Anti-Lecture Hall Compositions

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Abstract

We study the set A_k of integer sequences $\lambda = (\lambda_1, \ldots, \lambda_k)$, defined by

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \frac{\lambda_k}{k} \geq 0,$$

and show that the generating function is

$$\sum_{\lambda \in A_k} q^{|\lambda|} = \prod_{i=1}^{k} \frac{1+q^i}{1-q^{i+1}},$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_k$. We establish this by giving a bijective proof of the following refinement:

$$\sum_{\lambda \in A_k} q^{|\lambda|} u^{|\lfloor \lambda \rfloor|} v^{o(\lfloor \lambda \rfloor)} = \prod_{i=1}^{k} \frac{1+uvq^i}{1-u^2 q^{i+1}}.$$

To our knowledge this is a new result that complements the family of the Lecture Hall Theorems [3, 4, 5].

1 Introduction

For a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of integers, define the *weight* of λ to be $\lambda_1 + \cdots + \lambda_k$ and call each λ_i a *part* of λ. If λ has all parts nonnegative, we call it a *composition* and if, in addition, λ is a nonincreasing sequence, we call it a *partition*.

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In [3], Bousquet-Mélou and Eriksson considered *lecture hall partitions*, specifically, the set L_k of partitions, λ, into at most k parts satisfying

$$\frac{\lambda_1}{k} \geq \frac{\lambda_2}{k-1} \geq \ldots \geq \frac{\lambda_{k-1}}{2} \geq \frac{\lambda_k}{1} \geq 0,$$

and proved the following surprising result, known as the *Lecture Hall Theorem*:

$$\sum_{\lambda \in L_k} q^{|\lambda|} = \prod_{i=0}^{k-1} \frac{1}{1-q^{2i+1}}. \tag{1}$$

Subsequently, partition analysis was used by Andrews in [1] to give an analytical proof (with combinatorial aspects) and by Andrews, Paule, Riese, and Strehl, in [2], to construct a bijection. An elegant bijective proof of the Lecture Hall Theorem, due to Yee, appears in [6].

A different approach of Bousquet-Mélou and Eriksson in [5] led to the *Refined Lecture Hall Theorem*:

$$\sum_{\lambda \in L_k} q^{|\lambda|} u^{\lceil \lambda \rceil} v^{o(\lceil \lambda \rceil)} = \prod_{i=1}^{k} \frac{1+uvq^i}{1-u^2 q^{k+i}}, \tag{2}$$

where for a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, $o(\lambda)$ is the number of odd parts of λ and $\lceil \lambda \rceil$ denotes the partition $(\lceil \lambda_1/k \rceil, \lceil \lambda_2/(k-1) \rceil, \ldots, \lceil \lambda_{k-1}/2 \rceil, \lceil \lambda_k/1 \rceil)$. Setting $u = v = 1$ in (2) gives (1). Yee gave also a beautiful bijective proof [7] of this result.

In this paper, we consider a new twist on these results by studying the set A_k of *compositions* into at most k parts satisfying

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \frac{\lambda_k}{k} \geq 0.$$

We refer to these as *anti-lecture hall compositions* and show the following.

Theorem 1 *(The Anti-lecture Hall Theorem)*

$$\sum_{\lambda \in A_k} q^{|\lambda|} = \prod_{i=1}^{k} \frac{1+q^i}{1-q^{i+1}}. \tag{3}$$

In fact, we prove the following refinement of (3):

Theorem 2 *(The Refined Anti-lecture Hall Theorem)*

$$\sum_{\lambda \in A_k} q^{|\lambda|} u^{|\lfloor \lambda \rfloor|} v^{o(\lfloor \lambda \rfloor)} = \prod_{i=1}^{k} \frac{1+uvq^i}{1-u^2 q^{i+1}}, \tag{4}$$

where $\lfloor\lambda\rfloor = (\lfloor\lambda_1/1\rfloor, \lfloor\lambda_2/2\rfloor, \ldots, \lfloor\lambda_k/k\rfloor)$ *and* $o(\lambda)$ *denotes the number of odd parts of a composition* λ*.*

The bijective proof we give follows the idea of Yee's proof [7] of the Refined Lecture Hall Theorem (2).

2 Proof of Main Result

Define the generating function :

$$H_k(u,v,q) = \sum_{\lambda \in A_k} q^{|\lambda|} u^{|\lfloor\lambda\rfloor|} v^{o(\lfloor\lambda\rfloor)},$$

where, as in Theorem 2, $\lfloor\lambda\rfloor = (\lfloor\lambda_1/1\rfloor, \lfloor\lambda_2/2\rfloor, \ldots, \lfloor\lambda_k/k\rfloor)$ and $o(\lambda)$ denotes the number of odd parts of the composition λ.

Given an anti-lecture hall composition $\lambda \in A_k$, we can write λ as $((l_1, \ldots, l_k), (r_1, \ldots, r_k))$ where $\lambda_i = il_i + r_i$, with $0 \le r_i \le i-1$ for $1 \le i \le k$. Then $(l_1, \ldots, l_k) = \lfloor\lambda\rfloor$. Note that $\lambda \in A_k$ if and only if (i) $l_1 \ge l_2 \ge \ldots l_k \ge 0$ and (ii) $r_i \ge r_{i+1}$ whenever $l_i = l_{i+1}$. Moreover $|\lambda| = \sum_{i=1}^k r_i + il_i$.

Proof of Theorem 2. Let D_k be the set of the partitions into distinct parts less than or equal to k. Let E_k be the subset of A_k consisting of those λ for which every l_i is even. To prove Theorem 2 we give two bijections :

- A bijection between A_k and $D_k \times E_k$ such that if (α, β) is the image of λ then $|\alpha| + |\beta| = |\lambda|$, $|\lfloor\beta\rfloor| + L(\alpha) = |\lfloor\lambda\rfloor|$ and $L(\alpha) = o(\lfloor\lambda\rfloor)$, where $L(\alpha)$ is the number of positive parts of α.
- A bijection between E_k and the set P_k, of partitions into parts in the set $\{2, 3, \ldots, k+1\}$, such that if μ is the image of λ then $|\mu| = |\lambda|$ and $L(\mu) = \sum_{i=1}^k l_i/2$.

The first bijection will show that :

$$H_k(u,v,q) = \prod_{i=1}^{k} (1 + uvq^i) E_k(u,q),$$

where

$$E_k(u,q) = \sum_{\beta \in E_k} q^{|\beta|} u^{|\lfloor\beta\rfloor|}.$$

The second bijection will show that :

$$E_k(u,q) = \prod_{i=2}^{k+1} \frac{1}{1-u^2q^i},$$

completing the proof of (4).

The first bijection (between A_k and $D_k \times E_k$). Given $\lambda \in A_k$ we construct $\alpha \in D_k$ and $\beta \in E_k$ with the following algorithm :

> $\alpha \leftarrow$ empty partition
>
> $\beta \leftarrow \lambda$
>
> While one of the l_i is odd do
>
> > $d \leftarrow \max\{i \mid l_i \text{ odd}\}$
> >
> > $i \leftarrow \min\{j \geq d \mid j = k \text{ or } l_d > l_{j+1} + 1 \text{ or } r_d \geq r_{j+1}\}$
> >
> > (Note $l_d > l_{d+1}$ by choice of d.)
> >
> > $l_d \leftarrow l_d - 1$
> >
> > $c \leftarrow r_d$
> >
> > For j from d to $i-1$ do
> >
> > > $r_j \leftarrow r_{j+1} - 1$
> >
> > $r_i \leftarrow c$
> >
> > $\alpha \leftarrow \alpha \cup i$

Consider the j^{th} iteration of the "while" loop. Let d_j and i_j the indices chosen during that iteration. Then l_{d_j}, the last remaining odd l_i, is decreased by 1 and no other l_i is altered during this iteration. Thus $d_j > d_{j+1} \geq 1$ and each iteration decreases by 1 the number of odd l_i. Iteration j preserves the property $\beta \in A_k$, so at termination $\beta \in E_k$.

During iteration j, l_{d_j} is decreased by 1 and $i_j - d_j$ of the r_i are decreased by 1, so the weight of β is decreased by $d_j + (i_j - d_j) = i_j$. Since the part i_j is added to α, the weight $|\lambda| = |\alpha| + |\beta|$ is preserved. For each iteration of the loop, we decrease one odd part, l_{d_j}, by 1 and add one part, i_j, to α, so $|\lfloor\beta\rfloor| + L(\alpha) = |\lfloor\lambda\rfloor|$ and $L(\alpha) = o(\lfloor\lambda\rfloor)$.

Clearly any part i_j added to α satisfies $1 \leq i_j \leq k$, but we must verify that the parts of α are distinct. We show $i_{j+1} < i_j$. At the beginning of iteration j, if $t < d_j$ and l_t is odd, then either (i) $l_t > l_{d_j}$ or (ii) $l_t = l_{d_j}$ and $r_t \geq r_{d_j} = c$. Then by choice of i_{j+1}, in case (i), $i_{j+1} < d_j \leq i_j$. Since $r_{i_j} = c$ at the end of iteration j, in case (ii) we also have $i_{j+1} < i_j$.

Example. Starting with $\lambda = ((7,6,4,3,3,2),(0,1,1,2,2,3))$ we apply the first bijection and get

- α empty and $\beta = ((7,6,4,3,3,2),(0,1,1,2,2,3))$ (initially)

- $d = 5, i = 6,\ \alpha = (6)$ and $\beta = ((7,6,4,3,2,2),(0,1,1,2,2,2))$ (iteration 1)
- $d = 4, i = 4,\ \alpha = (6,4)$ and $\beta = ((7,6,4,2,2,2),(0,1,1,2,2,2))$ (iteration 2)
- $d = 1, i = 2,\ \alpha = (6,4,2)$ and $\beta = ((6,6,4,2,2,2),(0,0,1,2,2,2))$ (iteration 3)

We now define the reverse bijection. Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ be a partition in D_k and β a composition in E_k. Then we apply the following algorithm to create $\lambda \in A_k$:

$\lambda \leftarrow \beta$

For j from l downto 1 do

  $i \leftarrow \alpha_j$

  $d \leftarrow \max\{t \le i | t = 1 \text{ or } l_{t-1} > l_t\}$

  $c \leftarrow r_j$

  For t from $d+1$ to j do

    $r_t \leftarrow r_{t-1} + 1$

  $r_d \leftarrow c$

  $l_d \leftarrow l_d + 1$

Note that at the end of each iteration,

- d is the largest index for which l_d is odd,
- $l_d > l_{d+1} = \ldots = l_i \ge l_{i+1}$,
- $r_{d+1} \ge \ldots \ge r_i > r_d$, and
- if $l_d = l_{i+1} + 1$ then $r_d \ge r_{i+1}$.

Thus i is the smallest index greater than or equal to d such that $l_{i+1} < l_d + 1$ or $l_{i+1} = l_d + 1$ and $r_d \ge r_{i+1}$. Therefore the mapping is reversible and we have defined a bijection.

The second bijection (between E_k and P_k). For $\lambda \in E_k$, write λ as $((l_1, \ldots, l_k), (r_1, \ldots, r_k))$ where $\lambda_i = il_i + r_i$ and $0 \le r_i \le i - 1$ for $1 \le i \le k$. (Then $r_1 = 0$.) We construct $\mu \in P_k$ by specifying the multiplicity in μ, $m_\mu(i)$, of each $i \in \{2, 3, \ldots, k+1\}$:

$$\begin{aligned} m_\mu(k+1) &= r_k + \frac{l_k}{2}k; \\ m_\mu(i) &= r_{i-1} - r_i + \frac{l_{i-1} - l_i}{2}(i-1),\ 2 \le i \le k. \end{aligned}$$

It is easy to see that we can reconstruct λ from μ.

Note that $m_\mu(i)$ is always nonnegative as $r_{i-1} < i$ and if $r_{i-1} < r_i$ then $l_{i-1} > l_i$, that is, $l_{i-1} > l_i + 1$ (since each l_i is even) and $(l_{i-1} - l_i)(i-1)/2 \geq i - 1$. Now we must show that $|\mu| = |\lambda|$:

$$
\begin{aligned}
|\mu| &= \sum_{i\geq 1} i m_\mu(i) \\
&= r_k(k+1) + \sum_{i=2}^{k} i(r_{i-1} - r_i) + k(k+1)l_k/2 + \sum_{i=2}^{k}(i-1)i(l_{i-1} - l_i)/2 \\
&= \sum_{i=1}^{k}(r_i + l_i((i+1)i - i(i-1))/2) \\
&= \sum_{i=1}^{k}(r_i + il_i) = |\lambda|.
\end{aligned}
$$

Finally, we show that the number of positive parts of μ, $L(\mu)$, is half the sum of the l_i.

$$
\begin{aligned}
L(\mu) &= \sum_{i=2}^{k+1} m_\mu(i) \\
&= r_k + \sum_{i=2}^{k}(r_{i-1} - r_i) + kl_k/2 + \sum_{i=2}^{k} = (i-1)(l_{i-1} - l_i)/2 \\
&= \sum_{i=1}^{k} l_i/2
\end{aligned}
$$

Therefore the second bijection satisfies the required conditions and Theorem 2 is proved. □

Example. Starting from $\lambda = ((6,6,4,2,2,2),(0,0,1,2,2,2))$ we apply the second bijection and get $\mu = (7,7,7,7,7,7,7,7,7,4,4,3))$.

3 Concluding Remarks

A refinement of the Lecture Hall Theorem (1), different from (2) was proved in [3]:

$$
\sum_{\lambda \in L_k} x^{|\lambda|_o} y^{|\lambda|_e} = \prod_{i=0}^{k-1} \frac{1}{1 - x^{i+1}y^i},
$$

where $|\lambda|_o = \lambda_1 + \lambda_3 + \lambda_5 + \ldots$, and $|\lambda|_e = \lambda_2 + \lambda_4 + \lambda_6 + \ldots$. (This was further generalized in [4].)

We have a conjecture as to the generating function for the corresponding refinement of the Anti-lecture Hall Theorem which we hope to be able to prove in ongoing work.

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