Hamilton-Connected Derangement Graphs on S_n

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Abstract

We consider the *derangements graph* in which the vertices are permutations of $\{1 \ldots n\}$. Two vertices are joined by an edge if the corresponding permutations differ in *every* position. The derangements graph is known to be hamiltonian and it follows from a recent result of Jung that *every* pair of vertices is joined by a Hamilton path. We use this result to settle an open question, by showing that it *is* possible, for any n and k satisfying $2 \leq k \leq n$ and $k \neq 3$, to generate permutations of $\{1 \ldots n\}$ so that successive permutations differ in k *consecutive* positions. In fact, the associated k-consecutive derangements graph is also Hamilton-connected.

1 Introduction

The problem of generating permutations has received much attention, classically, because of its importance in algorithms, and, more recently, because of its connection with some open problems for Cayley graphs.

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Given a group G and a set $X \subseteq G$, the (undirected) Cayley graph of G with respect to X is the graph $C[G;X]$ with vertex set G in which vertices u and v are joined by an edge (labeled x) if and only if $ux = v$ or $vx = u$ for some $x \in X$. The graph $C[G;X]$ is connected if and only if X is a set of generators for G. It is an open question whether every connected Cayley graph is hamiltonian or even has a Hamilton path. The latter is a special case of the more general conjecture of Lovàsz that every connected, undirected, vertex-transitive graph has a Hamilton path [Lo]. It is even unknown whether every connected Cayley graph of S_n is hamiltonian. Some results on Cayley graphs are surveyed in [WiGa].

It was established independently by Johnson [Jo] and Trotter [Tr] that it is possible to generate all permutations of $1 \ldots n$, each exactly once, so that successive permutations (as well as the first and last) differ only by one swap of two elements in adjacent positions, that is, by an *adjacent transposition*. This gives an efficient algorithm for generating permutations and at the same time shows that the Cayley graph of S_n with respect to the generating set $\{(j \;\; j+1) : 1 \le j < n\}$ is hamiltonian. Techniques for still more efficient permutation generation are surveyed in [Se]. In this paper we focus on two other Cayley graphs of S_n: the derangements graph and a generalization, the k-consecutive derangements graph.

We will use both straight line notation and cycle notation for permutations. In straight line notation, $\langle a_1 a_2 \ldots a_n \rangle$ denotes the permutation $\pi \in S_n$ defined by $\pi(i) = a_i$ for $1 \le i \le n$. In cycle notation, the cycle $\sigma = (b_1 \ldots b_k) \in S_n$ is defined by

$$\begin{aligned} &\sigma(b_k) = b_1, \\ &\sigma(b_i) = b_{i+1} \;\; \text{for} \;\; 1 \le i \le k-1, \text{ and} \\ &\sigma(j) = j \;\; \text{if} \;\; j \notin \{b_1, \ldots, b_k\}. \end{aligned}$$

We perform composition of permutations from right to left so, for example,

$$(3\ 2) \cdot \langle 24531 \rangle = \langle 2\ 1 \rangle$$

whereas

$$\langle 24531 \rangle \cdot (3\ 2) = \langle 25431 \rangle.$$

A permutation $\pi \in S_n$ is a *derangement* if π has no fixed points. For σ, $\sigma' \in S_n$, we say that σ is a derangement of σ' if $\sigma = \sigma' \cdot \pi$ for some derangement, π. In this case, note that $\sigma(i) \neq \sigma'(i)$ for $1 \leq i \leq n$.

The set of derangements is a generating set of S_n for $n \neq 3$. This follows for $n \geq 4$, since $\{(j \;\; j+1) : 1 \leq j < n\}$ is a generating set for S_n and for $1 \leq i < n$, the transposition $(j \;\; j+1)$ is the product of the two derangements:

$$(j \;\; j+1) = [(1 \ldots n)^2] \;\cdot\; [(n \ldots 1)^2 \;\cdot\; (j \;\; j+1)].$$

The *derangements graph*, for given n, is the Cayley graph of S_n with respect to the generating set of derangements. The question as to whether the derangements graph is hamiltonian was posed in [Rab] and [Wi1]. A constructive solution due to Eggleton and Wallis is described in [EgWa] (the construction as described does not work when $n = 7$, but can be easily patched) and another, due to Yarbrough is described in [RaSl]. Existence of a Hamilton cycle was shown in [Me] and [Wi2] to follow from Jackson's theorem that every d-regular, 2-connected graph with at most $3d$ vertices is hamiltonian [Ja]. We will show in Section 2 that a generalization of Jackson's theorem, due to Jung [Ju], can be applied to show that for $n \geq 4$ the derangements graph is also Hamilton-connected, that is, every pair of distinct vertices is joined by a Hamilton path.

To simultaneously generalize the problems of generating permutations, at one extreme, by adjacent transpositions and, at the other extreme, by derangements, we considered generating permutations by k-derangements in [Sa]. Call a permutation $\pi \in S_n$ a *k-derangement* if π has exactly $n-k$ fixed points. For $k \neq 3$ and $2 \leq k \leq n$, the set of k-derangements forms a generating set for S_n. For example, consider the adjacent transposition $(j \;\; j+1)$. Choose an i satisfying $1 \leq i \leq n-k+1$ and $j,\ j+1 \in \{i, \ldots, i+k-1\}$. Then $(j \;\; j+1)$ is the product of the two k-derangements:

$$(j \;\; j+1) = [(i+k-1 \ldots i+1 \;\; i)^2] \;\cdot\; [(i \;\; i+1 \ldots i+k-1)^2 \;\cdot\; (j \;\; j+1)].$$

It was shown independently in [Sa] and [Pu1] that the Cayley graph of S_n with respect to the generating set of k-derangements is hamiltonian. (This does not follow from Jackson's theorem unless $k = n$ or $k = n-1$, as noted in [Sa].) However, it was left open whether this result would still hold if one only considers the subset of k-derangements in which the non-fixed points are *consecutive*.

Call a permutation $\pi \in S_n$ a *k-consecutive derangement* if the non-fixed points of π are $i, i+1, \ldots, i+k-1$ for some $i : 1 \leq i \leq n-k+1$. The argument above shows that the set of k-consecutive derangements is also a generating set of S_n for $k \neq 3,\ 2 \leq k \leq n$. We call the Cayley graph of S_n with respect to this generating set the *k-consecutive derangements graph* and it was asked in [Sa] whether this graph is hamiltonian. The answer was shown to be *yes* for *even* values of k in [Pu2]. We show in Section 2 that the k-consecutive derangements graph is hamiltonian for $k \neq 3,\ \ 2 \leq k \leq n$. In fact, it is Hamilton-connected for all $k \geq 4$. This becomes the Hamilton-connectivity of the derangements graph when $k = n$.

Finally, recall that any permutation can be written as the product of transpositions and is even or odd, according to whether the number of transpositions is even or odd. For $\pi \in S_n$, sign(π)= 1 if π is even; otherwise, sign(π) $= -1$.

2 The k-Consecutive Derangements Graph

Let $G(n,k)$ denote the Cayley graph of S_n with respect to the generating set of k-consecutive derangements. For $k = 2$, $G(n,k)$ is the Cayley graph $C[S_n; B]$, where $B = \{(1\ 2), (2\ 3), \ldots (n-1\ \ n)\}$, otherwise known as the adjacent transposition graph. The adjacent transposition graph is bipartite and the following theorem from [Tc] establishes that $G(n,2)$ is Hamilton-laceable for $n \geq 4$. (A bipartite graph is *Hamilton-laceable* if any two vertices from different parts of the bipartition are joined by a Hamilton path.)

Theorem 1 (Tchuente) *For* **any** *generating set B of transpositions for S_n, $C[S_n; B]$ is Hamilton-laceable for $n \geq 4$.*

Proof. See [Tc], Thm. 1, p. 117. □

When $k = 3$ and $n \geq k$, $G(n,k)$ is not connected, since any 3-consecutive derangement is a product of two 2-cycles and therefore permutations of opposite sign cannot be joined by a path in $G(n,3)$. For $n = k$, $G(n,k)$ is the derangements graph and the following generalization of Jackson's theorem [Ja], due to Jung [Ju], can be used to show the derangements graph is Hamilton-connected for $n \geq 4$.

Theorem 2 (Jung) *Let G be a d-regular, 3-connected graph with at most $3d-1$ vertices. If G is not isomorphic to the graph $H(d)$ in Figure 1, then G has the hamiltonian property, i.e., G is Hamilton-laceable if G is bipartitite, and Hamilton-connected otherwise.*

Proof. See [Ju], Thm. 1, p. 282. □

To establish that $G(n,n)$ is 3-connected, we will use the following theorem.

Theorem 3 (Watkins) *If G is a connected, vertex transitive graph with vertex degree d, then the connectivity of G is at least $2d/3$.*

Proof. See [Wa], Thm. 3, p. 28. □

Corollary 1 *The derangements graph $G(n,n)$ is Hamilton-connected for $n \geq 4$.*

Proof. The graph $G(n,n)$ is a Cayley graph and all Cayley graphs are vertex transitive. Thus, $G(n,n)$ cannot be isomophic to $H(d)$ of Figure 1 for any d. $G(n,n)$ is connected since the derangements form a generating set, as shown in Section 1. The degree of a vertex is just the number of derangements of $\langle 1 \ldots n \rangle$, which is (see, e.g., [Ha], pp. 9-10)

$$n!(1 - 1/1! + 1/2! - 1/3! + \ldots + (-1)^n/n!).$$

For $n \geq 4$ this is at least

$$n!(1 - 1/1! + 1/2! - 1/3! + 1/4! - 1/5!) = 11n!/30 > (n! + 1)/3.$$

So by Theorem 2, it remains to show that $G(n,n)$ is 3-connected. By Theorem 3, the connectivity of $G(n,n)$ is at least $(2/3)(11n!/3)$ which, for $n \geq 4$, is at least 3. □

We now show that $G(n,k)$ is Hamilton-connected for all n and k satisfying $n \geq k \geq 4$ using a modification of the construction of Tchuente. Note that Theorem 2 does not apply for $k < n-1$: the k-consecutive derangements graph is a subgraph of the k-derangements graph in which each vertex degree is ([Ha], p.9)

$$(n!/(n-k)!)(1 - 1/1! + 1/2! - 1/3! + \ldots + (-1)^k/k!)$$

which for $n - 2 \geq k \geq 2$ is at most

$$(n!/2)(1 - 1/1! + 1/2!) = n!/4 < n!/3.$$

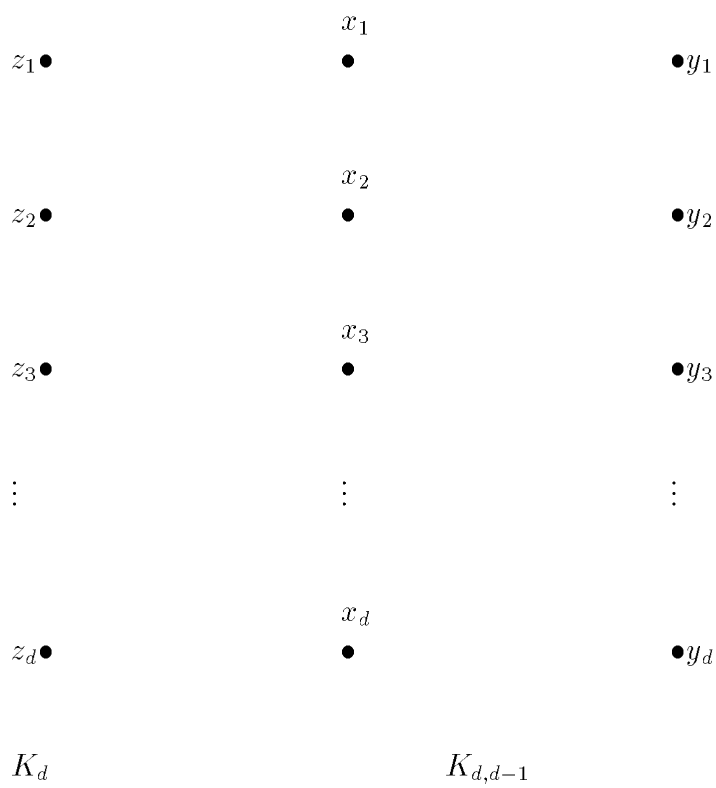

Figure 1: The graph $H(d)$ of Theorem 2.

Theorem 4 *For $n \geq k \geq 4$, the k-consecutive derangements graph $G(n,k)$ is Hamilton-connected.*

Proof. We use induction on the difference $n-k$. If $n-k=0$ then $G(n,k) = G(n,n)$ which is Hamilton-connected by Corollary 1. Otherwise, let $n > k \geq 4$ and assume inductively that $G(n-1,k)$ is Hamilton-connected. Since Cayley graphs are vertex-transitive, it suffices to show that $\langle 1 \ldots n \rangle$ is joined by a Hamilton path in $G(n,k)$ to every $\pi \in S_n$.

For $1 \leq r \leq n$, let $G_r(n,k)$ denote the subgraph of $G(n,k)$ induced by those vertices $\sigma \in S_n$ with $\sigma(n) = r$. Then $G_r(n,k) \cong G(n-1,k)$. Given distinct $\sigma, \sigma' \in S_n$ and a list $T = t_1, \ldots, t_m$, $2 \leq m \leq n$, of distinct elements from $\{1 \ldots n\}$ satisfying $\sigma(n) = t_1$ and $\sigma'(n) = t_m$, define a *k-admissible sequence* for (σ, σ', T) to be a sequence of $2m$ distinct permutations $A_1 = \sigma, A'_1, A_2, A'_2, \ldots, A_m, A'_m = \sigma'$ satisfying, for $1 \leq i \leq m-1$,

(i) $A_1 = \sigma$, $A'_m = \sigma'$,

(ii) $A_i, A'_i \in G_{t_i}(n,k)$ for $1 \leq i \leq m$, and

(iii)A'_i and A_{i+1} are adjacent in $G(n,k)$ for $1 \leq i \leq m-1$.

Clearly a k-admissible sequence exists for every such (σ, σ', T). For $\pi \in S_n$, to construct a Hamilton path from $\langle 1 \ldots n \rangle$ to π, we consider two cases.

Case 1. If $\pi(n) \neq n$, let $T = t_1, t_2, \ldots, t_n$ be a listing of the elements of $\{1, \ldots, n\}$ with $t_1 = n$ and $t_n = \pi(n)$. Let $A_1, A'_1, \ldots, A_n, A'_n$ be a k-admissible sequence for $(\langle 1 \ldots n \rangle, \pi, T)$. By induction, for $1 \leq i \leq n$, A_i and A'_i are joined by a Hamilton path p_i in $G_{t_i}(n,k)$. Concatenation of these paths $p_1 p_2 \ldots p_n$ gives a Hamilton path from $\langle 1 \ldots n \rangle$ to π in $G(n,k)$.

Case 2. If $\pi(n) = n$, by induction there is a Hamilton path q from $\langle 1 \ldots n \rangle$ to π in $G_n(n,k)$. Let α, β be consecutive vertices on q satisfying $\alpha(n-1) \neq \beta(n-1)$. Rotate the last k positions of α and β to get $\alpha' = \alpha \cdot (n \;\; n-1 \;\ldots\; n-k+1)$ and $\beta' = \beta \cdot (n \;\; n-1 \;\ldots\; n-k+1)$. Let $T = t_1, \ldots, t_{n-1}$ be a sequence of $n-1$ distinct elements from $\{1, \ldots, n\}$ with $t_1 = \alpha'(n) = \alpha(n-1)$ and $t_{n-1} = \beta'(n) = \beta(n-1)$ and let $A_1, A'_1, \ldots, A_{n-1}, A'_{n-1}$ be a k-admissible sequence for (α', β', T). By induction, there is a Hamilton path p_i in $G_{t_i}(n,k)$ for $1 \leq i \leq n-1$ joining A_i and A'_i. Replacing the edge joining α and β in q by the path $\alpha p_1 \dots p_{n-1}\beta$ gives the required Hamilton path in $G(n, k)$.

3 Concluding Remarks

A different generalization of the derangements graph was considered in [RaSl]. Given a graph G with n vertices, consider only the *G-derangements*, that is, those derangements ρ satisfying: $\rho(i)$ is adjacent to i in G for $1 \le i \le n$. Let $D[G]$ be the Cayley graph of S_n with respect to these G-derangements. Then $D[K_n]$ is the derangements graph which is known to be hamiltonian. But in fact, not all of the edges of K_n were needed. Using a variation on Yarbrough's construction, Rall and Slater have found a graph G with only $2n + 1$ edges for which $D[G]$ is hamiltonian. They ask whether any graph G with fewer edges can have this property. Recently, Jacobson and West have noted that when n is odd, the Cayley graph of S_n, with respect to the basis consisting of the two derangements $(1\ \dots\ n)^2$ and $(1\ \dots\ n)(n-1\ n)$, is hamiltonian. There is a graph G with only $n + 2$ edges which will allow these two permutations as G-derangements. One can pose similar questions for the property of Hamilton-connectivity.

What other families of generating sets give rise to hamiltonian Cayley graphs of S_n (or A_n)? Examples appear in [GoRo], [Co], and [Pu2]. Perhaps the most general result concerns the Cayley graphs of S_n with respect to *any* basis of transpositions. These were shown to be hamiltonian in [KoLi] and in [Sl] (by a much simpler argument) and Hamilton-connected in [Tc]. Suppose one considers generating sets of permutations π which are involutions, that is, $\pi^2 = 1$? Not every involution is a transposition.

Finally, is every connected (undirected) *non-bipartite* Cayley graph, with minimum vertex degree at least 3, Hamilton-connected? According to Brian Alspach, no counterexamples are known.

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References

[Co] C. Compton, "Hamilton cycles in the Cayley graph of S_n and a doubly adjacent Gray code," Ph.D. Thesis, Department of Mathematics, University of California, San Diego (1990).

[EgWa] R. B. Eggleton and W. D. Wallis, "Problem 86: Solution I," *Mathematics Magazine* 58 (1985) 112-113.

[GoRo] R. J. Gould and R. Roth, "Cayley digraphs and $(1, j, n)$ sequencings of the alternating group A_n," *Discrete Mathematics* 66 (1987) 91-102.

[Ha] M. Hall, Jr., *Combinatorial Theory*, second edition, John Wiley, New York, 1986.

[Ja] B. Jackson, "Hamilton cycles in regular 2-connected graphs," *Journal of Combinatorial Theory, Series B* 29 (1980) 27-46.

[Jo] S. M. Johnson, "Generation of permutations by adjacent transpositions," *Math. Comp.* 17 (1963) 282-285.

[Ju] H. A. Jung, "Longest ab-paths in regular graphs," *Annals of Discrete Mathematics* 41 (1989) 281-298.

[KoLi] V. L. Kompel'makher and V. A. Liskovets, "Sequential generation of arrangements by means of a basis of transpositions," *Kibernetica* 3 (1975) 17-21.

[Lo] L. Lovász, Problem 11 in *Combinatorial Structures and their Applications*, Gorden and Breach, London 1970.

[Me] J. Metzger, "Problem 1186," *Mathematics Magazine* 58 (1985) 113-114.

[Pu1] T. C. Putnam, "A Gray code variant: sequencing permutations via fixed points," manuscript (1989).

[Pu2] T. C. Putnam, "Combinatorial Gray code and a generalization of the Johnson-Trotter algorithm to contiguous k-cycles," manuscript, presented at the *Fifth SIAM Conference on Discrete Mathematics*, Atlanta, June 1990.

[Rab] S. Rabinowitz, "Problem 1186," *Mathematics Magazine* 57 (1984) 109.

[RaSl] D. F. Rall and P. J. Slater, "Generating all permutations by graphical derangements" (1987), unpublished manuscript.

[Sa] C. Savage, "Generating permutations with k-differences," *SIAM Journal on Discrete Mathematics* 3 (1990) 561-573.

[Se] R. Sedgewick, "Permutation generation methods," *Computing Surveys* 9 (1977) 137-164.

[Sl] P. J. Slater, "Generating all permutations by graphical transpositions," *Ars Combinatoria* 5 (1978) 219-225.

[Tc] M. Tchuente, "Generation of permutations by graphical exchanges," *Ars Combinatoria* 14 (1982) 115-122.

[Tr] H. F. Trotter, "PERM (Algorithm 115)", *Communications of the ACM* 5 (1962) 434-435.

[Wi1] H. S. Wilf, "Generalized Gray codes", invited talk, *SIAM Conference on Discrete Mathematics*, San Francisco, June 1988.

[Wi2] H. S. Wilf, *Combinatorial Algorithms: An Update*, SIAM, Philadelphia (1989).

[Wa] H. E. Watkins, "Conectivity of transitive graphs," *Journal of Combinatorial Theory, Series B* 8 (1970) 23-29.

[WiGa] D. Witte and J. A. Gallian, "A survey - hamiltonian cycles in Cayley graphs," *Discrete Mathematics* 51 (1984) 293-304.