Question 1 (transfer functions)
Consider the filter,
\[ y(n) = 0.8y(n-1) + ax(n). \]

(a) Calculate the transfer function, \( H(\omega) = Y(\omega)/X(\omega). \)

**Solution:** Taking the Fourier transform of the difference equation, \( Y(\omega) = 0.8Y(\omega)e^{-j\omega} + aX(\omega). \) Rearranging terms, \( Y(\omega)[1 - 0.8e^{-j\omega}] = aX(\omega), \) and so
\[ H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{a}{1 - 0.8e^{-j\omega}}. \]

(b) Determine \( a \) so that \( H(0) = 1. \)

**Solution:** We want
\[ 1 = H(0) = \frac{a}{1 - 0.8e^{-j\omega}} = \frac{a}{1 - 0.8 \cdot 1} = \frac{a}{0.2}. \]
To satisfy this constraint, \( a = 0.2. \)

(c) Compute a frequency, \( \omega, \) for which \( |H(\omega)|^2 = \frac{1}{21}. \) (If you are unsure about the value of \( a, \) you may assume that \( a = -0.2). \)

**Solution:** We want
\[ |H(\omega)|^2 = \left| \frac{0.2}{1 - 0.8e^{-j\omega}} \right|^2 = \frac{0.2^2}{|1 - 0.8 \cos(\omega) - j0.8 \sin(\omega)|^2} = \frac{0.2^2}{1 + 0.64 \cos^2(\omega) - 2 \cdot 0.8 \cos(\omega) + 0.64 \sin^2(\omega)} = \frac{0.2^2}{1.64 - 1.6 \cos(\omega)} = \frac{1}{21}. \]
Because the numerator is $0.2^2 = 0.04$, the denominator must be 21 times larger, which is 0.84,

$$1.64 - 1.6 \cos(\omega) = 0.84.$$ 

Therefore, $\cos(\omega) = 0.5$, meaning that $\omega = \pm \pi/3$. Finally, using the “you may assume that” route, the numerator is $(-0.2)^2 = 0.04$, which is the same as before. Therefore, $\omega = \pm \pi/3$.

(d) Is this filter lowpass, bandpass, or highpass? Make sure to justify your answer.

**Solution:** In the $z$ domain, the transfer function is $H(z) = \frac{0.2}{1-0.8z^{-1}} = \frac{0.2z}{z-0.8}$ with a zero at the origin ($z = 0$) and pole at 0.8. The pole at 0.8 is close to $\omega = 0$ on the unit circle ($z = e^{j\omega} = 1$), hence it amplifies lower frequencies. This is a lowpass filter.
A moving average filter is used to reduce noise and smooth out data from one value to the next. We define a moving average filter as

$$y(n) = \frac{1}{N+1}(x(n) + x(n-1) + \ldots + x(n-N)).$$

(Note that this definition is somewhat different from that in the slides.) Let $N = 3$. The filter averages over $N + 1 = 4$ samples, and the relationship between the input $x(n)$ and output $y(n)$ is given by the difference equation,

$$y(n) = \frac{x(n) + x(n-1) + x(n-2) + x(n-3)}{4}.$$

(a) Express the impulse response of the averaging filter (use $N = 3$). Plot the impulse response.

**Solution:** We can see from the difference equation that $h(n) = \frac{1}{4}$ for $n \in \{0, 1, 2, 3\}$, else $h(n) = 0$. This impulse response can be expressed as $h = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$. A plot of the impulse response would label the horizontal and vertical axes, and also provide a sense of scale (labeling $n \in \{0, \ldots, 3\}$ and $h(n) = \frac{1}{4}$).

(b) What is the transfer function, $H(z)$, in the $z$-domain? Please include the region of convergence.

**Solution:** The transfer function obeys

$$H(z) = \frac{1}{4}[1 + z^{-1} + z^{-2} + z^{-3}].$$

The limitation in terms of ROC is when $z$ cannot be inverted, meaning $z = 0$. Therefore, $ROC = \{z \in \mathbb{C} : z \neq 0\}$.

(c) What is the output of the moving average filter when the input is $x(n) = \cos(\pi n/2 + 0.3)$? (Hint: your answer should be simple, for example $y(n) = 0.5^n$, and not a complicated formula.)

**Solution:** Note that $x(n)$ is a sinusoidal input. Therefore, $y(n)$ is amplified by $H(\omega)$, where $\omega = \pi/2$,

$$H(\pi/2) = H(z = \exp(j\pi/2) = j) = \frac{1}{4}[1 + j^{-1} + j^{-2} + j^{-3}] = 0.$$

We can see that $x(n)$ is multiplied by zero, hence $y(n) = 0$. 
A resonance filter boosts frequencies by placing a pair of complex conjugate poles near the unit circle. The angular location where the poles are placed, \( \theta \), and distance from the origin, \( r \), control what frequencies get boosted and by how much. In this question, we will design a resonance filter and discuss some of its properties.

(a) We place 2 poles at locations \( p_1 = r \exp(j\theta) \) and \( p_2 = r \exp(-j\theta) \). The poles are placed at conjugate locations in order for the impulse response to be real-valued. We also place two zeros at locations \( z_1 = -1 \) and \( z_2 = +1 \). In this part, you will help us derive \( H(z) = Y(z)/X(z) \), the transfer function of the resonance filter. In the derivation below, \( \alpha \), \( \beta \), and \( \gamma \) are missing; please specify these expressions. (For your convenience, these missing parts are underlined and in red font. Note that \( \alpha \) is missing twice.)

\[
H(z) = \frac{G(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}
\]
\[
= \frac{G}{(z - r \exp(j\theta))(z - r \exp(-j\theta))}
\]
\[
= \frac{G}{z^2 - 2rz \cos(\theta) + \gamma}
\]

where \( G \) is a gain constant.

**Solution:** The missing expresions are

\[
\alpha = z^2 - 1,
\]
\[
\beta = rz(\exp(j\theta) + \exp(-j\theta)),
\]
\[
\gamma = r^2.
\]

(b) Explain possible advantages of placing the zeros at locations \( z_1 = -1 \) and \( z_2 = +1 \).

**Solution:** The locations of the zeros, \( z_1 \) and \( z_2 \), correspond to frequencies \( \omega_1 = 0 \) and \( \omega_2 = \pi \) on the unit circle. These locations ensure that the filter rejects high (\( \omega_2 \)) and low (\( \omega_1 \)) frequencies well.
(c) Instead of $z_1 = -1$ and $z_2 = +1$, an alternative is to place both zeros at the origin, i.e., $z_1 = z_2 = 0$. The advantage of this approach is that all points along the unit circle are equi-distant from the origin, hence $H(\omega)$ is easier to analyze. Explain what must change in the derivation of part (a).

**Solution:** Instead of $z^2 - 1$, we now have $z^2$ in the numerator,

$$\hat{H}(z) = G \frac{z^2}{z^2 - 2rz \cos(\theta) + r^2}.$$ 

(d) The gain constant, $G$, was not specified in part (a). Using the transfer function from part (a) (not the modified one from part (c)), compute $G$ that ensures that $|H(\omega = \pi/2)| = 1$. In your calculation, use the values $r = 0.8$ and $\theta = \pi/2$. (If you are unsure about $H(z)$ from part (a), you may assume that $H(z) = G \frac{z^2 + 3}{z^2 + z \cos(\theta) + 2r}$.)

**Solution:** We want

$$\left| \frac{G \frac{z^2 - 1}{z^2 - 2rz \cos(\theta) + r^2}}{z^2 - 1} \right| = 1.$$ 

Moreover, $\omega = \pi/2$ corresponds to $z = \exp(j\pi/2) = j$, and so

$$|G| = \left| \frac{z^2 - 2rz \cos(\theta) + r^2}{z^2 - 1} \right|$$

$$= \frac{|(0.8^2 - 1) + j(-2 \cdot 0.8 \cos(\pi/2))|}{| - 1 - 1 |}$$

$$= \frac{\sqrt{(1 - 0.8^2)^2 + (2 \cdot 0.8 \cdot 0)^2}}{2}$$

$$= \frac{\sqrt{1 - 0.64)^2}}{2}$$

$$= \frac{1 - 0.64}{2}$$

$$= 0.18.$$ 

For the “you may assume” part,

$$1 = \left| G \frac{z^2 + 3}{z^2 + z \cos(\theta) + 2r} \right|.$$ 

Therefore,

$$|G| = \left| \frac{z^2 + z \cos(\theta) + 2r}{z^2 + 3} \right| = \left| \frac{-1 + j \cdot 0 + 2 \cdot 0.8}{-1 + 3} \right| = \left| \frac{-1 + 1.6}{2} \right| = 0.3.$$