Please remember to justify your answers carefully.

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Consider a system $H$ that processes an input $x(n)$ as follows, $y(n) = 3^n x(n + 2)$. We will evaluate different properties of this system.

(a) Is this a causal system? Why?

**Solution:** Because the output $y$ at time index $n$ depends on the input at time index $n + 2$, which is a future time index, the system is not causal.

(b) Is the system bounded input bounded output (BIBO) stable or not? Why?

**Solution:** Consider the input $x(n) = u(n)$. That is, we have a step input. The output can be shown to blow up as $3^n$ as $n$ increases. Therefore, we have a bounded input, and yet the output is unbounded. Therefore, the system is not BIBO stable.
Question 2

Consider a sampling frequency of $F_s = 10\, kHz$.

(a) What is the highest range of analog frequencies that can be sampled and then perfectly reconstructed with $F_s = 10\, kHz$?

**Solution:** The Nyquist rate is equal to twice the highest frequency. Hence, the highest frequency is $10/2 = 5\, kHz$.

(b) Find two different continuous-time signals that will produce the sequence

$$x(n) = \cos(0.25\pi n)$$

when sampled with $F_s = 10\, kHz$.

**Solution:** We want $x_a(t)$ such that $x(n) = x_a(t = n/F_s) = \cos(0.25\pi n)$. Consider $x_a(t) = \cos(2\pi F t)$, then $x(n) = \cos(2\pi F n/F_s)$. One simple solution is that $2F/F_s = 0.25$, yielding $F = 0.125F_s = 1,250$ Hz. In particular, $x_{a1}(t) = \cos(2500\pi t)$ is a possible solution.

A second solution involves aliasing. We will have $2.25\pi$ radians per sample, which will be aliased to $0.25\pi$ radians per sample. This requires $2\pi F/F_s = 2.25\pi$, and we conclude that $F = 1.125F_s = 11,250$ Hz or $x_{a2}(t) = \cos(22500\pi t)$.

(c) Let $x_a(t) = \cos(3500\pi t) + 2\sin(12000\pi t)$. What are the discrete time frequencies of the sinusoids in the sampled sequence when $F_s = 10\, kHz$?

**Solution:** The discrete time frequency $\omega$, for $\Omega = 3500\pi$ radians/second = $0.35\pi$ radians/sample.

The discrete time frequency $\omega$, for $\Omega = 12000\pi$ radians/second = $1.2\pi$ radians/sample. However, due to aliasing, $\sin(1.2\pi n)$ will become $\sin(-0.8\pi n)$. And because $\sin(-x) = -\sin(x)$, we have $x(n) = \cos(0.35\pi n) - 2\sin(0.8\pi n)$.
Question 3

A digital filter obeys the following difference equation,

\[ y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + \frac{7}{8}x(n). \]

The input of the filter is an impulse, i.e., \( x(n) = \delta(n) \), and the initial conditions are \( y(-1) = -1 \) and \( y(-2) = 1 \).

We will compute the output \( y(n) \) in steps using the one sided z transform.

(a) Express the one-sided z transforms of \( y(n-1) \) and \( y(n-2) \) using \( Y^+(z) \), which is the one-sided z transform of \( y(n) \), and the initial conditions.

**Solution:** The one sided transforms obey

\[
Z^+\{y(n-1)\} = z^{-1}[Y^+(z) + y(-1)z] = z^{-1}Y^+(z) - 1,
\]

\[
Z^+\{y(n-2)\} = z^{-2}[Y^+(z) + y(-1)z + y(-2)z^2] = z^{-2}Y^+(z) - z^{-1} + 1.
\]

(b) What is the one-sided transform of \( x(n) \)?

**Solution:** Because \( x(n) = \delta(n) \),

\[
X^+(z) = \sum_{n=0}^{+\infty} x(n)z^{-n} = x(0)z^{-0} = 1.
\]

(c) Using your results from parts (a) and (b), express the difference equation in terms of one-sided z transforms. (Your answer should look something like \( Y^+(z) = c_1z^{-k_1}Y^+(z) + c_2z^{-k_2} + c_3z^{k_3} \); it will be an expression in the one-sided z domain.)

If you did not solve parts (a) and (b), you can assume that \( Z^+\{y(n-1)\} = z^2Y^+(z), Z^+\{y(n-2)\} = 2z^{-2}Y^+(z), \) and \( X^+(z) = 3z \).

**Solution:** Substituting the expressions from (a) and (b),

\[
Y^+(z) = \frac{3}{4}\{z^{-1}Y^+(z) - 1\} - \frac{1}{8}\{z^{-2}Y^+(z) - z^{-1} + 1\} + \frac{7}{8}.
\]

This can be rewritten as

\[
Y^+(z)[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}] = \frac{1}{8}z^{-1}.
\]

The “if you did not solve” part yields

\[
Y^+(z) = \frac{3}{4}z^2Y^+(z) - \frac{1}{8}2z^{-2}Y^+(z) + \frac{3\cdot 7}{8}z.
\]

(d) Express \( Y^+(z) \) using partial fractions. (Your answer should look something like \( Y^+(z) = \frac{13}{1+z^{-1}} - \frac{2}{1-4z^{-1}} \).)
If you did not solve part (c), you can assume $Y^+(z)[1 - 1.5z^{-1} + 0.5z^{-2}] = 7z^{-1}$.

**Solution:** Solving for $Y^+(z)$,

$$Y^+(z) = \frac{1}{8} \left( \frac{z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{5}z^{-2}} \right).$$

Solving by partial fraction expansion,

$$Y^+(z) = \frac{1}{8} \left( \frac{z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} \right) = \frac{a}{1 - \frac{1}{2}z^{-1}} + \frac{b}{1 - \frac{1}{4}z^{-1}}.$$

We have

$$\frac{1}{8}z^{-1} = a(1 - \frac{1}{4}z^{-1}) + b(1 - \frac{1}{2}z^{-1}) = (a + b) + z^{-1}(-\frac{1}{4}a - \frac{1}{2}b).$$

It can be seen that $a + b = 0$, which implies that $b = -a$. Moreover, $-\frac{1}{4}a - \frac{1}{2}b = -\frac{1}{4}a + \frac{1}{2}a = \frac{1}{8}$. Therefore, $a = \frac{1}{2}$ and $b = -\frac{1}{2}$. Substituting these values,

$$Y^+(z) = \frac{\frac{1}{2}}{1 - \frac{1}{2}z^{-1}} - \frac{\frac{1}{2}}{1 - \frac{1}{4}z^{-1}}.$$

The “if you did not solve” part proceeds as follows,

$$Y^+(z) = \frac{7z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}} = \frac{7z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})} = \frac{a}{1 - z^{-1}} + \frac{b}{1 - 0.5z^{-1}}.$$

Next,

$$a(1 - 0.5z^{-1}) + b(1 - z^{-1}) = (a + b) + z^{-1}(-0.5a - b) = 7z^{-1},$$

and similar to before $b = -a, -0.5a - b = -0.5a + a = 0.5a = 7$, yielding $a = 14$ and $b = -14$. Therefore,

$$Y^+(z) = \frac{14}{1 - z^{-1}} - \frac{14}{1 - 0.5z^{-1}}.$$

(e) Compute the output $y(n)$ using the inverse one-sided $z$ transform.

If you did not solve part (d), you can assume that $Y^+(z) = \frac{\frac{1}{2}}{1 - \frac{3}{2}z^{-1}} - \frac{\frac{1}{2}}{1 - \frac{1}{4}z^{-1}}$.

**Solution:** Taking the inverse transform,

$$g(n) = \left[ \frac{1}{2} \left( \frac{1}{2} \right)^n - \frac{1}{2} \left( \frac{1}{4} \right)^n \right] u(n).$$
Question 4

A linear time invariant system is described by the following difference equation,
\[ y(n) = 0.5y(n-1) + bx(n), \]
where \( b \) is a constant that will be determined soon. This difference equation can be shown to be a low pass filter, and we will evaluate its properties.

(a) Find the transfer function in the Fourier domain, \( H(\omega) = \frac{Y(\omega)}{X(\omega)}. \)

**Solution:** It may be convenient to look at the z domain where
\[ Y(z) = b \frac{z}{1 - 0.5z^{-1}}Y(z) + bX(z), \]
which yields \( Y(z)[1 - 0.5z^{-1}] = bX(z). \) Therefore, the transfer function satisfies
\[ H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1 - 0.5z^{-1}}. \]

We convert to the Fourier domain using \( z = e^{j\omega}, \) and so
\[ H(\omega) = \frac{b}{1 - 0.5e^{-j\omega}}. \]

(b) What is the squared magnitude of the transfer function, \( |H(\omega)|^2? \)

**Solution:** The denominator of \( H(\omega) \) can be expressed as \( 1 - 0.5e^{-j\omega} = 1 - 0.5\cos(-\omega) - j0.5\sin(-\omega) = 1 - 0.5\cos(\omega) + j0.5\sin(\omega). \) The squared magnitude of \( H(\omega) \) is given by
\[
|H(\omega)|^2 = \frac{b^2}{|1 - 0.5\cos(\omega) + j0.5\sin(\omega)|^2} = \frac{b^2}{(1 - 0.5\cos(\omega))^2 + (0.5\sin(\omega))^2} = \frac{b^2}{1 - \cos(\omega) + 0.25\cos^2(\omega) + 0.25\sin^2(\omega)}.
\]
Because \( \cos^2(x) + \sin^2(x) = 1, \) we have
\[
|H(\omega)|^2 = \frac{b^2}{1.25 - \cos(\omega)}.
\]

(c) Find the value of \( b \) such that \( |H(\omega)| = 1 \) at frequency \( \omega = 0. \)
If you did not solve part (b), you can assume that \( |H(\omega)|^2 = \sin(\omega) + b\cos(\omega). \)

**Solution:** Because \( \omega = 0 \) and \( \cos(0) = 1, \) \( |H(\omega)|^2 = \frac{b^2}{0.25}. \) We know that \( |H(\omega)| = 1, \) and so
\[
\frac{b^2}{0.25} = 1
\]
Solving for $b$, we have $b = \pm 0.5$.

The “if you did not solve” part proceeds as follows,

$$ |H(\omega)|^2 = \sin(\omega) + b\cos(\omega) = b = 1. $$

(d) Find the half power frequency, i.e., the frequency at which $|H(j\omega)|^2$ is equal to one half of its peak value. It is fine to leave your answer in a form as follows, $4\sin(\omega) - 7\cos(\omega) = 1$.

If you did not solve part (b), you can use the same expression for $|H(\omega)|^2$ as in part (c).

**Solution:** To find the half power frequency, we use the following equation,

$$ \frac{0.25}{1.25 - \cos(\omega)} = 0.5. $$

We need $\cos(\omega) = 0.75$, and so

$$ \omega = \cos^{-1}(0.75). $$

The “if you did not solve” part proceeds as follows,

$$ |H(\omega)|^2 = \sin(\omega) + b\cos(\omega) = \frac{1}{2}b = \frac{1}{2}. $$

Therefore, $\sin(\omega) + \cos(\omega) = \frac{1}{2}$. 

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