Performance Regions in Compressed Sensing from Noisy Measurements

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Main Idea
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Noiseless Compressed Sensing

Input $x$ is sparse

- Measurements are taken by a matrix $\Phi \in \mathbb{R}^{M \times N}$, $M < N$
- Exploit the sparsity of the signal, resulting in fewer measurements
Application of Compressed Sensing

- Medical imaging
- CDMA
- Seismic imaging
- Financial prediction
Crucial Reality

- Noise Noise Noise!!!
- For CS, much less attention has been focused on noisy case
Additive Noise Channel

Input $x_i \sim f_X(x_i)$ i.i.d.
Measurement matrix $\Phi \in \mathbb{R}^{M \times N}$ has unit norm rows, $\Phi_{ij} \sim \mathcal{N}(0, 1/N)$
$\gamma$ is inverse noise level, related to SNR
Measurement rate $R = \frac{M}{N}$
Conventional Signal Reconstruction

In vector channel $y = \sqrt{\gamma} \Phi x + z$, increased $\gamma$ leads to reduction in error.

Not the case for CS!
Surprise [Krzakala et al. ’12] had similar results
Background and Problem Setting
Decoupling Theorem [Guo & Wang 08]

- Large system limit
  \[
  \lim_{N \to \infty} \frac{M}{N} = R > 0
  \]

- Decouple vector channel
  \[
  y = \sqrt{\gamma} \Phi x + z
  \]

  into scalar channel
  \[
  \tilde{y}_i = \sqrt{\gamma \eta} R x_i + \tilde{z}_i, \quad i \in \{1, 2, \ldots, N\}
  \]

- \(\tilde{y}_i, \tilde{z}_i\) sufficient statistics for \(y, z\)
- \(\eta\) degradation of the channel; the bigger the better
- Easier to analyze
- Want to compute minimum mean square error (MMSE)
Tanaka’s Fixed Point Equation [Tanaka 02]

- Fundamental information theoretical limit for scalar channel
  \[ \frac{1}{\eta} = 1 + \gamma \cdot \text{MMSE}(\eta) \]

- Let
  \[ F_1(\eta) = \eta; \quad F_2(\eta) = 1 - \eta \gamma \cdot \text{MMSE}(\eta) \]

- Solution for $\eta$ is fixed point of equation
Decide Correct Fixed Point

- May have multiple fixed points (solutions for $\eta$)
- Correct one minimizes free energy (from statistical physics)

$$E(\eta) = I(x_i; \sqrt{\gamma \eta} R x_i + \tilde{z}_i) + \frac{R}{2} [(\eta - 1) \log_2(e) - \log_2(\eta)]$$
Sparse Gaussian distribution with sparsity $p$

$$f_{X_i}(x_i) = p \cdot \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2} + (1 - p) \cdot \delta_0(x_i), \quad i \in \{1, 2, ..., N\}$$

Gaussian noise

$$f_{Z_i}(z_i) = \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2}, \quad i \in \{1, 2, ..., N\}$$

Can obtain expression for MMSE, solve for $\eta$
Main Results
Main Results

Different Regions and Thresholds

[Image of graph showing different regions and thresholds]

[Wu & Verdú '11]

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Performance Regions in CS

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Main Results

Different Regions and Thresholds

\[ \gamma \rightarrow \infty \]

\begin{align*}
\text{Region 1}_{\text{low}} & : \gamma_{\text{low}} \\
\text{Region 1}_{\text{high}} & : \gamma_{\text{high}} \\
\text{Region 3}_{\text{low}} & : \gamma_{\text{low}} \\
\text{Region 3}_{\text{high}} & : \gamma_{\text{high}} \\
\end{align*}
Main Results

Different Regions and Thresholds

\[ \gamma (\text{dB}) \]

Region 4

Region 1\(_{\text{high}}\)

Region 1\(_{\text{low}}\)

\[ R \]

γ→∞
Different Regions and Thresholds

\[ \gamma \rightarrow \infty \]

Region 3_

Region 3_

Region 2

\[ \gamma \rightarrow \infty \]

Region 3_

Region 3_

0 10 20 30 40 50 60 70 80 90

0.10

0.12

0.14

0.16

0.18

0.20

0.22

0

0.01

0.02

0.03

0.04

0.05

0.06

0.07

0.08

0.09

0.1

0

0.1

0.09

0.08

0.07

0.06

0.05

0.04

0.03

0.02

0.01

0
Main Results

Different Regions and Thresholds

\[ \gamma (\text{dB}) \]

\[ R_c(\gamma) \]

\[ R_l(\gamma) \]

\[ R_{bp}(\gamma) \]

\[ \gamma \to \infty \]
**Realistic Case** Sparse Gaussian example \((p = 0.1)\)

Need to operate above \(R_i(\gamma)\)
Realistic Case in log Scale  Sparse Gaussian example ($p = 0.1$)

- MMSE in Region 3 decreases exponentially with $\gamma$.
The correct $\eta$ jumps, leading to the discontinuity at $R_l(\gamma)$. 

**Main Results**

**Fixed Point v.s. Free Energy** [Krzakala et al. ’12]

![Graph showing the relationship between Free Energy and $\eta$ for different $\gamma$ values. The graph illustrates the discontinuity at $R_l(\gamma)$.](image)
Main Results

Fixed Point v.s. Free Energy [Krzakala et al. ’12]

The correct $\eta$ jumps, leading to the discontinuity at $R_l(\gamma)$
Fixed Point v.s. Free Energy [Krzakala et al. ’12]

The correct $\eta$ jumps, leading to the discontinuity at $R_l(\gamma)$
Fixed Point v.s. Free Energy [Krzakala et al. ’12]

- The correct $\eta$ jumps, leading to the discontinuity at $R_l(\gamma)$
Main Results

Thresholds for Possible CS Reconstruction

- **Noiseless case**
  - $\ell_1$ minimization: $R \gtrsim -p \log(p)$  
    [Donoho & Tanner 06]
  - $\ell_0$ minimization: $R > p$

- **Low noise case:** $R > R_r = p$  
  [Wu & Verdú ’11]

- **Noisier case**
  - With ultimate reconstruction method: $R > R_l(\gamma)$  
    NEW!
  - With BP: ???
Evaluation of BP Performance
BP Results by Running GAMP [Rangan ’10]
Sparse Gaussian input, $p = 0.1$, $N = 10^4$, 100 repetitions

- Suboptimal in Region 3
- BP operates at smallest $\eta$
- Still have room for improvement
In realistic case, BP must operate above $R_{bp}(\gamma)$. 
Summary
Summary

- For CS, reconstruction error behaves differently in different regions.
- BP does not yield satisfactory performance below $R_{bp}(\gamma)$.
- Still have room for improvement in Region 3.
- Noisier case:
  - With ultimate reconstruction method: $R > R_i(\gamma)$.
  - With BP: $R > R_{bp}(\gamma)$.
Thank you!