1 Administrative Instructions

1. The project should be submitted in pairs or triples.

2. For any clarification or doubts, please contact the TAs, Hangjin Liu (email: hliu25 AT ncsu DOT edu).

3. You should submit an electronic copy via Moodle by midnight the day that the project is due.

4. Your report should describe any mathematical derivations, responses to questions, results including any plots, and your MATLAB code. Please justify your answers carefully.

2 Background

The goal of spectrum analysis is to determine the frequency content of an analog (continuous-time) signal. This is often accomplished by sampling the analog signal, windowing (truncating) the data, and computing and plotting the magnitude of its discrete Fourier transform (DFT). The DFT will be studied by the time the project is due; our explanations in this project document will help you make progress even before the DFT is covered in class.

2.1 Discrete-Time Fourier Transform (DTFT)

The Discrete-Time Fourier Transform (DTFT) is the primary theoretical tool for understanding the frequency content of a discrete-time (sampled) signal. The DTFT and the inverse DTFT (IDTFT) are defined as

\[ X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad (1) \]

\[ x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega \quad (2) \]

The DTFT is useful for theory and analysis, but is not practical for numerically computing a spectrum digitally, because infinite time samples in (1) require infinite computation and infinite delay, and (2) requires the evaluation of an integral.
2.2 Discrete Fourier Transform (DFT)

The discrete Fourier transform (DFT) is a sampled version of the DTFT, hence it is better suited for numerical evaluation on computers. The DFT and the inverse DFT (IDFT) are defined as

\[ X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \]  

\[ x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \]  

Here \( X(k) \) is an \( N \) point DFT of \( x(n) \). Note that \( X(k) \) is a function of a discrete integer \( k \), where \( k \) ranges from 0 to \( N - 1 \).

2.3 Relationships Between DFT and DTFT

The DTFT usually cannot be computed precisely, because the sum in \( e^{-j\omega_k n} \) is infinite. However, the DTFT may be approximately computed by truncating the sum to a finite window. When a discrete-time sequence happens to equal zero for all samples except for those between 0 and \( N - 1 \), the infinite sum in the DTFT equation \( e^{-j\omega_k n} \) becomes the same as the finite sum from 0 to \( N - 1 \) in the DFT equation (3). By matching the arguments in the exponential terms, we observe that the DFT values coincide with the DTFT for specific DTFT frequencies \( \omega_k = \frac{2\pi k}{N} \). That is, the DFT samples the DTFT at \( N \) equally spaced frequencies \( \omega_k = \frac{2\pi k}{N} \),

\[ X\left(\omega_k = \frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega_k n} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}} = X(k) \]  

2.4 Relationships Between DFT and Original Analog Signal

Let us relate DFT frequency samples to the original signal’s analog frequency content. Suppose that the analog signal is bandlimited and that the sampling frequency exceeds twice that; we are sampling above the Nyquist rate, and there is no aliasing. Therefore, the relationship between the continuous-time Fourier frequency \( \Omega \) (in radians) and DTFT frequency \( \omega \) imposed by sampling is \( \omega = \Omega T \), where \( T \) is the sampling period. Through the relationship \( \omega_k = \frac{2\pi k}{N} \) between the DTFT frequency \( \omega \) and the DFT frequency index \( k \), the correspondence between the DFT frequency index and the original analog frequency can be found:

\[ \Omega = \frac{2\pi k}{NT} \]  

or in terms of analog frequency \( f \) in Hertz (cycles per second rather than radians)

\[ f = \frac{k}{NT} \]  

where constraining \( k \) to the range between 0 and \( \frac{N}{2} \) ensures that we are sampling above the Nyquist rate.

* It is possible that we define things somewhat differently in this project than material covered in class. Please use the definitions and notations introduced here. While this might be somewhat inconvenient, self-learning is something that most of you will do during your careers. Resources such as the “standard textbook” are mostly available for mature material, and the world is constantly changing.
2.5 Zero-Padding and Picket-Fence Effect

The spacing between samples of the DTFT is determined by the number of points in the DFT. This can lead to surprising results when the number of samples is too small. To get a clear picture about the DTFT, zero-padding can be used to interpolate the spectrum. In particular, if $LN$ DTFT samples are desired of a length-$N$ sequence, we can compute the length-$LN$ DFT of a length-$LN$ zero-padded sequence as

$$z(n) = \begin{cases} 
  x(n) & \text{if } 0 \leq n \leq N - 1 \\
  0 & \text{if } N \leq n \leq LN - 1
\end{cases} \quad (8)$$

$$X\left(\omega_k = \frac{2\pi k}{LN}\right) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{LN}} = \sum_{n=0}^{LN-1} z(n) e^{-j\frac{2\pi nk}{LN}} = \text{DFT}_{LN} [z(n)] \quad (9)$$

Figure 1 shows the effects of zero-padding and spectral analysis. The left column of Figure 1 shows the magnitude of the DFT values corresponding to the non-negative frequencies of a real-valued length-64 DFT of a length-64 signal. You can see that the signal has a single dominant frequency component. In the middle column of Figure 1, 64 zero values are appended to the signal to derive a length-128 DFT; we call this factor-2 zero-padding, because the number of samples increased 2X from 64 to 128. The signal is now comprised of at least two narrowband frequency components; the notch or gap between them falls between DFT samples in the left column of Figure 1 (the 2X finer spectral resolution creates this gap), resulting in a misleading view of the spectral content. The line graph (panel e) is still a bit rough and the peak magnitudes and frequencies may not be precisely captured. The right column of Figure 1 provides a somewhat smoother rendition of the DFT using factor-16 zero-padding.

This phenomenon of missing information between samples in spectrum is called the **picket-fence effect**, and is a result of insufficient frequency sampling. While factor-2 zero-padding reveals more structure, it is unclear whether the peak magnitudes are reliably rendered, and the jagged linear interpolation in the line graph (panel e) does not yet reflect the smooth, continuously-differentiable spectrum of the DTFT of a finite-length truncated signal. Errors in the apparent peak magnitude due to insufficient frequency sampling are sometimes referred to as scalloping loss.

2.6 Windowing and Spectral Leakage

In Section 2.3, we assumed that the DTFT may be approximately computed by truncating the sum to a finite window. Let $w(n)$ be a rectangular window of length $N$:  

$$w(n) = \begin{cases} 
  1 & 0 \leq n \leq N - 1 \\
  0 & \text{else}
\end{cases} \quad (10)$$

We define a truncated version of the signal, $x_{tr}$, as

$$x_{tr}(n) = w(n) x(n) \quad (11)$$

Applying the DTFT multiplication property

$$\hat{X}(\omega_k) = \sum_{n=-\infty}^{\infty} x(n)w(n)e^{-j\omega_k n} = \frac{1}{2\pi} X(\omega_k) * W(\omega_k) \quad (12)$$

We find that the DFT of the windowed (truncated) signal does not produce samples of the true (desired) DTFT spectrum $X(\omega)$, but of a smoothed version $X(\omega_k) * W(\omega_k)$. We want the DFT of the windowed signal to resemble $X(\omega)$ as closely as possible, and so $W(\omega)$ should be as close to an
impulse as possible. The simple truncation window has a periodic sinc DTFT, as shown in Figure 2. The main lobe represents the desired frequency component, while the side lobes are the newly created frequency components due to windowing. This phenomenon of leaking energy from peak value to other samples is called spectral leakage.

The window \( w(n) \) needs not be a simple truncation (rectangle) window; other shapes can also be used as long as they limit the sequence to at most \( N \) consecutive non-zero samples, e.g., Hann window, Hamming window, etc. We will revisit windowing techniques toward the end of the semester; for now the project can give you a sneak peak at them.

3 Tasks

3.1 Spectral Analysis

Given the analog signal

\[
x_a(t) = 3 \cdot e^{-53.12t} \cdot \left[ \sin (1600\pi t + \frac{\pi}{4}) \right] + e^{-45.97t} \cdot \left[ \sin (2600\pi t + \frac{\pi}{2}) \right]
\]

for \( t = 0 \) to 100 years, you need to estimate the magnitude spectrum \( X_a[\Omega] \).

1. Sample \( x_a(t) \) with sampling frequency \( F_s = \frac{1}{T} = 12 \) kHz to obtain the discrete signal \( x(n) \). Provide the expression for \( x(n) \) and plot \( x(n) \) for \( n = 0, 1, \ldots, 127 \).

2. Using the MATLAB function \( \text{fft}(\cdot) \), compute the \( N \)-point DFT \( x(k) \), \( k = 0, 1, \ldots, N \). Plot the quantity \( 20 \log_{10} |x(k)| \), which is dB scale, for the following cases:
Figure 2: Length-64 truncation (rectangle, also called boxcar) window and its magnitude DFT spectrum.

i) \( N = 16 \), using the samples \( x(0), x(1), \ldots, x(15) \).

ii) \( N = 128 \), using the samples \( x(0), x(1), \ldots, x(15) \) plus 112 appended zeros. (This is factor-8 zero-padding.)

iii) \( N = 128 \), using the samples \( x(0), x(1), \ldots, x(63) \) plus 64 appended zeros. (This is factor-2 zero-padding.)

iv) \( N = 128 \), using the samples \( x(0), 0, x(1), 0, x(2), \ldots, x(62), 0, x(63), 0 \). (This is known as up-sampling.)

v) \( N = 128 \), using the samples \( x(0), x(1), \ldots, x(127) \). (The original signal.)

Use the same horizontal axis to represent the underlying range \( 0 \leq \omega \leq 2\pi \) in all cases. Recall that \( \omega = \frac{2\pi k}{N} \). Compare and discuss the results obtained. Comment on the concepts of spectral resolution and spectral leakage. Why do the obtained spectra seem different? Do you see any aliasing effects?

4 Mathematical Problem

The following problem is unrelated to the project, and involves working out some math, which will allow us to provide you with feedback. (This is problem 7.18 from the textbook.)

A linear time-invariant system with frequency response \( H(\omega) \) is excited with the periodic input

\[
x(n) = \sum_{k=-\infty}^{+\infty} \delta(n-kN)
\]

Suppose that we compute the \( N \)-point DFT, \( Y(k) \), of the samples \( y(n), 0 \leq n \leq N-1 \) of the output sequence. How is \( Y(k) \) related to \( H(\omega) \)?
Examples

Figure 3: Time domain

Figure 4: N=16
Figure 5: N=128 (factor-8 zero-padding)

Figure 6: N=128 (factor-2 zero-padding)
Figure 7: N=128 (up-sampling)

Figure 8: N=128 (original signal)