Solving Tanaka’s Fixed Point Equation

Dror Baron

This supplement provides more details about solving Tanaka’s fixed point equation.

**Fixed point equation:** Recall our derivation for Tanaka’s fixed point equation,

\[
\frac{1}{\eta} = 1 + \frac{\gamma}{\delta} \text{MMSE}(P_U, \gamma\eta, \epsilon),
\]

(1)

where \(\eta \in (0, 1)\) is the degradation of the equivalent scalar channel, \(\delta = M/N > 0\) is the aspect ratio of the measurement matrix, MMSE(·) is the minimum mean squared error obtained by a denoiser, \(P_U\) is the probability density function (pdf) for non-zeros in the signal, \(\gamma > 0\) is the signal to noise ratio (SNR), and \(\epsilon\) is the sparsity rate, which is the probability of non-zeros in the signal.

This fixed point equation (1) was originally derived for code division multiple access (CDMA) communication systems by Tanaka in 2002. It was revisited by Guo and Verdu in 2005, and by multiple other authors over the next several years. These derivations rely on the symmetry assumption of the replica method. While this method is commonly used in statistical physics, it is not rigorous. More recently, two research groups (Reeves and Pfister, and Krzakala et al.) provided rigorous proofs of Tanaka’s equation.

**Solutions of Tanaka’s equation:** A fixed point equation has the property that the variable (in this case \(\eta\)) appears on both the right- and left-hand sides (RHS and LHS, respectively), but due to complicated mathematical expressions on the LHS, RHS, or both, it is difficult to solve \(\eta\) directly.

One intuitive approach to solving fixed point equations involves plotting the LHS and RHS as functions of \(\eta\), and evaluating where the curves intersect. If there is a single intersection point, then the corresponding degradation \(\eta^*\) characterizes the equivalent scalar channel, and the MMSE of the original linear inverse problem, \(y = \sqrt{\gamma}Ax + z\), can be computed from \(\eta^*\). Moreover, for one fixed point, the MSE performance of approximate message passing (AMP) is optimal in the large system limit.

**Multiple fixed points:** In some configurations, there are multiple fixed points that solve Tanaka’s equation, and AMP achieves the worst (lowest or most severe degradation \(\eta\)) fixed point. Other algorithms based on belief propagation (BP) will also achieve the MSE performance corresponding to the worst fixed point. In some cases, this worst fixed point is indeed the correct one, and BP-based approaches such as AMP are once again asymptotically optimal in the sense that they achieve the best-possible MMSE. However, in other cases one another fixed point is the correct one, and it is possible that future algorithms will
yield lower MSEs than AMP in this case. Guo and Verdu showed in their work from 2005 how to determine which fixed point is correct. Discussions about the MSE performance of BP-based approaches in regions where there are multiple fixed points appear in Krzakala et al. (2012) and Zhu and Baron (2013). Recent work by Zhu, Krzakala, and Baron (2017) has extended these results involving the possibility of multiple fixed points to reconstruction problems involving multiple correlated signals.

**Numerical solution:** Again, it is infeasible to solve Tanaka’s equation (1) directly. Instead, we plot the LHS and RHS, and check where they intersect. Such an evaluation can be performed graphically, or in an automated fashion using a numerical solver in software, for example with Matlab. In principle, computing the RHS can be carried out by calculating the MSE function of the denoiser for the equivalent scalar channel in closed form. That is, the scalar channel is $V = X + W$ (here $X$ is a scalar random variable and not a length-$N$ random vector), the MMSE-achieving denoiser is conditional expectation, $\hat{X} = E[X|V]$, and the MMSE itself obeys

$$\text{MMSE} = E[(X - E[X|V])^2],$$

where expectation is over all possible values of the scalar random variables, $X$ and $V$.

For some types of random variables, the MSE is easily computed. For others, it is more difficult to do so. A simple alternative presented in class is to simulate many (millions of) instances of the signal $X$ and scalar channel output $v$, denoise to $\hat{x}$, and compute the empirical MSE. Finally, the RHS is computed using this empirical MSE.

\footnote{For some distributions $P_X$, it is possible to achieve the MMSE using an algorithm whose computational complexity is exponential in $N$. However, such algorithms are computationally intractable for problems of practical interest.}