MAE 308 Sample Exam 2

Fluid Mechanics

1) Given a steady incompressible flow field: \( \vec{V} = 2xy \hat{i} - y^2 \hat{j} \). Assume gravity is aligned with the \( z \)-direction.
   a.) Show this is a possible incompressible flow?
   b.) Is the flow rotational or irrotational?
   c.) Find an expression for the pressure gradient in the \( y \)-direction.

2) Consider the lift force for flow past an aircraft,
   a) What is the similarity for inviscid flows?
   b) What is the similarity for incompressible and inviscid flows?

3) Water (assumed frictionless and incompressible) flows steadily from a large tank and exits through a vertical, constant diameter pipe as shown. The air in the tank is pressurized to 50 kN/m². Determine (a) the height \( h \) to which the water jet rises, (b) the water velocity in the pipe and (c) the pressure in the horizontal part of the pipe. (Hint: For Part a) at the maximum height, \( h \), the velocity is assumed to be zero. Using the Bernoulli equation between different points.)

4) Consider a constant pressure, gravity driven flow in which a layer of fluid runs down a declined vertical wall. The wall is very tall and wide. The layer of fluid has a thickness \( h \). The fluid has viscosity \( \mu \) and density \( \rho \). Assuming no resistance between the air and the fluid, answer the following:
   a) Label this figure with the appropriate coordinate system and directions.
   b) What assumptions can be made about this flow scenario?
   c) Write out the full N.S. equation governing this flow, and reduce this equation to a second order ordinary differential equation.
   d) Find the solution to this ODE using appropriate Boundary Conditions. (Hint: The velocity along the direction of the wall at a distance of \( h \) from the wall can be considered constant.)
   e) Draw the velocity profile.
f) Where is the location of maximum velocity for this flow solution? Using these parameter values what is the maximum velocity of the flow?

\[ \rho = 1000 \frac{kg}{m^3}, \quad \mu = 0.00122 \frac{kg}{ms}, \quad h = 0.001m, \quad \theta = 30^\circ \]
Solution to problem 3

Mark four sections (points) as shown

a) Apply the Bernoulli's equation at points 1 and 2

\[
\frac{P_1}{\gamma} + \frac{V_1^2}{2} + g\delta_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2} + g\delta_2
\]

Note that \( V_1 = 0, \ P_1 = P_a + P_{\text{gage}}, \ \delta_1 = 2 \text{ m} \)

\( V_2 = 0, \ P_2 = P_a, \ \delta_2 = 4 \text{ m} \)

\[
\Rightarrow \quad V_2 = \sqrt{2 \left[ \frac{P_1 - P_2}{\gamma} + g (\delta_1 - \delta_2) \right]} = \sqrt{2 \left[ \frac{50 \times 10^3}{1000} + 9.8 \times (-2) \right]} = 7.08 \text{ m/s}
\]

b) Apply Bernoulli's equation at 3 and 2

\[
\frac{P_1}{\gamma} + \frac{V_1^2}{2} + g\delta_1 = \frac{P_3}{\gamma} + \frac{V_3^2}{2} + g\delta_3
\]

Note that \( V_1 = 0, \ P_1 = P_a + P_{\text{gage}}, \ \delta_1 = 2 \text{ m} \)

\( V_3 = 0, \ P_3 = P_a, \ \delta_3 = 2 \text{ m} \)

\[
\Rightarrow \quad P = \delta_1 + \frac{P_1 - P_3}{\gamma} = 2 + \frac{50 \times 10^3}{1000 \times 9.8} = 7.1 \text{ m}
\]
c) The pressure is constant in the horizontal part, which can be shown by using Bernoulli's equation on any two points in that section.

Apply Bernoulli's equation at points D and E:

\[
\frac{P_4}{\gamma} + \frac{V_4^2}{2} + g \delta_4 = \frac{P_1}{\gamma} + \frac{V_1^2}{2} + g \delta_1
\]

Note that \( V_1 = 0 \), \( \delta_1 = 2 \), \( P_1 = P_a + P_{gauge} \), \( V_4 = V_2 = 7.8 \text{ m/s} \), \( \delta_4 = 0 \),

\[
\Rightarrow P_4 = P_1 + \gamma \left[ \frac{V_1^2 - V_4^2}{2} + g (\delta_1 - \delta_4) \right]
\]

\[
P_4 = P_a + 50 \times 10^3 + 1000 \left[ -\frac{7.8^2}{2} + 9.8 \times 2 \right]
\]

\[
P_{4, \text{gauge}} = P_4 - P_a = 39,180 \text{ kPa}
\]
Solution to problem 4.

a) as illustrated in the Figure.

b) Assumptions:
   1) 2D steady flow
   2) Incompressible flow
   3) No pressure gradient in x-direction
   4) Flow is fully developed
      \[ V_x = V_x(y) \Rightarrow \frac{\partial V_x}{\partial x} = 0 \]
   5) Y-component velocity is zero \( V_y = 0 \)
   6) No slip at the wall: \( V_x |_{y=0} = 0 \)
   7) No shear stress at the air-fluid interface
      \[ \frac{dV_x}{dy} \bigg|_{y=R} = 0 \]

Continuity equation:
\[ \frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \Rightarrow \nabla \cdot \mathbf{V} = 0 \Rightarrow \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \]

x-momentum equation:
\[ \frac{3}{3} \frac{DV_x}{Dt} = -\frac{\partial p}{\partial x} + \nabla \cdot \mathbf{V} + \mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} \right) \]
\[ \Rightarrow g \left( \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_x}{\partial y^2} \right) = -\frac{\partial V_x}{\partial x} + g \frac{\partial V_x}{\partial x} + \mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} \right) \]

\[ \Rightarrow \quad g \frac{\partial V_x}{\partial x} + \mu \frac{\partial^2 V_x}{\partial y^2} = 0 \]

\[ \begin{align*}
\frac{\partial V_x}{\partial x} &= \frac{g \sin \theta}{\mu} \\
\frac{\partial^2 V_x}{\partial y^2} &= \frac{5g \sin \theta}{\mu} \\
\end{align*} \]

\( a \) ) Apply the boundary conditions:

at wall \( V_x \bigg|_{y=0} = 0 \)

at free surface \( \frac{dV_x}{dy} \bigg|_{y=R} = 0 \)

Integrate: \( \frac{d^2 V_x}{dy^2} = -\frac{5g \sin \theta}{\mu} \)

\( \frac{dV_x}{dy} = -\frac{5g \sin \theta}{\mu} y + C_1 \)

at \( y = R \), \( \frac{dV_x}{dy} = 0 \) \( \Rightarrow C_1 = \frac{5g \sin \theta}{\mu} R \)

Integrate one more time \( \Rightarrow \)

\( V_x = -\frac{5g \sin \theta}{\mu} \left( \frac{y^2}{2} - Ry \right) + C_2 \)

at \( y = 0 \), \( V_x = 0 \) \( \Rightarrow C_2 = 0 \)
Therefore

\[ v_x(y) = -\frac{5g\sin\theta}{2\mu} (y^2 - 2Ry) \]

e) as shown in the Figure

f) At \( y = R \), \( \frac{dv_x}{dy} = 0 \Rightarrow \text{Maximum Velocity} \)

\[ v_x|_{y=R} = -\frac{5g\sin\theta}{2\mu} (R^2 - 2R^2) = \frac{5g\sin\theta R}{2\mu} \]

\[ v_x|_{y=R} = \frac{1000 \times 9.8 \times \sin 30^\circ \times 0.001^2}{2 \times 0.00122} = 2 \text{ m/s} \]