Signal Reconstruction in Linear Mixing Systems with Different Error Metrics

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Motivation
Multiuser communication (CDMA)

- Users transmit according to columns of matrix $\Phi$
  - most users quiet; few users active $\rightarrow$ sparse
  - we do not know which users active
  - under-determined system
- Receiver sees noisy version $y = \Phi x + z$
- Cellphones estimate $x$ from $y$ and $\Phi$
Pervasive in Science and Engineering

- Medical imaging (tomography)
- Multiuser communication (CDMA)
- Financial prediction
- Electromagnetic scattering
- Seismic imaging (oil industry)
- Compressed Sensing (underdetermined systems)
Problem Model
Measurements $y$ depend linearly on unknown input $x$

Always contains errors
Goal: estimate $x$ from $y$ and $\Phi$
Error Metric

- Additive error metric:
  \[ D(\hat{x}, x) = \sum_{i=1}^{N} d(\hat{x}_i, x_i) \]

- Want to minimize expected error:
  \[ E[D(\hat{x}, x)] \]
What Next?

- Need correct error metric
- Need posterior information

Problem Model

What if a non-standard metric is desired?
Metric-optimal Algorithm
Posterior Information

- Decoupling Principle [Guo & Verdú, 2005; Guo & Wang, 2007]
- Relaxed Belief Propagation (BP) [Rangan, 2010]

\[ w = \Phi x \]

Linear mixing channels

\[ q_i = x_i + v_i, \quad v_i \sim \mathcal{N}(0, \mu) \]

Scalar Gaussian channels
Metric-optimal Algorithm

Scalar posterior from relaxed BP
User-defined additive error metric
Combining posterior and error metric $\Rightarrow$ expected error
Pointwise minimization of expected error
Properties
Claim 1
The proposed algorithm
- is asymptotically optimal as the signal dimension $N \to \infty$
- has the lowest possible error

Metric-optimal, achieves the best possible performance
I want $|\hat{x}_i - x_i|^2 + \frac{1}{2} |\hat{x}_i - x_i|^{3.5}$

How about $|\hat{x}_i - x_i|^{0.77}$

Claim 2

The minimum mean user-defined error (MMUE) is given by

$$\text{MMUE}(f, \mu) = \int_{R(q)} \left( \int_{R(x)} D(\hat{x}_{\text{opt}}, x) f(x|q) dx \right) dq$$
Numerical Results
Comparison with Other Algorithms \((N = 10,000)\)

\[
D(\hat{\mathbf{x}}, \mathbf{x}) = \sum_{i=1}^{N} |\hat{x}_i - x_i|^{0.5}
\]

![Graph showing comparison of different algorithms with varying number of measurements and error metrics.](image-url)
Comparison with Other Algorithms \((N = 10,000)\)

\[
D(\hat{x}, x) = \sum_{i=1}^{N} |\hat{x}_i - x_i|
\]

![Graph showing error comparison between different algorithms.]
Comparison with Other Algorithms \((N = 10,000)\)

\[
D(\hat{x}, x) = \sum_{i=1}^{N} |\hat{x}_i - x_i|^{1.5}
\]
Comparison with Theoretical Limits \((N = 10,000)\)

### Minimum mean absolute error estimator

- **Theoretical limit**
- **Metric–optimal**

### Minimum mean support error estimator

- **Theoretical limit**
- **Metric–optimal**

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Reconstruction with Different Error Metrics

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The $\ell_\infty$ Norm Error
**Definition of $\ell_\infty$ norm**

The $\ell_\infty$ norm is defined as:

$$\|\hat{x} - x\|_\infty = \max_{i \in \{1, 2, \ldots, N\}} |\hat{x}_i - x_i|$$

- Want to use metric-optimal algo, but $\ell_\infty$ not additive...
Definition of $\ell_\infty$ norm

$$\|\hat{x} - x\|_\infty = \max_{i \in \{1, 2, \ldots, N\}} |\hat{x}_i - x_i|$$

- Want to use metric-optimal algo, but $\ell_\infty$ not additive...
Wiener Filter

Gaussian signal, Gaussian noise \( (q = x_G + v) \)
- Wiener filter optimal for \( \ell_\infty \) error [Sherman, 1958]
\[
\hat{x}_{\text{Wiener}} = \text{const} \times q
\]

Sparse Gaussian signal, Gaussian noise \( (q = x_{SG} + v) \)
- Wiener filter \textit{asymptotically} optimal

What if \( N \) is finite?
Wiener Filter

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Wiener Filter

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- Wiener filter optimal for $\ell_\infty$ error [Sherman, 1958]

$$\hat{x}_{\text{Wiener}} = \text{const} \times \mathbf{q}$$

Sparse Gaussian signal, Gaussian noise ($\mathbf{q} = \mathbf{x}_{SG} + \mathbf{v}$)
- Wiener filter asymptotically optimal

What if $N$ is finite?
$\ell_p$ Approximation to $\ell_\infty$

$$
\|\hat{x} - x\|_\infty = \max_{i \in \{1, 2, \ldots, N\}} |\hat{x}_i - x_i| = \lim_{p \to \infty} \|\hat{x}_i - x_i\|_p
$$

$$
D(\hat{x}, x) = \sum_{i=1}^{N} |\hat{x}_i - x_i|^p
$$
The $\ell_\infty$ norm error

Numerical Results ($M/N = 0.3$)

$$D(\hat{x}, x) = \sum_{i=1}^{N} |\hat{x}_i - x_i|^p$$

![Graph showing numerical results with different error metrics and signal dimension N.]
Summary
User-defined additive error metric
Scalar Gaussian channel, scalar estimation

Wiener filter is only optimal for $N \rightarrow \infty$
Heuristic $d(x_i, \hat{x}_i) = |x_i - \hat{x}_i|^{p}$ suitable for finite $N
Thank you!