Why Riding the Lightning?
Equilibrium Analysis for Payment Hub Pricing

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Outlines

01 Background and Motivation
02 System Model
03 Game Formulation and Analysis
04 Evaluation
05 Conclusion
Scalability Problem of Bitcoin

7 tx/s  VS  45,000 tx/s
Payment Channel

Blockchain

Payments through a payment channel involve transactions (tx1, tx2, tx3) and hash functions (H()).
Payment Channel

deposit tx

closing tx

tx1

tx2

tx3

...txn
Payment Channel Network

sender

recipient

fee

router

fee

fee
If routers can set fees freely, will this lead to selfish fee setting that will increase PCN fees to be comparable to on-chain transaction fees, thus canceling the PCN’s economic advantage?
Overview

- Equilibrium analysis
  - Two-hub model
  - Game between senders and routers
  - Existence of pure Nash Equilibriums (NEs)
  - Derive lower and upper bounds on the equilibrium revenue

- Algorithm to find all pure NE
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Price-Setting Game

- Generalized Bertrand competition
  - non-continuous demand curves
  - locked-in demand
  - capacity constraints
Strategy Space and Demand Function

- **Strategy Space**

  \[ Y = [RE, c_{\text{max}}] \]

- **Demand Function**

  \[ d_\tau(p) = \sum_{k \in S_\tau(p)} \delta_k \]

  \[ S_\tau(p) = \{ k \in \Omega_\tau | c_k \geq p \}, \ \tau \in \{1, 2, \Lambda\} \]

  - left-continuous and monotonically non-increasing step function

- **Reserved price**

- **Maximum valuation**

- **Demand of sender** \( k \)

- **Available user set**
Utility Function

\[ \Pi_i(p_i, p_{-i}) = \begin{cases} 
L_i(p_i) \equiv p_i \cdot \min\{t_i, d_i(p_i) + d_\Lambda(p_i)\}, & \text{if } p_i < p_{-i}, \\
\Phi_i(p) \equiv p \cdot \min\{t_i, d_i(p) + \phi_i d_\Lambda(p)\}, & \text{if } p_i = p_{-i} = p, \\
M_i(p_i) \equiv p_i \cdot \min\{t_i, d_i(p_i) + \psi_i\}, & \text{if } p_i > p_{-i}, 
\end{cases} \]

\[ \phi_i = \frac{\max\{0, t_i - d_i(p)\}}{\max\{0, t_i - d_i(p)\} + t_{-i}} \quad \psi_i = \max\{0, d_\Lambda(p_i) - t_{-i}\} \]
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Pure Nash Equilibrium (NE)

- Strategy Profile \((p_1^*, p_2^*) \in Y \times Y\)
  
  for \(\forall i \in \{1, 2\}\), \(p_i^*\) is a best response \(p_i^* \in BR_i(p_{-i}^*)\)

- No router can unilaterally change its price to an alternative pure strategy and get a higher payoff
Bounds on Equilibrium Revenue

- **Lower bound**
  \[ R^* \geq \sum_{i=1}^{2} \max_{p_i \in Y} \{ M_i(p_i) \} = R_{LB}^* \]

- **Upper bound**
  \[ R^* \leq \max_{p_1, p_2 \in Y} \{ \sum_{i=1}^{2} \Pi_i(p_i, p_{-i}) \} = R_{UB}^* \]
NE Analysis

**Theorem:**
Best responses and pure NEs can only exist when both routers set prices at the valuation or RE.

**Lemma:**
Given the other router’s price $p_{-i}$, the best response set is empty iff $\sup_{p_i \in Y} \{ \Pi_i(p_i, p_{-i}) \} = L_i(p_{-i}) > \Phi_i(p_{-i})$. 
Pure NE Searching

- Find the candidate best response sets of two routers.
- Add strategy profile to the pure NE set
  - the best response sets of both routers exist given each other’s price
  - prices of both routers are in their best response sets respectively
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Simulation Settings

- Lighting Network (LN) topology
- Routers and users
  - Choose the two most connected nodes as routers 1 and 2
    - 390 overlap users, 620 (496) locked-in users
  - Channel capacity: $10^6$ satoshi
- Demand
  - Sampling transactions from a real-world credit card dataset
- User distributions
  - Ratio, Overlap, Monopoly

Simulation Results

PoS LB(Ratio), PoS LB(Overlap), PoA/PoS(Monopoly), PoA UB(Ratio), PoA UB(Overlap)

Capacity↑, Competition↑
Simulation Results

PCN transaction fees can be driven down
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Our Conclusion

- The competitive nature of PCN will ultimately
  - make its transaction fee much lower than the blockchain
  - especially when the network capacity becomes larger and larger
Thank you very much!

Q&A?