ECE 421 – Introduction to Signal Processing
Project 1

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Due: January 27, 2020

Instructions:

1. The project can be submitted in pairs or individually.
2. For any clarification or doubts, please contact the TA, Weiqi Sun (email: wsun23 AT ncsu DOT edu).
3. You should submit an electronic copy via Moodle by midnight the day that the project is due.
4. Your report should describe any mathematical derivations, responses to questions, results including any plots, and your MATLAB code. Please justify your answers carefully.

Band pass sampling: This project demonstrates that aliasing can be used to our advantage in certain applications. Consider a baseband audio signal that occupies the frequency range 0-20 kHz. (A baseband signal occupies frequencies near zero. Note also that voice signals are often band-limited up to 3 kHz or so, and 20 kHz bandwidth corresponds to high-fidelity audio.)

In amplitude modulation (AM), before transmission an audio signal is modulated (frequency shifted) by a carrier frequency, $F_c$. The frequency shift is achieved by a frequency mixer circuit that multiplies the input signal by $\cos(2\pi F_c t)$ generated by a local oscillator.

![Ideal frequency mixer](image)

Figure 1: Ideal frequency mixer

In traditional AM demodulation, frequency mixing is followed by applying a low pass filter (LPF). The frequency mixing shifts the modulated signal back to its baseband frequency range. The demodulated audio signal can be digitized by sampling, and subsequent digital processing is possible. Sampling can be represented as

$$x(n) = x_a(n/F_s),$$

where $x(n)$ is the digital sequence, $x_a(t)$ is the analog signal, and $F_s$ is the sampling frequency.

Note that the entire demodulation process can be performed digitally if we have an analog to digital (A/D) convertor that samples quickly. However, sampling at high frequencies is expensive, it requires a
fast A/D converter. An alternate approach is to sample the AM channel at a much lower frequency such that aliasing causes all the folded frequencies to fall into the baseband frequency range. As an illustration, a sinusoidal signal \( \cos(2\pi \times 5 \times 10^3 \times t) \) sampled at 4 kHz will become \( \cos(2\pi \times 1 \times 10^3 \times t) \), and a sinusoidal signal \( \cos(2\pi \times 3 \times 10^3 \times t) \) sampled at 4 kHz will become \( \cos(2\pi \times -1 \times 10^3 \times t) = \cos(2\pi \times 1 \times 10^3 \times t) \). For detailed explanations about folding, see page 24 in the text book. This scheme allows us to avoid frequency mixing electronics and low pass filters, which are analog.

**Alias system design and simulation:** Consider an audio signal with baseband \([0,20 kHz]\) modulated by a carrier frequency, \( F_c \) using the frequency mixer circuit shown in Figure 1. This creates a double side band signal, with a lower side band \([F_c - 20 kHz, F_c]\), which is the flipped version of the baseband, and an upper side band \([F_c, F_c + 20 kHz]\). The double side band is represented as \([F_l, F_h] = [F_c - 20 kHz, F_c + 20 kHz]\).

1. Derive the frequency mixer output in the AM modulator, \( x_{fm1}(t) \) if the input signal is \( x_{in}(t) = \cos(2\pi F t) \), and the local oscillator signal is \( x_{os}(t) = \cos(2\pi F_c t) \). That is, express the product as a sum of cosines.
2. Derive the frequency mixer output of the AM demodulator, \( x_{fm2}(t) \) if the input signal is \( x_{in}(t) = x_{fm1}(t) \), and the local oscillator signal is \( x_{os}(t) = \cos(2\pi F_c t) \). That is, express the output as a sum of cosines. Explain which cosines should be removed by low pass filtering in order to recover the original signal, \( x_{in}(t) \).
3. Provide pseudo code or a formula to find the lowest sampling frequency, \( F_{sl} \) that folds the entire modulated frequency band \([F_l, F_h]\), into the baseband \([0,20 kHz]\).

**NOTE:** Sampling at \( F_{sl} = 40 kHz \) irrespective of the position of \([F_l, F_h]\) may cause undesirable folding. For example, if \([F_l, F_h] = [30 kHz, 70 kHz]\), and \( F_{sl} = 40 kHz \), the frequency band \([50 kHz, 60 kHz]\) is mapped to \([10 kHz, 20 kHz]\) instead of \([0 kHz, 10 kHz]\). Table 1 shows examples of correct \( F_{sl} \)s for given \([F_l, F_h]\).

<table>
<thead>
<tr>
<th>Modulated frequency range</th>
<th>Lowest sampling frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>([20 kHz, 60 kHz])</td>
<td>40 kHz</td>
</tr>
<tr>
<td>([30 kHz, 70 kHz])</td>
<td>50 kHz</td>
</tr>
<tr>
<td>([40 kHz, 80 kHz])</td>
<td>60 kHz</td>
</tr>
<tr>
<td>([60 kHz, 100 kHz])</td>
<td>40 kHz</td>
</tr>
<tr>
<td>([70 kHz, 110 kHz])</td>
<td>45 kHz</td>
</tr>
<tr>
<td>([140 kHz, 180 kHz])</td>
<td>40 kHz</td>
</tr>
<tr>
<td>([135 kHz, 165 kHz])</td>
<td>30 kHz</td>
</tr>
<tr>
<td>([130 kHz, 170 kHz])</td>
<td>50 kHz</td>
</tr>
</tbody>
</table>

Table 1: Examples for \([F_l, F_h]\) vs \( F_{sl} \)

4. Using the pseudo code or formula designed above, find the lowest sampling frequency, \( F_{sl} \) for frequency bands
   a. \([F_l, F_h] = [100 kHz, 140 kHz]\)
   b. \([F_l, F_h] = [110 kHz, 150 kHz]\)
5. For real systems, the sampling frequency is enough to implement a band pass sampling system using an A/D converter. However, to simulate band pass sampling in MATLAB, we can use the technique of downsampling, which is defined as
\[ x_{fm2}(m) = x_{fm1}(n = km), m \in \mathbb{Z}, n \in \mathbb{Z}, \]
where \( x_{fm2}(m) \) is the downsampled sequence, \( x_{fm1}(n) \) is the sequence sampled from \( x_{fm1}(t) \) with sampling frequency, \( F_{sa} \) that preserves the whole analog band pass signal, and \( k \) is the downsampling factor. In the analog domain, the above equation is equivalent to
\[ x_{fm2}\left(\frac{m}{F_{s_l}}\right) = x_{fm1}\left(\frac{n}{F_{sa}}\right). \]

Let \( F_{sa} = 1200 \text{ kHz} \), compute the downsampling factor, \( k \) for \([F_l, F_h] = [100 \text{ kHz}, 140 \text{ kHz}]\).

6. Consider \([F_l, F_h] = [100 \text{ kHz}, 140 \text{ kHz}]\), and \( F_{sa} = 1200 \text{ kHz} \). Test the aliasing system in MATLAB using 3 modulated sinusoidal signals defined below.
   a. \( s_1(t) = \cos(2\pi*135*10^3*t) \)
   b. \( s_2(t) = \cos(2\pi*125*10^3*t) \)
   c. \( s_3(t) = \cos(2\pi*115*10^3*t) \)

Measure the frequency of each sinusoid after aliasing with the designed downsampling factor, \( k \). Plot the test signals and aliased signals.

The next question is mathematical in nature, and unrelated to the previous bandpass part of the project. The question is intended to help you practice the mathematical aspects of ECE421 with questions that are hopefully a bit more interesting than those offered in Webwork.

7. In this problem, some characteristics of systems are examined. For each part, propose a counter example of the property being considered, or prove the property. Note that input signals can be complex valued, and \( \text{Im}\{x\} \) means Imaginary part of \( x \). As a hint, you’ll need to come up with more counter examples than proofs.

   a) Determine whether the following systems are linear or non-linear.
      i. \( y(n) = \text{Im}\{x(n)\} \)
      ii. \( y(n) = x(n) + 2 \)
   b) Determine whether the following systems are Time-variant or Time-invariant.
      i. \( y(n) = x(-n) \)
      ii. \( y(n) = \begin{cases} 2x\left(\frac{n}{2}\right) & n \text{ is even} \\ x\left(\frac{n+1}{2}\right) & n \text{ is odd} \end{cases} \)
   c) Determine whether the following systems are BIBO or not.
      i. \( y(n) = \begin{cases} \frac{1}{x(n)-1} & n \geq 0 \\ \frac{1}{x(n)} & n < 0 \end{cases} \)
      ii. \( y(n) = e^{\frac{1}{x(n)}} \)
Discussion of project
Because the project contains plenty of notations and concepts, here is some material that can help you focus on the main ideas. First, the structure of the AM system:

1. A sinusoidal input $x_{in}(t)$ enters a modulator.
2. AM modulation involves multiplying by $x_{os}(t)$, which is another cosine at the carrier frequency, and results in $x_{fm1}(t)$, the output of the first stage.
3. This output, $x_{fm1}(t)$, becomes the input $x_{in}(t)$ of the AM demodulator, which once again multiplies by $x_{os}(t)$, which is the same sinusoid as before, resulting in $x_{fm2}(t)$, the output of the second stage.
4. That same output from the first stage, $x_{fm1}(t)$, is sampled at rate $F_{sa}$ resulting in a discrete time $x_{fm1}(n)$.
5. The discrete time $x_{fm1}(n)$ is down-sampled by a factor $k$, resulting in $x_{fm2}(m)$. Down sampling means that only every $k$'th sample of $x_{fm1}$ appears in $x_{fm2}$. Another way of thinking about the down sampling is that the ration between $F_{sa}$ and $F_{sl}$ is $k$.

And some comments about part 3. The range $[F_l,F_h]$ will be occupied by the signal. We want to sample the signal such that the middle of the frequency range, $F_{mid}=(F_l+F_h)/2$, gets aliased / folded to DC, which is the zero frequency. Therefore, we want $F_s$ to be one among the frequencies $F_{mid}$, $F_{mid}/2$, $F_{mid}/3$, and so on. Sampling at these rates will “fold” the middle of the band at $F_{mid}$ to DC. At the same time, we also want the sampling rate $F_s$ to be fast enough to be above Nyquist, meaning that $F_s$ is at least as large as $F_h-F_l$. These two considerations allow to solve part 3.