \) Hz. What are the discrete time frequencies, \( f_1 \) and \( f_2 \), that correspond to \( F_1 \) and \( F_2 \)? (Make sure that \( f_1, f_2 \in [-0.5, +0.5) \) cycles per sample.)
Question 2 (properties of signals)

Consider the following discrete time system $T$,

$$T[x(n)] = (x(n))^2.$$ 

Is $T$ linear or non-linear? Time variant or time invariant? Causal? Make sure to justify your answer carefully.
Consider a linear time invariant (LTI) discrete time system $H$ whose impulse response $h$ satisfies,

$$h(n) = 0.5^{|n|} + \delta(n) = 0.5^n u(n) + 2^n u(-n - 1) + \delta(n).$$

(a) Determine the $z$-transform, $H(z)$.

(b) Is $H$ causal?

(c) What is the region of convergence (ROC), $ROC_H$?
(d) Is $H$ stable in the bounded input bounded output (BIBO) sense? Please answer in *two different ways.*
Consider the block diagram below. A discrete time input signal, $x(n)$, is processed by a first block, resulting in $q(n)$, which is then processed by a second block, resulting in the discrete time output signal, $y(n)$. We also have a parameter, $k$, which controls the amplification of $q(n - 1)$ and $q(n - 2)$.

(a) Derive the transfer function, $H(z) = Y(z)/X(z)$. (Hint: it may be convenient for some of you to first derive a transfer function corresponding to the first block, $Q(z)/X(z)$, and then to the second block, $Y(z)/Q(z)$.)

(b) Suppose that the parameter $k$ satisfies $k = -4$. Where are the poles of $H(z)$? (Hint: if you are not sure about your answer in part (a), you may assume that $H(z) = \frac{k + z^{-2}}{1 + kz^{-2}}$.)
Question 5 (z-transform)
In this question, you will derive the z-transform of the ramp signal, \( u_r(n) = nu(n) \). To do so, you will work through several parts.

(a) Consider a discrete time system whose input is the step function, \( u(n) \), and whose output satisfies,
\[
y(n) = y(n - 1) + u(n),
\]
where the output is initially at rest, meaning that \( y(n) = 0 \) for \( n < 0 \). Express \( y(n) \) in the form,
\[
y(n) = \alpha u_r(n) + \beta u(n) + \gamma \delta(n).
\]
That is, compute the constants \( \alpha, \beta, \) and \( \gamma \). (Hint: compute \( y(n) \) for several values of \( n \).)

(b) Based on part (a), \( Y(z) = \alpha U_r(z) + \beta Z\{u(n)\} + \gamma Z\{\delta(n)\} \). Compute the \( z \)-transforms of \( \delta(n) \) and \( u(n) \).
(c) Derive $Y(z)$, the $z$-transform of $y(n)$. To do so, recall that $y(n)$ is the output of the difference equation given in (1), and the $z$-transform of its input was already derived in part (b).

(d) You can now combine the above information to compute the $z$-transform of $u_r(n)$. Because there are many opportunities for glitches in this question, this part will not be graded.