Gray Code Results for Acyclic Orientations

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Abstract

Given a graph G, the *acyclic orientation graph of* G, denoted AO(G), is the graph whose vertices are the acyclic orientations of G, and two acyclic orientations are joined by an edge in AO(G) iff they differ by the reversal of a single edge. A hamilton cycle in AO(G) gives a Gray code listing of the acyclic orientations of G. We prove that for certain graphs G, AO(G) is hamiltonian, and give explicit constructions of hamilton cycles or paths. This work includes Gray codes for listing the acyclic orientations of trees, complete graphs, odd cycles, chordal graphs, odd ladder graphs, and odd wheel graphs. We also give examples of graphs whose acyclic orientation graph is *not* hamiltonian. We show that the acyclic orientations of even cycles, some complete bipartite graphs, even ladder graphs, and even wheel graphs cannot be listed by the defined Gray code.

1 Introduction

A Gray code for a set S is a listing of the elements of S such that successive elements differ by a small amount. The classic example has the set S being the binary strings of length n and the small amount being the complementation of a single bit. Gray codes exist for a wide variety of combinatorial sets, including permutations [Tr62, Jo63], binary trees [LRR], integer partitions [Sa89, RSW], and linear extensions of some posets [Ru92, PR91].

In this paper, we study Gray codes for acyclic orientations of graphs. Let $G = (V(G), E(G))$ be an undirected graph. If we replace each edge $(u, v) \in E(G)$ with either the arc $u \to v$ or the arc $v \to u$, we get an *orientation* of G. We will denote orientations by Greek letters, $\alpha, \beta, \ldots$. An *acyclic orientation* of G is an orientation

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which contains no directed cycles. Given G, let $A(G)$ denote the set of acyclic orientations of G. We wish to list the elements of $A(G)$ in such a way that successive elements differ by the reversal of a single edge. We turn this into a hamilton cycle problem in the following way: Given G, define a new graph, $\mathrm{AO}(G)$, where $V(\mathrm{AO}(G)) = A(G)$ and $(\alpha_i, \alpha_j) \in E(\mathrm{AO}(G))$ if α_i and α_j are acyclic orientations which differ by a single edge reversal. A hamilton cycle in $\mathrm{AO}(G)$ gives a cyclic Gray code for the acyclic orientations of G. We refer to $\mathrm{AO}(G)$ as the the acyclic orientation graph of G, or the ao-graph of G.

Given a graph G, let σ be an orientation and $e_1, \ldots, e_m$ be a listing of the edges of G. We can represent any orientation α of G relative to σ as a bit string, $b_1 \ldots b_m$, where $b_i = 0$ if e_i is oriented identically in both α and σ, and $b_i = 1$ otherwise. Given G and a fixed orientation σ of G, we let $g_\sigma(\alpha)$ denote this bit string for the orientation α. Using this representation, it is easy to see that ao-graph is a subgraph of the m-cube, and is thus a bipartite graph. Relative to σ, each orientation α can be classified as even or odd based upon the number of ones in $g_\sigma(\alpha)$. We denote the set of even acyclic orientations of G relative to σ as $E_\sigma(G)$. Analogously, we denote the set of odd acyclic orientations of G relative to σ as $O_\sigma(G)$. We let $|g_\sigma(\alpha)|$ represent the number of ones in $g_\sigma(\alpha)$, or the number of arcs of σ one must reverse to get α.

We will study the ao-graph for special classes of graphs. Definitions of acyclic graphs, trees, cycles, and complete graphs can be found in Bondy & Murty [BM76]. We use C_n and K_n to denote the cycle and complete graph on n vertices, respectively. Given disjoint sets, X and Y, the *complete bipartite graph* with partite sets X and Y has vertex set $X \cup Y$, and edges (x, y) for all $x \in X$ and $y \in Y$. When $|X| = m$ and $|Y| = n$, we denote this graph as $K_{m,n}$. A *ladder graph* is the Cartesian product of a 2-path and an n-path. In particular, L_n consists consists of paths $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ plus the edges (x_i, y_i) for $1 \leq i \leq n$. The *wheel graph*, W_n, is formed from C_n by adding a new vertex adjacent to all the others.

Others have previously studied similar structures based on different types of orientations of graphs. Given G, Pretzel [Pr86] created a graph $\mathrm{PD}(G)$, where the vertices of $\mathrm{PD}(G)$ are the acyclic orientations of G and $(\alpha, \beta) \in E(\mathrm{PD}(G))$ if we can get β from α by reversing all edges from a source vertex (a vertex of indegree 0). This operation is called a *push-down*. Pretzel examined necessary and sufficient conditions for there to be a path from α to β in $\mathrm{PD}(G)$. Note that unlike $\mathrm{AO}(G)$, $\mathrm{PD}(G)$ is not necessarily connected. Pretzel also extends some of the results for acyclic orientations to general orientations by modifying the adjacency condition of $\mathrm{PD}(G)$.

Similar work was done on the strong orientations of a graph (also called "strongly connected orientations," meaning orientations in which there is a directed path from each vertex to every other vertex). Given G, Donald and Elwin [DE93] create a graph $\mathrm{Strong}(G)$, where the vertices of $\mathrm{Strong}(G)$ are the strong orientations of a graph, and two strong orientations are adjacent in $\mathrm{Strong}(G)$ if they differ by a *simple transformation*. The simple transformation consists of reversing the edges of a directed cycle, or of particular directed paths. The main result of Donald and Elwin is that $\mathrm{Strong}(G)$ is itself strongly connected. They also give a polynomial algorithm for finding a path between any two strong orientations in $\mathrm{Strong}(G)$.

The rest of the paper is organized as follows. In Section 2 we give two examples of graphs where a hamilton cycle in the ao-graph corresponds to some previously defined Gray codes, and give a proof on the connectedness of $\mathrm{AO}(G)$. Section 3 contains examples of graphs whose ao-graphs are not hamiltonian, and therefore have no Gray code listing as defined in this work. This section contains proofs that when n is even, C_n, L_n, W_n, and $K_{m,n}$ $(m > 1)$ cannot be listed in Gray code sequence. In Section 4, we construct a Gray code for C_n, L_n, and W_n when n is odd. We also give a Gray code for chordal graphs in this section. Section 5 summarizes our work and indicates areas of future research.

2 Two Simple Examples

Two of the more basic combinatorial sets are binary strings and permutations. There are well known Gray codes for listing both of these sets. The original binary reflected Gray code [Gr53] lists all binary strings of length m such that adjacent strings differ by a single bit. There is a Gray code for permutations [Tr62, Jo63] which lists all permutations of m items such that adjacent permutations differ by a single transposition. Both of these Gray codes are cyclic in that the first and last elements listed also differ in the prescribed way. In this section, we show that both of these Gray codes are generalized by the problem of listing the acyclic orientations of a graph such that successive orientations differ by a single edge reversal.

Let G be an acyclic graph on m edges. Every orientation of G is acyclic. Hence, by examining the $g_\sigma(\alpha)$ representation of orientations, $A(G)$ is easily seen to contain all bit strings of length m. For any orientations σ and α, reversing an edge in α is equivalent to complementing a bit in $g_\sigma(\alpha)$. Two orientations will differ by a single edge reversal iff their bit string representations differ by a single bit. Therefore, listing the acyclic orientations of an acyclic graph G on m edges is equivalent to listing bit strings of length m such that adjacent strings differ by a single bit. This listing is given by the binary reflected Gray code for bit strings.

Let $G = K_m$, the complete graph on m vertices, and let α be an acyclic orientation of G. Given α, we can list the vertices of G in order of decreasing out-degree. This gives a permutation of the vertices of G. Conversely, given any permutation Π on $\{1, \ldots, m\}$, we can get an acyclic orientation of G by orienting the edge (i, j) as $i \to j$ iff $\Pi^{-1}(i) < \Pi^{-1}(j)$. This gives a bijection between the acyclic orientations of K_m and the permutations of $\{1 \ldots m\}$. We can therefore consider the vertices of $\mathrm{AO}(K_m)$ to be permutations.

We now need to determine the edge set of $\mathrm{AO}(K_m)$. Given an acyclic orientation α, which arcs in α can have their direction reversed while keeping the orientation acyclic? The arc $i \to j$ can be reversed iff there are no other directed paths from i to j. This occurs when, in the permutation associated with α, i and j are adjacent. Reversing the arc $i \to j$ will transpose the elements i and j in the associated permutation. Therefore, a Gray code for the acyclic orientations of K_m is equivalent to listing permutations of m elements such that adjacent permutations differ by a single transposition.

These two examples demonstrate the power of the ao-graph. It captures two of the more famous Gray codes as special cases for certain classes of graphs. We will see in Section 4 that other classes of graphs give different structures to the ao-graph. In the next section, we prove that certain ao-graphs are not hamiltonian. However, as we show below, ao-graphs are always connected.

Theorem 2.1 *G is a simple graph iff $AO(G)$ is connected.*

Proof. G is not simple iff it has a loop or parallel edges. If G has a loop, then AO(G) has no vertices and is therefore not a graph. If G has parallel edges e_1 and e_2, let α be any acyclic orientation of G. In α, e_1 and e_2 must be oriented identically. Since we can not reverse the direction of e_1 or e_2 without creating a cycle, α cannot be connected to any acyclic orientation having e_1 and e_2 oriented in the opposite direction. Therefore, if AO(G) is connected, G must be simple.

We now prove that if G is simple, then AO(G) is connected. We give a proof by induction on $n = |V(G)|$. If $n = 1$, the ao-graph is empty, and the theorem is vacuously true. Assume G has $n > 1$ vertices, and that for all simple graphs G' on $n - 1$ vertices, the ao-graph of G' is connected.

Let $\alpha, \beta \in A(G)$, and x be a vertex with outdegree 0 in α. Let α' and β' be the acyclic orientations of $G' = G - x$ induced by α and β, respectively. By induction, we can get from α' to β' in AO(G') by a sequence of single edge reversals. If we start from α and perform the same sequence of edge reversals, we do not form a cycle because x has outdegree 0. Let γ be the acyclic orientation obtained from α and this sequence of reversals. Note that γ differs from β only in the direction of arcs to x. Let $v_1, \ldots, v_{n-1}$ be a topological sort of β'. To obtain β from γ, examine the vertices of G' in the order $v_{n-1}, \ldots, v_1$. When examining v_i, if β contains the arc $x \rightarrow v_i$, then reverse this arc in γ. These reversals will not create a cycle because G is simple and $i < j$ implies that there is no directed path from v_j to v_i. Hence, for any $\alpha, \beta \in A(G)$, there is a path connecting α and β in AO(G). □

3 Non-Hamiltonian Acyclic Orientation Graphs

Given that a Gray code for acyclic orientations generalizes some well-known combinatorial Gray codes, we would like to know whether or not every graph can have its acyclic orientations listed in Gray code sequence. Unfortunately, not all graphs give rise to a hamiltonian ao-graph. This is the subject of the current section.

The bit string representation of an orientation tells us that for all G with m edges, AO(G) is a subgraph of the m-cube. As such, it is bipartite. If we fix σ, the two partite sets are $E_\sigma(G)$ and $O_\sigma(G)$. We will use the following lemma to show that AO(G) is not hamiltonian for certain G. If $\alpha \in A(G)$, $\overline{\alpha}$ denotes the acyclic orientation obtained by reversing the direction of every arc of α. We will use $a(G), e_\sigma(G)$, and $o_\sigma(G)$ to denote the sizes of the sets $A(G), E_\sigma(G)$, and $O_\sigma(G)$, respectively.

Lemma 3.1 *Let G be a graph with an even number of edges. If $AO(G)$ is hamiltonian, then $4|a(G)$.*

Proof. Let G be as stated, and $\alpha \in E_\sigma(G)$. Then $\overline{\alpha} \in E_\sigma(G)$ also. So $2|e_\sigma(G)$. Similarly, $2|o_\sigma(G)$. Since AO(G) is bipartite and hamiltonian, $e_\sigma(G) = o_\sigma(G)$. But $a(G) = e_\sigma(G) + o_\sigma(G)$, so $4|a(G)$. $\square$

By the above lemma, if G has an even number of edges and we can determine $a(G)$ is not equivalent to $0 \pmod 4$, we can conclude that AO(G) is not hamiltonian. For example, when n is even, C_n has an even number of edges. There are only two cyclic orientations of C_n, so $a(C_n) = 2^n - 2$. When n is even, this quantity is not divisible by four. We can conclude that AO(C_n) is *not hamiltonian* for even n.

Counting the number of acyclic orientations of a graph, however, is not a simple endeavor. Linial [Li86] shows that it is a #**P**-complete problem. For general graphs, computing $a(G)$ modulo 4 does not appear any easier. But for some classes of graphs which have a well defined structure, we can use the following idea of Stanley [St73] to determine $a(G)$. By *contracting* an edge (u, v) of G, we mean forming a new graph G' by replacing the two vertices u and v of G with a single vertex uv in G' such that if (x, u) or (x, v) is an edge of G ($x \neq u, v$), then (x, uv) is an edge of G'. We let $G - e$ denote the graph G minus the edge e, and $G \cdot e$ denote the graph G contracted on the edge e.

Lemma 3.2 (Stanley [St73]) *Let G be a graph. Then*

$$a(G) = a(G - e) + a(G \cdot e).$$

This lemma enables us to show that certain wheel graphs and certain ladder graphs have ao-graphs which are not hamiltonian.

Theorem 3.1 *For even n, $AO(W_n)$ is not hamiltonian.*

Proof. Let w_n denote the number of acyclic orientations of W_n. Let G_n be obtained from W_n by deleting one of the outer edges, and let g_n denote the number of acyclic orientations of G_n. Using repeated applications of Lemma 3.2, we get the following recurrence relations:

$$\begin{aligned} g_3 &= 18 & \qquad w_3 &= 24 \\ g_n &= 3g_{n-1} & \qquad w_n &= w_{n-1} + g_n \end{aligned}$$

Solving these relations, we see that $g_n = 2 \cdot 3^{n-1}$ and $w_n = 3^n - 3$. When n is even, 4 does not divide w_n, so by Lemma 3.1, AO(W_n) is not hamiltonian. $\square$

Theorem 3.2 *When n is even, $AO(L_n)$ is not hamiltonian.*

Proof. Let l_n denote the number of acyclic orientations of L_n. By repeated applications of Lemma 3.2, we get the following recurrence:

$$\begin{aligned} l_1 &= 2 \\ l_n &= 7l_{n-1} \end{aligned}$$

Hence $l_n = 2 \cdot 7^{n-1}$. This number is never divisible by four, so when L_n has an even number of edges, $\mathrm{AO}(L_n)$ is not hamiltonian. L_n has an even number of edges iff n is even. □

There are times when Lemma 3.2 is not strong enough to allow us to develop an equation for the number of acyclic orientations, even just modulo 4. However, if we examine the structure of the acyclic orientations of a specific graph, we can often find another way of computing $a(G)$. We demonstrate by determining sufficient conditions to prevent $\mathrm{AO}(K_{m,n})$ from being hamiltonian.

Let $S(n,k)$ denote the Stirling numbers of the second kind [SW86]. $S(n,k)$ counts the number of *set partitions* of an n-set into k blocks. Note that for $n > 1$, $S(n,1) = 1$ and $S(n,2) = 2^{n-1} - 1$. The formula $k!S(n,k)$ counts the number of *ordered set partitions*, which are set partitions in which the order of the blocks is important. We denote an ordered set partition of an n-set into k blocks as $P_{n,k}$. There is an interesting relationship between Stirling numbers of the second kind and acyclic orientations of $K_{m,n}$ which is given in the following lemma.

Lemma 3.3 *There exists a bijection between acyclic orientations of $K_{m,n}$ and triples of the form $(P_{m,k}, P'_{n,k+\epsilon}, \delta)$, where $\epsilon \in \{0, \pm 1\}$, $0 \le k \le m$, $k + \epsilon \le n$, and if $\epsilon = 0$ then $\delta \in \{0,1\}$, else $\delta = 0$.*

Proof. Let $(P_{m,k}, P'_{n,k+\epsilon}, \delta)$ be such a triple. We get an orientation of $K_{m,n}$ from this triple in the following way. We will alternate the blocks of $P_{m,k}$ and $P'_{n,k+\epsilon}$ to form a linear arrangement of all blocks, and direct any edges of $K_{m,n}$ from earlier blocks to later blocks. If $\epsilon = 0$, the number of blocks in each ordered set partition is the same, so we can start with either the first block of $P_{m,k}$ ($\delta = 0$), or the first block of $P'_{n,k}$ ($\delta = 1$). If $\epsilon \neq 0$, there are more blocks in one ordered set partition than the other. Begin the sequence of alternating blocks with the first block of the partition which has more blocks. In this way, each such triple gives an orientation of $K_{m,n}$.

Conversely, given an acyclic orientation α of $K_{m,n}$ we can reverse the process to get such a triple. We refer to the two partite sets of $K_{m,n}$ as X and Y. Since α is acyclic, there exists a vertex of indegree zero. Note that since every $x \in X$ is adjacent to every $y \in Y$, the vertices of indegree zero must all be in X or in Y, never in both. WLOG, we can assume the vertices of indegree zero are in X. We let our first block of $P_{m,k}$ be the l vertices of indegree zero in X. Remove all such vertices from $K_{m,n}$. We are left with an acyclic orientation of $K_{m-l,n}$. Now all vertices of indegree zero are in Y, and we can define our first block of $P_{n,k+\epsilon}$ to be this collection of vertices. If we continue forming a block out of the vertices of indegree zero, we will terminate with a sequence of alternating blocks of the X and Y vertices. These blocks give ordered set partitions of m and n. If the number of blocks in each partition is the same, we set $\epsilon = 0$ and $\delta = 0, 1$ depending whether the sequence of blocks started with a block of X or Y, respectively. Otherwise, the number of blocks differs by 1, so we set $\delta = 0$ and $\epsilon = -1, 1$ depending if there are more blocks of X or Y, respectively. This gives a bijection between such triples and acyclic orientations of $K_{m,n}$. □

So the acyclic orientations of $K_{m,n}$ correspond to pairs of ordered set partitions, under appropriate restrictions. We know that the number of ordered set partitions of n into k blocks is equal to $k!S(n,k)$. Using this information, we can prove that certain AO($K_{m,n}$) are not hamiltonian.

Theorem 3.3 *Let m and n be such that $m, n > 1$ and $2|mn$. Then AO($K_{m,n}$) is not hamiltonian.*

Proof. Lemma 3.3 gives a way of counting the number of acyclic orientations of $K_{m,n}$.

$$a(K_{m,n}) = \sum_{i=1}^{m} S(m,i)i! \left[S(n,i-1)(i-1)! + 2S(n,i)i! + S(n,i+1)(i+1)!\right]$$

If $i \geq 3$ it is easy to see that all terms in the above equation are divisible by four by just looking at the factorial terms. Hence

$$\begin{aligned} a(K_{m,n}) &\equiv S(m,1)\left[2S(n,1) + 2S(n,2)\right] + \\ & \quad 2S(m,2)\left[S(n,1) + 4S(n,2) + 6S(n,3)\right] \pmod 4 \\ &\equiv 2 + 2S(n,2) + 2S(m,2) \pmod 4 \\ &\equiv 2 + (2^n - 2) + (2^m - 2) \pmod 4 \\ &\equiv 2 \pmod 4 \end{aligned}$$

Note that when $2|mn$, the number of edges of $K_{m,n}$ is even. Therefore, when m and n are as in the theorem, the ao-graph of $K_{m,n}$ cannot be hamiltonian. □

Note that if m and n are both odd, Lemma 3.1 no longer applies, so we cannot conclude that AO($K_{m,n}$) is not hamiltonian. Also, when $m = 1$, $K_{1,n}$ is a tree, and therefore AO($K_{1,n}$) is hamiltonian. Thus far, we have been unable to determine the hamiltonicity of AO($K_{3,3}$).

We have given several examples of graphs whose ao-graph is not hamiltonian. In all of our proofs we use the fact that AO(G) is a bipartite graph and for it to be hamiltonian, it must have the two sets in its bipartition be of equal size. We actually know a little more than what is stated in the theorems. The theorems say that for certain G, AO(G) does not have a hamilton *cycle*. The proofs also show that AO(G) cannot have a hamilton *path*. When G has an even number of edges, both $e_\sigma(G)$ and $o_\sigma(G)$ are even for any σ. If $e_\sigma(G) \neq o_\sigma(G)$, then they must differ by at least two, and hence AO(G) cannot even have a hamilton path.

4 Hamiltonian Acyclic Orientation Graphs

Section 3 showed that in certain cases, there is a parity problem in AO(G) which prevents it from being hamiltonian. In this section, we look at some of the same classes of graphs when there is no parity problem, and see that, in many cases, we can construct a hamiltonian cycle in AO(G).

We first need to introduce some notation. In the following definitions, G and H denote edge disjoint graphs (not necessarily vertex disjoint), and G' a subgraph of G. Let α be an acyclic orientation of G and β be an acyclic orientation of H. Since G and H are edge disjoint, we write $\alpha\beta$ to represent the orientation of $G \cup H$ where the edges of G are oriented as in α and the edges of H are oriented as in β. If $P = \alpha_1, \ldots, \alpha_N$ then $P^R = \alpha_N, \ldots, \alpha_1$ and $\overline{P} = \overline{\alpha_1}, \ldots, \overline{\alpha_N}$. If P_1 and P_2 are lists, then $P = P_1, P_2$ denotes the concatenation of P_1 and P_2. If P is a list of orientations of H and α is an orientation of G, then αP is the list of orientations of $G \cup H$ where the edges of G are fixed as in α, and the orientation of the edges of H changes as it does in P. Finally, if β is an acyclic orientation of G', we denote by $f_G(\beta)$ the set of acyclic orientations $G - E(G')$ which, when combined with β, are still acyclic. That is

$$f_G(\beta) = \{\delta | \delta \in A(G - E(G')), \beta\delta \in A(G)\} \tag{1}$$

We sometimes refer to the set $f_G(\beta)$ as the set of acyclic extensions of β to G because $\beta f_G(\beta)$ are the acyclic orientations of G which are equivalent to β on G'.

4.1 A Hamilton Cycle in AO(G) for Chordal G

The first set of graphs we examine are *chordal*, or triangulated, graphs. Let G be a graph. A cycle $v_1, \ldots, v_k$ of G is *chordless* if G does not contain an edge of the form (v_i, v_j) where $|i - j| \neq 1$ modulo k. We call G a *chordal* graph if it has no chordless cycles of length greater than 3. We use two facts about chordal graphs from Golumbic [Go80]. First, if G is chordal, then there exists a vertex x such that the neighborhood of x is a clique. Secondly, all induced subgraphs of a chordal graph are chordal.

Theorem 4.1 *If G is chordal, then $AO(G)$ is hamiltonian.*

Proof. We give a proof by induction on $n = |V(G)|$. If $n = 1$, the theorem is vacuously true. Let G be a chordal graph, $|V(G)| = n > 1$, and assume the theorem for all chordal graphs on $n - 1$ vertices. By our first fact on chordal graphs, we know there exists a vertex v_0 such the neighborhood of v_0 is a clique. Let $G' = G - v_0$. Let $v_1, \ldots, v_k$ be the neighbors of v_0 in G, and let E' denote the set of edges $\{(v_0, v_i) | 1 \leq i \leq k\}$. We claim that for each $\alpha \in A(G')$, there exists a Gray code for $f_G(\alpha)$ which begins at the orientation having each edge of E' oriented $v_0 \rightarrow v_i$, and ends at the orientation having each edge of E' oriented $v_i \rightarrow v_0$.

Recall that acyclic orientations of the complete graph correspond to permutations of the vertices. Therefore, to each $\alpha \in A(G')$, there corresponds a permutation of $1, \ldots, k$, denoted Π_α, determined by the orientation of the k-clique induced by $\{v_1, \ldots, v_k\}$ in α. When v_0 is made adjacent to all members of this clique, a $(k+1)$-clique is formed. Any orientation β of the edges E' such that $\alpha\beta$ is still acyclic, will induce a permutation $\Pi_{\alpha\beta}$ of $0, \ldots, k$. The relative order of $1, \ldots, k$ is the same in both Π_α and $\Pi_{\alpha\beta}$, so that $\Pi_{\alpha\beta}$ is obtained from Π_α by inserting 0 somewhere in Π_α. Conversely, inserting 0 after the i^{th} element of Π_α, $0 \le i \le k$, gives a distinct orientation of the $(k+1)$-clique induced by $\{v_0, \ldots, v_k\}$.

If Π_α is $i_1 i_2 \ldots i_k$, we can use the permutation representation to list the elements of $f_G(\alpha)$ in a Gray code sequence by:

$$0 i_1 i_2 \ldots i_k, \;\; i_1 0 i_2 \ldots i_k, \;\; \ldots, \;\; i_1 \ldots i_k 0.$$

Denote this sequence by $P(\alpha)$. Note that for all $1 \le i \le k$, the first element of $P(\alpha)$ includes the arc $v_0 \to v_i$, and the last element of $P(\alpha)$ includes the arc $v_i \to v_0$. Therefore, for any $\alpha_i, \alpha_j \in A(G')$, the first (last) element of $P(\alpha_i)$ is the same as the first (last) element of $P(\alpha_j)$.

Since G' is a subgraph of G, G' is also chordal. By induction, there exists a hamilton cycle $C = \alpha_1, \ldots, \alpha_N$ of AO(G'). Note N is even because we can pair with each orientation α_i the orientation $\overline{\alpha_i}$. Also, the last orientation in $\alpha_i P(\alpha_i)$ differs from the first orientation of $\alpha_{i+1} P^R(\alpha_{i+1})$ by a single edge reversal. We can conclude that

$$\alpha_1 P(\alpha_1), \; \alpha_2 P^R(\alpha_2), \; \ldots, \; \alpha_N P^R(\alpha_N)$$

is a hamilton cycle in AO(G). □

4.2 A Hamilton Cycle in AO(C_n) and AO(L_n)

Let G_1 and G_2 be graphs with hamiltonian ao-graphs, and let $C_i = \alpha_{i,1}, \ldots, \alpha_{i,N_i}$ denote a hamilton cycle in AO(G_i). If we form G by choosing a vertex v_i in each G_i and identifying v_1 with v_2, we can use the C_i to get a hamilton cycle C in AO(G), namely $C = \alpha_{1,1} C_2, \; \alpha_{1,2} C_2^R, \; \ldots, \; \alpha_{1,N_1} C_2^R$. In this way we can build graphs with hamiltonian ao-graphs from smaller graphs with hamiltonian ao-graphs.

If we try to generalize this idea to identifying pairs of vertices from each G_i, it does not always succeed. For example, suppose G_1 is a 4-path (a path of 4 vertices), and G_2 is an edge. Both of these graphs have hamiltonian ao-graphs. If we try to identify the ends of the 4-path G_1 with the ends of the 2-path G_2, we get a 4-cycle. We have already seen that AO(C_4) is not hamiltonian. Although, in general, this idea does not work, we will give some conditions on the graphs to be combined, and on the vertices to be identified, which allow it to succeed. But first we must present a few more definitions.

Definition 4.1 *Let G be a graph, $\sigma \in A(G)$, and $(i,j) \notin E(G)$. We say the arc $i \to j$ is* allowable *for the orientation σ of G if there is no directed j to i path in σ.*

Definition 4.2 *Let G be a graph, $e \in E(G)$. We say the ao-graph of G is* strongly hamiltonian with respect to e *if it has a hamilton cycle C such that the orientation of e is reversed only twice in C.*

An allowable arc for an acyclic orientation can be added to the orientation while leaving it acyclic. A strongly hamiltonian ao-graph with respect to (u, v) has a hamilton cycle $C = P_1, P_2$, where P_1 lists all orientations that include the arc $u \to v$ and P_2 lists all orientations that include the arc $v \to u$. A condition more restrictive than strongly hamiltonian is given by the following definition.

Definition 4.3 *Let G be a graph, $(i, j) \notin E(G)$, $e \in E(G)$. We call the triple $\langle G, (i, j), e\rangle$* acyclicly adjoinable *if there exists a hamilton cycle C in $AO(G)$ such that:*

- *e reverses direction only twice in C.*
- *C can be decomposed into $C = P_1, P_2, P_3, P_4$ such that*
  - *P_i has an odd number of vertices for $i = 2, 4$. Since the length of C is even, this further implies that the parity of the length of P_1 is the same as the parity of the length of P_3.*
  - *For all $\sigma \in A(G)$, $\sigma \in P_1 \Leftrightarrow j \to i$ is not allowable for σ, and $\sigma \in P_3 \Leftrightarrow i \to j$ is not allowable for σ.*

We now more precisely define the operation previously described of combining two graphs by identifying a pair of vertices from each graph.

Definition 4.4 *Let G_1 and G_2 be graphs, $(i_1, j_1) \in E(G_1)$, $(i_2, j_2) \notin E(G_2)$. Define*

$$H = \langle G_1, (i_1, j_1)\rangle \oplus \langle G_2, (i_2, j_2)\rangle$$

such that H is the graph formed by identifying i_1 with i_2, j_1 with j_2, and which has $E(H) = E(G_1) \cup E(G_2)$.

The following lemma tells us we can combine a strongly hamiltonian graph with an acyclicly adjoinable graph to get another graph whose ao-graph is strongly hamiltonian.

Lemma 4.1 *Let G_1 be a graph such that $AO(G_1)$ is strongly hamiltonian with respect to the edge (i_1, j_1). Let G_2 be a graph, $e \in E(G_2)$ and $(i_2, j_2) \notin E(G_2)$. If $\langle G_2, (i_2, j_2), e\rangle$ is acyclicly adjoinable and*

$$H = \langle G_1, (i_1, j_1)\rangle \oplus \langle G_2, (i_2, j_2)\rangle,$$

then $AO(H)$ is strongly hamiltonian with respect to e.

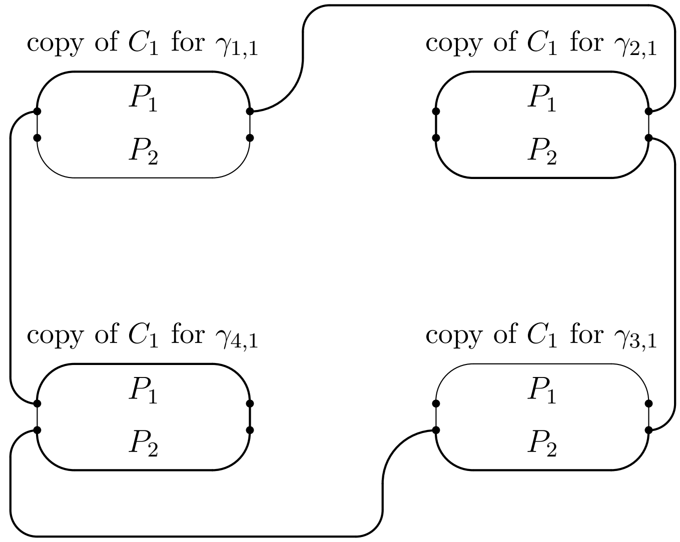

Figure 1: Patching cycles together.

Proof. Since the ao-graph of G_1 is strongly hamiltonian with respect to (i_1, j_1), we can find a hamilton cycle $C_1 = P_1, P_2$ of $\mathrm{AO}(G_1)$, where P_1 is a path through all acyclic orientations of G_1 that include the arc $i_1 \to j_1$, and P_2 is a path through all the acyclic orientations of G_1 which include the arc $j_1 \to i_1$.

Since $\langle G_2, (i_2, j_2), e\rangle$ is acyclicly adjoinable, there exists a hamilton cycle $C_2 = Q_1, Q_2, Q_3, Q_4$ in $\mathrm{AO}(G_2)$ such that (1) the arc $j_2 \to i_2$ is allowable for all orientations not listed in Q_1 (2) the arc $i_2 \to j_2$ is allowable for all orientations not listed in Q_3 (3) the length of Q_i is odd for $i = 2, 4$, and (4) e reverses direction only twice in C. Denote the list Q_i as $\gamma_{i,1}, \ldots, \gamma_{i,n_i}$.

Given any acyclic orientation α of H, we can write it as $\alpha = \beta\gamma$, where β is an acyclic orientation of G_1 and γ is an acyclic orientation of G_2. Similarly, let β be an acyclic orientation of G_1 and γ be an acyclic orientation of G_2. Then $\alpha = \beta\gamma$ is an acyclic orientation of H as long as the orientation of the edge (i_1, j_1) does not cause a cycle for G_2. Equivalently, if the arc $i_1 \to j_1$ is in β, then $\beta\gamma$ is acyclic if and only if $i_2 \to j_2$ is allowable for γ.

The idea behind constructing a hamilton cycle in $\mathrm{AO}(H)$, which will be made more formal in the following paragraphs, is this: consider a collection of copies of C_1, one for each acyclic orientation of G_2. For each acyclic orientation of G_2, traverse the applicable parts of C_1 (those orientations of G_1 which won't form a cycle with the current orientation of G_2). Figure 1 gives an example of this process when $C_2 = \gamma_{1,1}, \gamma_{2,1}, \gamma_{3,1}, \gamma_{4,1}$, ie where Q_i consists of the single orientation $\gamma_{i,1}$. Note P_1 contains arc $i_1 \to j_1$, so $\gamma_{i,1}P_1$ is acyclic except when $i = 3$, because $i_2 \to j_2$ is not allowable for only $\gamma_{3,1}$. Similarly, $\gamma_{i,1}P_2$ is acyclic except when $i = 1$.

More generally, the hamilton cycle of AO(H) is (assuming n_1 and n_3 odd):

$$\begin{aligned} C \;=\; & P_1\gamma_{1,1},\ P_1^R\gamma_{1,2},\ \ldots,\ P_1\gamma_{1,n_1}, \\ & (P_1^R, P_2^R)\gamma_{2,1},\ (P_2, P_1)\gamma_{2,2},\ \ldots,\ (P_1^R, P_2^R)\gamma_{2,n_2}, \\ & P_2\gamma_{3,1},\ P_2^R\gamma_{3,2},\ \ldots,\ P_2\gamma_{3,n_3}, \\ & (P_2^R, P_1^R)\gamma_{4,1},\ (P_1, P_2)\gamma_{4,2},\ \ldots,\ (P_2^R, P_1^R)\gamma_{4,n_4}. \end{aligned}$$

If n_1 and n_3 were even, the first row would end with $P_1^R\gamma_{1,n_1}$, the second would begin with $(P_1, P_2)\gamma_{2,1}$, etc.

To prove this is a hamilton cycle through the ao-graph of H, we must show that every acyclic orientation of H is included in the list, and that consecutive elements differ by the reversal of a single edge. The cycle contains all elements from the direct product $C_1 \times C_2$, except those orientations in $P_1 \times Q_3$ and $P_2 \times Q_1$. If β is listed in P_1, then β contains the arc $i_1 \rightarrow j_1$. If γ is listed in Q_3, then $i_2 \rightarrow j_2$ is not allowable for G_2. When we form H we identify i_1 with i_2 and j_1 with j_2, so the orientation $\beta\gamma$ is a *cyclic* orientation of H. Similarly any orientation in $P_2 \times Q_1$ is a cyclic orientation of H. Therefore, our list contains all of the acyclic orientations of H.

It is clear that elements within $P_i\gamma$ differ by the reversal of an edge because the elements of P_i differ by a single edge reversal. Let β be the last orientation in P_1. Then since $\gamma_{i,j}$ and $\gamma_{i,j+1}$ differ by a single reversal, so do $\beta\gamma_{i,j}$ and $\beta\gamma_{i,j+1}$. Therefore, the last element in $P_1\gamma_{i,j}$ differs from the first element in $P_1^R\gamma_{i,j+1}$ by a single reversal. Similarly the last element of $P_1\gamma_{i,n_i}$ differs from the first of $P_1^R\gamma_{i+1,1}$ by a single reversal. Hence, we have a hamilton cycle in AO(H).

Note that the orientation of e is reversed only twice in the given cycle (whenever it is reversed in C_2), so that the ao-graph of H is strongly hamiltonian with respect to e. □

It was shown in Section 3 that when n is even, neither C_n nor L_n have hamiltonian ao-graphs. When n is odd, however, the previous lemma can be used to construct a hamilton cycle in both AO(C_n) and AO(L_n).

Theorem 4.2 *If n is odd, then AO(C_n) is strongly hamiltonian with respect to any edge.*

Proof. Let G_1 the graph consisting of a single edge e. The ao-graph of G_1 is strongly hamiltonian with respect to e. Let G_2 be the n-path with $V(G_2) = \{1, \ldots, n\}$ and $E(G_2) = \{e_1, \ldots, e_{n-1}\}$, where $e_i = (i, i+1)$. Let $\sigma \in A(G_2)$ be the orientation consisting of the arcs $i \rightarrow i+1$ for $1 \leq i < n$. We can denote any $\alpha \in A(G_2)$ by $g_\sigma(\alpha)$, where the i^{th} bit of $g_\sigma(\alpha)$ is 0 iff α contains the arc $i \rightarrow i+1$. As in Section 2, a Gray code for all bit strings of length $n-1$ gives a hamilton cycle in AO(G_2). The binary reflected Gray code [Gr53] is defined by: $S_n = S_{n-1}0, S_{n-1}^R 1$, with $S_2 = 0, 1$.

The sequence S_n gives a hamilton cycle in AO(G_2) which proves AO(G_2) is strongly hamiltonian with respect to the edge whose direction is represented by the final bit, which in this case is the edge $(n-1, n)$. The arc $n \to 1$ is allowable for every element in $A(G_2)$ except for the orientation σ. Similarly, the arc $1 \to n$ is allowable except for the orientation $\overline{\sigma}$. Both arcs are allowable for all other orientations. We can write S_n as $S_n = \sigma, P_2, \overline{\sigma}, P_4$. When n is odd, both P_2 and P_4 are of odd length. Therefore, $\langle G_2, (1,n), (n-1,n) \rangle$ is acyclicly adjoinable.

Let $H = \langle G_1, e \rangle \oplus \langle G_2, (1,n) \rangle$. By Theorem 4.1 AO$(H)$ is strongly hamiltonian with respect to edge $(n-1,n)$. H is just C_n. Since C_n is an edge transitive graph, AO(C_n) is strongly hamiltonian with respect to *any* edge (a graph G is *edge transitive* if for every pair of distinct edges e' and e'' there exists an automorphism of G which maps e' to e''). Hence the theorem. □

The following corollary will be necessary for Section 4.3. If Π is a permutation of $\{1 \ldots, m\}$ and $b = b_1 \ldots b_m$ is an m-bit string, then we let $\Pi(b)$ denote the string $b_{\Pi(1)} \ldots b_{\Pi(m)}$.

Corollary 4.2 *If n is odd, and if $\gamma_1, \gamma_2 \in A(C_n)$ differ by a single edge reversal, then there exists a hamilton path in $AO(C_n)$ starting at γ_1 and ending at γ_2.*

Proof. Let $C = \alpha_1, \ldots, \alpha_N$ be the hamilton cycle of AO(C_n) given by Theorem 4.2, and σ be a cyclic orientation of C_n. Since $|g_\sigma(\gamma_1)|$ and $|g_\sigma(\gamma_2)|$ differ by one, C must contain an edge between an orientation with $|g_\sigma(\gamma_2)|$ ones in its bit string representation and an orientation with $|g_\sigma(\gamma_1)|$ ones in the bit string representation. Let i be such that $|g_\sigma(\alpha_i)| = |g_\sigma(\gamma_2)|$ and $|g_\sigma(\alpha_{i+1})| = |g_\sigma(\gamma_1)|$. We can find a permutation Π of $\{1, \ldots, m\}$ such that such that $\Pi(g_\sigma(\alpha_i)) = g_\sigma(\gamma_2)$ and $\Pi(g_\sigma(\alpha_{i+1})) = g_\sigma(\gamma_1)$. Then

$$P = \Pi(g_\sigma(\alpha_{i+1})),\ \ldots,\ \Pi(g_\sigma(\alpha_N)),\ \Pi(g_\sigma(\alpha_1)),\ \ldots,\ \Pi(g_\sigma(\alpha_i))$$

gives a Gray code (in the bit string representation) for AO(C_n) starting at γ_1 and ending at γ_2. □

We now give a results similar to Theorem 4.2 for ladder graphs.

Theorem 4.3 *If n is odd, then $AO(L_n)$ is hamiltonian.*

Proof. Assume the vertices of L_{n-2} are labeled as in Figure 2 and let G be the 6-vertex graph in Figure 2. Note that $L_n = \langle L_{n-2}, (2n-5, 2n-4) \rangle \oplus \langle G, (v_1, v_6) \rangle$. We prove by induction that L_n is strongly hamiltonian with respect to its rightmost edge (e_6). When $n = 1$ this is obviously true.

Assume $n \geq 3$. By induction, assume that the ao-graph of L_{n-2} is strongly hamiltonian with respect to e. If we can show that $\langle G, (v_1, v_6), e_6 \rangle$ is acyclicly adjoinable, then by Lemma 4.1, we can conclude AO(L_n) is strongly hamiltonian with respect to e_6.

Let σ be the acyclic orientation of G as given by the arrows in Figure 2, and list the edges of G in the order $e_1, \ldots, e_6$. Note that the only cyclic orientations α of G are of the form $g_\sigma(\alpha) = XX1000$ or $g_\sigma(\alpha) = XX0111$, where X can be a zero or one.

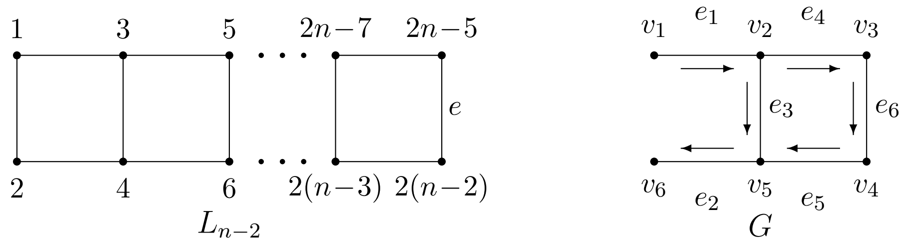

Figure 2: L_{n-2} and G for building L_n

Hence, there are $2^6 - 8 = 56$ acyclic orientations of G. We use both α and $g_\sigma(\alpha)$ to represent an orientation interchangeably.

Define $S = 110, 100, 000, 010, 011, 001, 101$. Let $P_1 = 000S$. Note that P_1 consists of all acyclic orientations of G such that arc $v_1 \to v_6$ is allowable but arc $v_6 \to v_1$ is not. If we let $P_3 = 111\overline{S} = \overline{P_1}$, then P_3 lists the acyclic orientations of G for which $v_6 \to v_1$ is allowable, but $v_1 \to v_6$ is not.

Define P_2 to be the following sequence of orientations of G:

$$
\begin{aligned}
P_2 \;=\; & 100101, 110101, 010101, \\
& 010001, 110001, 100001, \\
& 100011, 110011, 010011, \\
& 011011, 011111, 011101, 011001, \\
& 001001, 001101, 001111, 001011, \\
& 101011, 101111, 101101, 101001.
\end{aligned}
$$

P_2 consists of the acyclic orientations of G for which both arcs $v_1 \to v_6$ and $v_6 \to v_1$ are allowable, and for which e_6 is oriented $v_4 \to v_3$ (the last bit is always 1). Finally, let $P_4 = \overline{P_2}$.

Let $C = P_1, P_2, P_3, P_4$. Then C contains 56 orientations, which is the size of $A(G)$. It is easy to check that no orientation is listed twice in our P_i and that every orientation listed is acyclic. It is also easy to see that consecutive elements in C differ by a single reversal. Note that the last bit, signifying the direction of the edge e_6, is changed only twice. Also, P_2 and P_4 contain an odd number (21) of vertices. Hence $\langle G, (v_1, v_6), e_6 \rangle$ is acyclicly adjoinable. By Lemma 4.1 and the principle of induction, we conclude that for odd n, the ao-graph of the n-ladder is hamiltonian. More precisely, it is strongly hamiltonian with respect to the rightmost (or leftmost) edge. □

4.3 A Hamilton Path in AO(W_n)

We will now give a method for constructing a hamilton path in the ao-graph of W_n when n is odd. Recall that when n is even, by Theorem 3.1, AO(W_n) can have neither a hamilton path nor a hamilton cycle.

First, some preliminaries. We denote the n-cube by $\mathcal{B}_n$. A bipartite graph with partite sets X and Y, $|X| = |Y|$, is called *hamilton laceable* if, given any $x \in X$ and $y \in Y$, there exists a hamilton path beginning at x and ending at y.

Lemma 4.3 *$\mathcal{B}_n$ is hamilton laceable.*

Proof. Let x and y be given. For x and y to be in different sets of the bipartition, they must differ in an odd number of bit positions. We give a proof by induction on n and $d(x, y)$, the number of bits that differ in x and y. If $n = 1$, the proof is trivial. Assume $n > 1$. If $d(x, y) = 1$, x and y differ by a single bit. WLOG, we can assume $x = 0^n$ and $y = 0^{n-1}1$. The binary reflected Gray code [Gr53] provides a sufficient hamilton path.

Now, assume that $n > 1$ and $d(x, y) = k > 1$. Denote $x = x_1 \dots x_n$ and $y = y_1 \dots y_n$. WLOG, assume $x_i \neq y_i$ for $1 \leq i \leq k$. Let $\delta_x = \overline{x_2} x_3 \dots x_n$ and $\delta_y = y_2 \overline{y_3} \dots \overline{y_k} y_{k+1} \dots y_n$. Note that $x' = x_2 \dots x_n$ and δ_x differ by a single bit, while $y' = y_2 \dots y_n$ and δ_y differ by $k - 2$ bits. By induction, there exists a hamilton path P_x in $\mathcal{B}_{n-1}$ from x' to δ_x and a hamilton path P_y in $\mathcal{B}_{n-1}$ from y' to δ_y. Note further that $\delta_x = \delta_y$. If $P = x_1 P_x, y_1 P_y^R$, then P gives a hamilton path in $\mathcal{B}_n$ from x to y. □

We now consider the acyclic orientations of the wheel, W_n. WLOG, we can assume $V(W_n) = \{v_0, v_1, \dots, v_n\}$, where v_0 is the center of the wheel, and the v_i are numbered clockwise around the wheel. The edges of W_n can be divided into two disjoint sets: the spokes of W_n, which are the edges (v_0, v_i) for $1 \leq i \leq n$, are denoted by S_n, and the remaining edges are denoted by C_n, because they form a cycle. We can therefore write an orientation of W_n as $\alpha\beta$, where α is an orientation of S_n, and β is an orientation of C_n.

We will show that if α is an orientation of S_n, then there is a Gray code for the set $f_{W_n}(\alpha)$. Furthermore, we will be able to connect these individual Gray codes together to form a hamilton path in AO(W_n). Let $\sigma_{S_n} \in A(S_n)$ be the orientation of S_n containing the arcs $v_0 \to v_i$ for $1 \leq i \leq n$. Similarly, let σ_{C_n} be the orientation of C_n containing the arcs $v_i \to v_{i+1}$ for $1 \leq i \leq n$ (addition mod$n$).

Suppose $\alpha \in A(S_n)$, and let $g_{\sigma_{S_n}}(\alpha) = a_1 \dots a_n$, where a_i represents the direction of the edge (v_0, v_i) relative to σ_{S_n}. What is $f_{W_n}(\alpha)$? Well, further suppose that $a_i = 0$, and $a_{i+1} = 1$ for some i. Then for all $\beta \in f_{W_n}(\alpha) \subseteq A(C_n)$, if $g_{\sigma_{C_n}}(\beta) = b_1 \dots b_n$, where b_i represents the direction of the edge (v_i, v_{i+1}), it must be that $b_i = 1$. Similarly, if $a_i = 1$ and $a_{i+1} = 0$, then $b_i = 0$ for all $\beta \in f_{W_n}(\alpha)$. The orientation α *fixes* the orientation of the edge (v_i, v_{i+1}) in these cases. In this sense, we can classify all edges of C_n as *fixed* or *free* relative to any orientation α of S_n. A free edge can be oriented in either direction independently of the other cycle edges. The only exception to this is when *all* edges are free, in which case the free edges can be oriented in any way but the two cyclic orientations of C_n.

If $g_{\sigma_{S_n}}(\alpha) = a_1 \dots a_n$, define the *pattern* $p = p_1 \dots p_n$ for the elements in $f_{W_n}(\alpha)$ by

1. If $a_i = a_{i+1}$, $p_i = X$.
2. If $a_i = 0$, $a_{i+1} = 1$, then $p_i = 1$.
3. If $a_i = 1$, $a_{i+1} = 0$, then $p_i = 0$.

For example, $\alpha = 00101$ gives $p = X1010$. In this case, α allows 1 free edge on the cycle. The elements of $f_{W_n}(\alpha)$ will then be all n bit strings which match this pattern. So, in our example, $f_{W_n}(\alpha) = \{01010, 11010\}$. If p is a pattern with k X's, we denote by $p(b_1 \dots b_k)$ the bit string with b_i substituted for the i^{th} X in p. In our example, $p(1) = 11010$. Similarly, if P is a sequence of k-bit strings, $p(P)$ represents the sequence where the elements of P are substituted for the X's of p. So, in our example above, if $P = 0, 1$, $p(P) = 01010, 11010$. Define free(α) to be the number of cycle edges left free by α. In our example, free(α) $= 1$. We define the *pattern distance* from pattern p to pattern q, denoted by pd(p, q), to be the number of i such that $p_i = X$ and $q_i \in \{0, 1\}$. If there exists an i such that $p_i, q_i \in \{0, 1\}$, and $p_i \neq q_i$, then we define pd$(p, q) = \infty$. Since a pattern is determined by an orientation of S_n, we sometimes use the notation free(p) and pd(α, β) interchangeably with free(α) and pd(p, q).

If free$(\alpha) < n$ and p is the pattern of α, then a hamilton path P_k in $\mathcal{B}_k$ would give a Gray code P for $f_{W_n}(\alpha)$ by $P = p(P_k)$. Similarly, if free$(\alpha) = n$, a hamilton path in AO(C_n) gives a Gray code for $f_{W_n}(\alpha)$. We need to show that we can join these individual Gray codes together to form a hamilton path in AO(W_n). Note that the number of fixed edges, which are given by transitions between 0 and 1 in the bit string representation of an orientation, must be an even number. The following lemma says that, in most cases, it is possible to connect these separate hamilton paths of $f_{W_n}(\alpha_i)$ together in a number of ways.

Lemma 4.4 *Let $n > 3$ be odd, and let $\alpha_1, \alpha_2 \in A(S_n)$. Let β be an arbitrary element of $f_{W_n}(\alpha_1)$. If $1 <$ free$(\alpha_1) < n$ and pd$(\alpha_1, \alpha_2) <$ free(α_1), then there exists a Gray code for $f_{W_n}(\alpha_1)$ starting at β and ending at a $\beta' \in f_{W_n}(\alpha_1) \cap f_{W_n}(\alpha_2)$.*

Proof. Let $k =$ free(α_1). As noted previously, the number of fixed edges must be even, so if n is odd, k must be odd also. Hence, $3 \leq k \leq n - 2$.

Let $p = p_1 \dots p_n$ and $q = q_1 \dots q_n$ be the patterns for α_1 and α_2, respectively. Let $\delta \in \mathcal{B}_k$ be such that $p(\delta) = \beta$. Let $\delta' \in \mathcal{B}_k$ be such that δ and δ' differ in an odd number of bits and $p(\delta') \in f_{W_n}(\alpha_2)$. Such a δ' always exists because q fixes only pd$(p, q) < k$ of p's k free edges. By Lemma 4.3, there exists a hamilton path P in $\mathcal{B}_k$ starting at δ and ending at δ'. Then $p(P)$ is a Gray code for $f_{W_n}(\alpha_i)$ beginning with $p(\delta) = \beta$ and ending at $p(\delta') = \beta' \in f_{W_n}(\alpha_2)$. □

As an example, let $n = 5$, $\alpha_1 = 11011$, $\alpha_2 = 11001$. Then $p = X01XX$ and $q = X0X1X$. Let $\beta = 00100$. We need to find a hamilton path P of $\mathcal{B}_3$ starting at 000 and ending some bit string matching $X1X$ (the second free edge of α_1 is fixed to 1 by α_2). By Lemma 4.3, we could end our path at 111 or 010 (ie any orientation with an odd number of 1's).

When free$(\alpha_1) = 1$, Lemma 4.4 is not true. For example, let $\alpha_1 = 10101$, $\alpha_2 = 10100$, and $\beta = 01011$. Then $p = 0101X$ and $q = 010X1$. If we started at β, we would have to finish at 01010, which is not in $f_{W_n}(\alpha_2)$. The following lemma gives us a hamilton path in $\mathrm{AO}(S_n) = \mathcal{B}_n$ which controls the placement of the α with free$(\alpha) = 1$.

Lemma 4.5 *If $n > 1$ is odd, there exists a hamilton path P_n in $\mathcal{B}_n$ such that P_n can be written as $P_n = Q_1, P, Q_2$, where $\alpha \in P$ iff free$(\alpha) = 1$. Further, P is in one of the following forms:*

$$P = \alpha_1, \ldots, \alpha_{n-1}, 0(10)^{(n-1)/2}, \alpha_{n+1}, \ldots, \alpha_{2n-1}, 1(01)^{(n-1)/2} \quad (2)$$

or

$$P = 0(10)^{(n-1)/2}, \alpha_2, \ldots, \alpha_n, 1(01)^{(n-1)/2}, \alpha_{n+2}, \ldots, \alpha_{2n}. \quad (3)$$

For the unspecified α_i with $1 \le i \le n$, α_i begins with 0 and ends with 1. For the unspecified α_i with $n+1 \le i \le 2n$, α_i begins with 1 and ends with 0. Further, α_1 and α_{2n} are also adjacent. Also, 0^n is listed in Q_1 and 1^n is listed in Q_2. Finally, the number of elements in each Q_i is $2^{n-1} - n$.

Proof. Let $P_3 = 000, 001, 011, 010, 110, 100, 101, 111$. This is a hamilton path of $\mathcal{B}_3$ of form (2) satisfying all criteria of the lemma. In P_3, $Q_1 = 000$ and $Q_2 = 111$. Let $n \ge 3$ be odd and inductively assume that P_n is a hamilton path of $\mathcal{B}_n$ which satisfies the conditions of the lemma. We show how to construct P_{n+2}.

Write $P_n = Q_1, P, Q_2$, where $\alpha \in P$ iff free$(\alpha) = 1$, and $P = \alpha_1, \ldots \alpha_{2n}$. If $\alpha \in \mathcal{B}_{n+2}$ and free$(\alpha) = 1$, then there exists an i such that $\alpha = \alpha_i b_1 b_2$, where $b_j \in \{0, 1\}$. Hence, to list all $\alpha \in \mathcal{B}_{n+2}$ with free$(\alpha) = 1$, we can list the α_i and append all possible two bit combinations that keep free$(\alpha_i b_1 b_2) = 1$. For all α_i except $1(01)^{(n-1)/2}$ and $0(10)^{(n-1)/2}$, there is only one two bit extension that works (either 01 or 10 depending if in first or second half of list), while for the two exceptions there are three possible two bit extensions that work.

Let $S_1 = 00, 01, 11$; $S_2 = 00, 10, 11$; $S_3 = 10, 11, 01$; $S_4 = 01, 00, 10$; $S_5 = 00, 01, 11, 10$; and $S_6 = 11, 10, 00, 01$. In the case that the form of P is given by (2), we set

$$P'' = 0(10)^{(n-1)/2} S_3,\ \alpha_{n-1}01,\ \ldots,\ \alpha_1 01,\ 1(01)^{(n-1)/2} S_4,\ \alpha_{2n-1}10,\ \ldots,\ \alpha_{n+1}10.$$

Note P'' is now of form (3), and P'' contains all $\alpha \in \mathcal{B}_{n+2}$ with free$(\alpha) = 1$. P'' satisfies all properties given in the lemma. Set

$$\begin{aligned} Q_1' &= \alpha_1 S_2,\ \alpha_2 S_2^R,\ \ldots,\ \alpha_{n-1} S_2^R,\ 0(10)^{(n-1)/2}00 \\ Q_2' &= \alpha_{n+1} S_1^R,\ \alpha_{n+2} S_1,\ \ldots,\ \alpha_{2n-1} S_1,\ 1(01)^{(n-1)/2}11. \end{aligned}$$

The Q_j' list all $\alpha = \alpha_i b_1 b_2 \in \mathcal{B}_{n+2}$ ($1 \le i \le 2n$ and $b_1, b_2 \in \{0, 1\}$) with free$(\alpha) > 1$. If $Q_j = \gamma_{j,1} \ldots, \gamma_{j,n_j}$, note that $n_j = 2^{n-1} - n$ is odd. Define

$$\begin{aligned} Q_1'' &= \gamma_{1,1} S_5^R, \gamma_{1,2} S_5, \ldots, \gamma_{1,n_1} S_5^R, Q_1' \\ Q_2'' &= Q_2', \gamma_{2,1} S_6, \gamma_{2,2} S_6^R, \ldots, \gamma_{2,n_2} S_6. \end{aligned}$$

The Q''_j list all $\alpha \in \mathcal{B}_{n+2}$ which have free$(\alpha) > 1$. Note that 0^{n+2} is in Q''_1 and 1^{n+2} is in Q''_2. The length of each Q''_j is $4(2^{n-1} - n) + 3(n-1) + 1 = 2^{n+1} - (n+2)$.

Then $P_{n+2} = Q''_1, P'', Q''_2$ gives a hamilton path for $\mathcal{B}_{n+2}$ which satisfies the requirements of the lemma. A similar construction is possible when P is of the form (3). □

The above lemma gives us a hamilton path in AO(S_n) which controls where the α such that free$(\alpha) = 1$ are placed. If α and β are adjacent bit strings, then $|\text{free}(\alpha) - \text{free}(\beta)| \leq 2$. So Lemma 4.4 tells us, roughly, that when $1 < \text{free}(\alpha) < n$, we have the flexibility to start a Gray code for $f_{W_n}(\alpha)$ arbitrarily, and end it in $f_{W_n}(\beta)$ for an arbitrary β adjacent to α in AO(S_n). We have yet to deal with the case when free$(\alpha) = n$, which is done in the following lemma.

Lemma 4.6 *Let $n \geq 5$ be odd. Let $\alpha_1 \neq \alpha_2$ both be adjacent to 1^n in $AO(S_n)$, and let β be an arbitrary element of $f_{W_n}(\alpha_1)$. Then we can find a Gray code P_1 for $f_{W_n}(\alpha_1)$ beginning at β and terminating at a $\beta_1 \in f_{W_n}(\alpha_1) \cap f_{W_n}(1^n)$, and a Gray code P_2 for $f_{W_n}(1^n)$ starting at β_1 and ending at a $\beta_2 \in f_{W_n}(1^n) \cap f_{W_n}(\alpha_2)$.*

Proof. Note that free$(\alpha_i) = n-2$ for $i = 1, 2$ because free$(1^n) = n$ and α_i differs from 1^n by a single edge reversal. The proof will be based on two cases: pd$(\alpha_1, \alpha_2) = 2$, and pd$(\alpha_1, \alpha_2) = \infty$. Note that these are the only two possibilities for pd(α_1, α_2) because each α_i fixes only two edges. Suppose pd$(\alpha_1, \alpha_2) = 2$. Since free$(\alpha_1) = n - 2$ and $n \geq 5$, we know free$(\alpha_1) \geq 3 > \text{pd}(\alpha_1, \alpha_2)$. Hence, by Lemma 4.4, we can find a hamilton path P_1 of $f_{W_n}(\alpha_1)$ such that P_1 begins with β and ends at a $\beta_1 \in f_{W_n}(\alpha_2) \subset f_{W_n}(1^n)$. Since α_1 and α_2 each only fix 2 edges, and $n \geq 5$, there is at least one edge left free by both α_i. Let β_2 be obtained from β_1 by reversing the orientation of this edge. Note that $\beta_2 \in f_{W_n}(\alpha_2) \subset f_{W_n}(1^n)$. Corollary 4.2 says we can find a hamilton path P_2 of $f_{W_n}(1^n)$ beginning at β_1 and ending at β_2. Hence, if pd$(\alpha_1, \alpha_2) = 2$, the lemma is true.

A similar argument proves the lemma for pd$(\alpha_1, \alpha_2) = \infty$. □

We can now use the hamilton path in AO(S_n) of Lemma 4.5 to construct a hamilton *path* in AO(W_n).

Theorem 4.4 *For odd $n \geq 3$, there exists a hamilton path in $AO(W_n)$.*

Proof. Let $P_n = Q_1, P, Q_2$ be the hamilton path in AO(S_n) given by Lemma 4.5. We start building our hamilton path for AO(W_n) with P. Let $P = \alpha_1, \ldots, \alpha_{2n}$. It is simple to see that since α_i and α_{i+1} are adjacent, for all $1 \leq i < 2n$, there exists an orientation $\beta_i \in f_{W_n}(\alpha_i) \cap f_{W_n}(\alpha_{i+1})$. Each $f_{W_n}(\alpha_i)$ has only two elements (free$(\alpha_i) = 1$), namely β_i and β_{i-1}. Note that $\beta_{i-1} \neq \beta_i$ because $\alpha_{i-1} \neq \alpha_{i+1}$. Therefore,

$$P' = \alpha_1\beta_0,\ \alpha_1\beta_1,\ \alpha_2\beta_1,\ \alpha_2\beta_2,\ \ldots,\ \alpha_{2n}\beta_{2n-1},\ \alpha_{2n}\beta_{2n}$$

gives a Gray code for the set $\cup_i \alpha_i f_{W_n}(\alpha_i)$.

If $n = 3$, then $Q_1 = 000$ and $Q_2 = 111$. By Theorem 4.2 there exists a hamilton cycle C in $\mathrm{AO}(C_3)$. Let P_{111} be the Gray code for the set $f_{W_3}(111)$ obtained from C by shifting C until it begins with β_{2n}. Let $Q'_2 = 111P_{111}$. We similarly find $Q'_1 = 000P_{000}$. Then Q'_1, P', Q'_2 gives a hamilton path in $\mathrm{AO}(W_3)$.

Assume $n > 3$. For $j = 1, 2$, let $Q_j = \gamma_{j,1}, \ldots, \gamma_{j,n_j}$. The free edges of α_{2n} are contained in the free edges of $\gamma_{2,1}$, so we see $\beta_{2n} \in f_{W_n}(\gamma_{2,1})$. If $\gamma_{2,2} \neq 1^n$, then since $1 < \mathrm{free}(\gamma_{2,1}) < n$, by Lemma 4.4 we can find a Gray code $P_{2,1}$ for $f_{W_n}(\gamma_{2,1})$ starting at β_{2n} and ending at $\beta_{2,1} \in f_{W_n}(\gamma_{2,1}) \cap f_{W_n}(\gamma_{2,2})$. We continue this process of repeated applications of Lemma 4.4 until we reach i such that $\gamma_{2,i+1} = 1^n$, ie until the orientation *before* 1^n. Lemma 4.6 will give us Gray codes $P_{2,i}$ and $P_{2,i+1}$ for $f_{W_n}(\gamma_{2,i})$ and $f_{W_n}(\gamma_{2,i+1}) = f_{W_n}(1^n)$, respectively. By Lemma 4.6, $P_{2,i+1}$ will terminate at a $\beta_{2,i+1} \in f_{W_n}(\gamma_{2,i+2})$. We can then return to using repeated applications of Lemma 4.4 to find a Gray code $P_{2,j}$ for $f_{W_n}(\gamma_{2,j})$ as j goes from $i+2$ to n_2.

Note that

$$Q'_2 = \gamma_{2,1}P_{2,1},\ \ldots,\ \gamma_{2,n_2}P_{2,n_2}$$

gives a Gray code for the set $\cup_i \gamma_{2,i} f_{W_n}(\gamma_{2,i})$. We can similarly work backwards from α_1 to get a Gray code Q'_1 for the set $\cup_i \gamma_{1,i} f_{W_n}(\gamma_{1,i})$. Then Q'_1, P', Q'_2 gives a hamilton path in $\mathrm{AO}(W_n)$. □

Unfortunately, we have not been able to construct a hamilton *cycle* in $\mathrm{AO}(W_n)$ for odd n, although we strongly suspect one exists.

5 Conclusions

The ao-graph of a given graph may or may not be hamiltonian. There are certain graphs which give an ao-graph that has a parity problem, and cannot have a hamilton cycle. We have seen several such examples. In most of these cases, when we look at the same class of graphs when they do not have a parity problem, we have been able to construct a Gray code for the acyclic orientations.

A similar phenomenon occurs in Gray codes for linear extensions of posets. Ruskey [Ru92] conjectures that when there is no parity conflict in the linear extension problem, a Gray code always exists. We can make a similar conjecture about a Gray code for acyclic orientations.

In most of the cases presented here, this has proven to be true, the exception being the complete bipartite graph. When $m, n > 1$ and $2|mn$, Theorem 3.3 provides a proof that there exists a parity problem in $\mathrm{AO}(K_{m,n})$. We have yet to discover a hamilton path or cycle in $\mathrm{AO}(K_{m,n})$ when m and n are both odd, $m, n > 1$.

Although the linear extension problem did sometimes have a parity problem, Pruesse and Ruskey [PR91] were able to construct a less restrictive Gray code. Instead of requiring that successive elements differ by a single edge in the linear extension graph, successive elements were allowed to differ by a path of length *less than or equal to two*. This relaxation was enough for a Gray code to always exist. We are able to do a similar modification to the acyclic orientation problem. If we relax the adjacency criterion and define two acyclic orientations to be adjacent if they differ by one *or two* edge reversals, we are able to construct a Gray code for listing the acyclic orientations of *any* graph [Sq].

Pruesse and Ruskey [PR91] were able to implement their construction efficiently, giving the first algorithm for listing linear extensions in constant average time per extension. We are investigating ways of implementing our relaxed Gray code for acyclic orientations in the hope of obtaining an efficient algorithm for listing acyclic orientations.

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