Long Cycles in the Middle Two Levels of the Boolean Lattice

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Abstract

An intriguing open question is whether the graph formed by the middle two levels of the Boolean lattice of subsets of a $2k+1$-element set has a Hamilton path for all $k \geq 1$. We consider finding a lower bound on the length of the longest cycle in this graph. A result of Babai for vertex transitive graphs gives a lower bound of $(3N)^{1/2}$, where N is the total number of vertices in the middle two levels. In this paper we show how to construct a cycle of length N^c where $c \approx .836$.

1 Introduction

One of the most interesting classes of open problems in the area of Gray codes for sets, as well as Hamilton cycles in vertex transitive graphs, involves the problems about paths among certain levels of B_n, the Boolean lattice of subsets of a n-element set. Perhaps the best known is the *middle two levels* problem which is attributed in [KT] to Dejter, Erdős, and Trotter and by others to Hável and Kelley. This problem has been attacked by several researchers with no success. The question is whether it is possible to list all of the k-element and $k+1$-element subsets of the set $\{1,\ldots,2k+1\}$ in such a way that (i) each subset occurs exactly once, (ii) the k- and $k+1$-element subsets occur alternately, and (iii) consecutive sets on the list differ by exactly one element. Restated, the question is whether there is a Hamilton cycle or path in the middle two levels of B_{2k+1}. At first glance, it would appear that one could take a Gray code for k-subsets, that is, a cyclic listing of all the k-subsets,

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STOP

k-subsets	$k+1$-subsets formed by taking union of successive pairs
{1,2}	
	{1,2,3}
{2,3}	
	{1,2,3}
{1,3}	
	{1,3,4}
{3,4}	
	{2,3,4}
{2,4}	
	{1,2,4}
{1,4}	
	{1,4,5}
{4,5}	
	{3,4,5}
{3,5 }	
	{2,3,5}
{2,5}	
	{1,2,5}
{1,5}	
	{1,2,5}

Figure 1: Revolving door algorithm listing of the 2-subsets of {1,2,3,4,5} does not give Hamilton cycle in the middle two levels of B_5.

in which successive elements differ in one element (see, e.g., [NW], [BER], [EM].) Then, by taking unions of successive pairs of elements, create a list of $k+1$-sets. Alternating between the list of k-subsets and the corresponding list of $k+1$-subsets would satisfy properties (ii) and (iii), but, unfortunately, not property (i), at least not for any known listing of k-sets (see Figure 1.)

The largest value of k for which a Hamilton cycle is known to exist appears to be about $k = 11$, and this is attributed to David Moews and Mike Reid. It is intriguing, on running programs for small k, to find thousands of Hamilton cycles and the number of cycles appears to increase rapidly with k. However, the number of subsets grows so rapidly that it is possible to run such programs only for relatively small k.

The graph formed by the middle two levels is a connected, undirected, vertex transitive graph. Thus, either it has a Hamilton path, or it provides a counterexample to the Lovász conjecture that every connected, undirected, vertex transitive graph has Hamilton path [Lo]. One approach to this problem which has been considered is to try to form a Hamilton cycle as the union of two edge-disjoint matchings. In [DSW], it was shown that a Hamilton cycle in the middle two levels cannot be the union of two lexicographic matchings. However, other matchings may work. Motivated by this approach to the problem, Kierstead and Trotter [KT] define a large class of matchings, called *lexical* matchings in the middle two levels.

Is there at least a good lower bound on the length of the longest cycle in the bipartite graph formed by the middle two levels of the Boolean lattice? No lower bound of the form $c * N$ is known. Since this graph is vertex transitive, Babai's result [Ba] shows that there is a cycle of length at least $(3N)^{1/2}$, where N is the total number of vertices in the middle two levels. As we will show in the next section, a result of Dejter and Quintana can be used to construct a cycle of length at least N^t where $t = (\log 3)/(\log 4) \approx .793$ [DQ]. The main result of this paper is a technique for constructing a cycle of length $\Omega(N^t)$ where $t \approx .836$. Using variations of this technique, it would likely be possible to construct cycles of length $c * N^t$ for larger, constant, values of t, at the expense of increasing the intricacy of the construction.

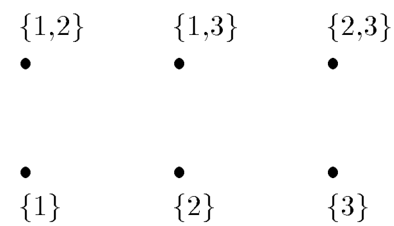
The long cycle construction is presented in Section 2. Possible extensions and variations are discussed in Section 3.

2 The Cycle Construction

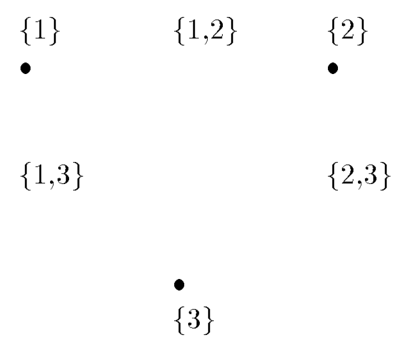
Let M_k denote the graph formed by the middle two levels of the Boolean lattice of subsets of a $2k+1$-element set, that is, the vertices of M_k are all of the k- and $k+1$-subsets of $\{1, \ldots, 2k+1\}$, and two sets are joined by an edge if one set can be obtained from the other by the addition of a single element (see Figure 2(a).) Then M_k has $2 * \binom{2k+1}{k}$ vertices.

Lemma 1 *The number of vertices in the middle two levels graph, M_k, satisfies*

$$|M_k| \leq \frac{3}{2}(4)^k.$$



(a) The middle two levels graph M_k of B_{2k+1} when $k = 1$.



(b) The graph $G_k(1, \ldots, 2k+1)$ when $k = 1$.

Figure 2: The graphs M_k and $G_k(1, \ldots, 2k+1)$ when $k = 1$.

Proof. If $k = 1$, this is clear, otherwise, assume the lemma is true for some $k \geq 1$. Then

$$|M_{k+1}| = 2 * \left(\begin{array}{c} (2(k+1)+1 \\ k+1 \end{array} \right) = |M_k| * \frac{(2k+3)(2k+2)}{(k+2)(k+1)} \leq \frac{3}{2}(4)^{k+1}.$$

□

In [DQ], a technique is presented for constructing, from a cycle, C_k, in the middle two levels graph, M_k, a cycle of length $3|C_k|$ in M_{k+1}. Since M_1 has a cycle of length 6 (namely, $\{1\}$, $\{1, 2\}$, $\{2\}$, $\{2, 3\}$, $\{3\}$, $\{1, 3\}$), this construction gives, for each $k \geq 1$, a cycle C_k of length $2 * 3^k$ in M_k. We show below that $|C_k|$ satisfies $|C_k| \geq |M_k|^{.79}$.

Theorem 1 *The middle two levels graph, M_k, has a cycle of length at least $|M_k|^{.79}$.*

Proof. Let C_k be the cycle of length $2 * 3^k$ in M_k constructed in [DQ]. Then by Lemma 1,

$$\frac{\ln |C_k|}{\ln |M_k|} \geq \frac{\ln 2 + k \ln 3}{\ln (3/2) + k \ln 4} \geq \frac{\ln 3}{\ln 4}$$

Thus, $|C_k| \geq |M_k|^{(\ln 3)/(\ln 4)} \geq |M_k|^{.79}$. □

Our strategy for constructing longer cycles in M_k will be, first of all, to recursively construct long cycles in three disjoint subgraphs of M_k, each isomorphic to M_{k-1}. These cycles are then linked together. Finally, the resulting cycle is enlarged to include additional vertices, in such a way as to asymptotically increase the rate at which the cycle length grows.

In Theorem 2, we show how to construct in M_k a cycle of length 3^k with certain additional properties which will be used in Theorem 3 to enlarge the cycle. It will be convenient to work with a variation of the graph M_k in which the vertices are the k-subsets and the edge labels are the $k+1$-subsets.

For distinct integers $x_1, \ldots, x_j$, let $G_i(x_1, \ldots, x_j)$ denote the edge-labeled graph whose vertex set is the set of all i-element subsets of $\{x_1, \ldots, x_j\}$ and in which vertices S and T are joined by an edge, labeled $S \cup T$ if and only if $|S \cap T| = i - 1$ (see Figure 2(b).) Note, then, that the edge labels are all of the $i+1$-element subsets of $\{x_1, \ldots, x_j\}$, with some edge labels occurring more than once.

In this context, the middle two levels problem is to find a Hamilton cycle in $G_k(x_1, \ldots, x_{2k+1})$ in which no edge label occurs more than once. (See Figure 3.) Furthermore, there is an obvious bijection between cycles of length t in $G_k(x_1, \ldots, x_{2k+1})$ and cycles of length $2t$ in the middle two levels graph, M_k, as illustrated in Figure 3.

If H is a graph whose vertices and edges are labeled by subsets of S, then for $y \notin S$, let $y \circ H$ be the graph obtained from H by adding element y to the label of each vertex and each edge of H. See Figure 4 for an example.

Theorem 2 *For $t \geq 1$, $G_t(x_1, \ldots, x_{2t+1})$ has a cycle of length 3^t in which all edge labels are distinct. Furthermore, this cycle contains each of the following edges:*

UV edge: $(\{x_1, x_3, \ldots, x_{2t-1}\}, \{x_3, x_5, \ldots, x_{2t+1}\})$ (when $t \geq 1$),

WX edge: $(\{x_1, x_3, \ldots, x_{2t-3}, x_{2t}\}, \{x_3, x_5, \ldots, x_{2t-1}, x_{2t}\})$ (when $t \geq 2$),

YZ edge: $(\{x_1, x_3, \ldots, x_{2t-5}, x_{2t-2}, x_{2t+1}\}, \{x_3, x_5, \ldots, x_{2t-3}, x_{2t-2}, x_{2t+1}\})$, (when $t \geq 3$).

Proof. When $t = 1$, $G_1(x_1, x_2, x_3)$ is a triangle with all edge labels distinct. The edge joining $\{x_1\}$ and $\{x_3\}$ is the UV edge.

For $t = 2$, the cycle $\{x_5, x_1\}, \{x_4, x_5\}, \{x_4, x_2\}, \{x_4, x_1\}, \{x_3, x_4\}, \{x_3, x_2\}, \{x_3, x_1\}, \{x_5, x_3\}, \{x_5, x_2\}$ in $G_2(x_1, \ldots, x_5)$ has length 9, distinct edge labels, and contains the UV-edge $(\{x_1, x_3\}, \{x_3, x_5\})$ and the WX-edge $(\{x_1, x_4\}, \{x_3, x_4\})$.

Assume the theorem is true for $t = k - 1 \geq 2$. Note that the following three graphs are vertex disjoint, edge-label-disjoint subgraphs of $G_k(x_1, \ldots, x_{2k+1})$, each isomorphic to $G_{k-1}(x_1, \ldots, x_{2k-1})$:

(1) $x_{2k+1} \circ G_{k-1}(x_1, \ldots, x_{2k-1})$

(2) $x_{2k} \circ G_{k-1}(x_1, \ldots, x_{2k-2}, x_{2k+1})$

(3) $x_{2k-1} \circ G_{k-1}(x_1, \ldots, x_{2k-2}, x_{2k})$.

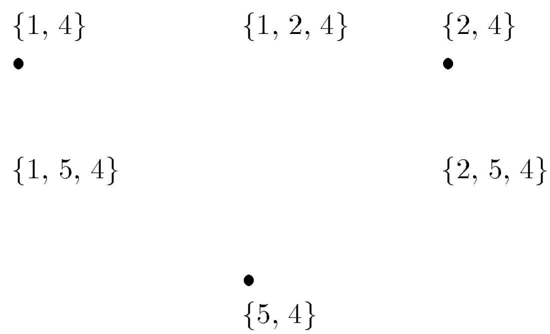
So, by induction, graphs (1), (2), and (3), contain cycles $C^{(1)}, C^{(2)}, C^{(3)}$, respectively, of length 3^{k-1}, satisfying the conditions of the theorem. For $i = 1, 2, 3$, let (U_i, V_i) be the UV-edge in $C^{(i)}$ and let (W_1, X_1) be the WX-edge in $C^{(1)}$. Specifically,
the UV-edge in $C^{(1)}$ joins

$$U_1 = \{x_{2k+1}\} \cup \{x_1, x_3, x_5, \ldots, x_{2k-3}\} \text{ to } V_1 = \{x_{2k+1}\} \cup \{x_3, x_5, \ldots, x_{2k-3}, x_{2k-1}\},$$

the UV-edge in $C^{(2)}$ joins

{1,4}		{1,4}	
	{1,4,5}		{1,4,5}
{4,5}		{4,5}	
	{2,4,5}		{2,4,5}
{2,5}		{2,5}	
	{1,2,5}		{1,2,5}
{1,5}		{1,5}	
	{1,3,5}		{1,3,5}
{1,3}		{1,3}	
	{1,2,3}		{1,2,3}
{1,2}		{1,2}	
	{1,2,4}		{1,2,4}
{2,4}		{2,4}	
	{2,3,4}		{2,3,4}
{2,3}		{2,3}	
	{2,3,5}		{2,3,5}
{3,5}		{3,5}	
	{3,4,5}		{3,4,5}
{3,4}		{3,4}	
	{1,3,4}		{1,3,4}
Cycle of length 10 in $G_2(1,2,3,4,5)$.		Corresponding cycle of length 20 in M_2	

Figure 3: Bijection between cycles of length t in $G_k(1,\ldots,2k+1)$ and cycles of length $2t$ in M_k.



The graph $4 \circ G_1(1,2,5)$

Figure 4: Example of notation $y \circ G_k(x_1, \ldots, x_{2k+1})$.

$$U_2 = \{x_{2k}\} \cup \{x_1, x_3, x_5, \ldots, x_{2k-3}\} \text{ to } V_2 = \{x_{2k}\} \cup \{x_3, x_5, \ldots, x_{2k-3}, x_{2k+1}\},$$

the UV-edge in $C^{(3)}$ joins

$$U_3 = \{x_{2k-1}\} \cup \{x_1, x_3, x_5, \ldots, x_{2k-3}\} \text{ to } V_3 = \{x_{2k-1}\} \cup \{x_3, x_5, \ldots, x_{2k-3}, x_{2k}\},$$

and the WX-edge in $C^{(1)}$ joins

$$W_1 = \{x_{2k+1}\} \cup \{x_1, x_3, x_5, \ldots, x_{2k-5}, x_{2k-2}\} \text{ to } X_1 = \{x_{2k+1}\} \cup \{x_3, x_5, \ldots, x_{2k-3}, x_{2k-2}\}.$$

Note that in $G_k(x_1, \ldots, x_{2k+1})$,

U_1 and V_2 are joined by an edge labeled $U_2 \cup V_2$,

U_2 and V_3 are joined by an edge labeled $U_3 \cup V_3$, and

U_3 and V_1 are joined by an edge labeled $U_1 \cup V_1$.

Thus, we can join $C^{(1)}, C^{(2)}$, and $C^{(3)}$ into a single cycle C by deleting edges $(U_1, V_1), (U_2, V_2)$, and (U_3, V_3) while adding edges $(U_1, V_2), (U_2, V_3)$, and (U_3, V_1). The resulting cycle has length $3 * 3^{k-1} = 3^k$.

Finally, the edge (U_3, V_1) of C becomes the UV-edge required by the theorem. The edge (U_2, V_3) is the required WX-edge, and the edge (W_1, X_1), originally on $C^{(1)}$, but now on C, is the required YZ-edge. □

Define a function $L(t)$ by

$$L(t) = \begin{cases} 3^t & \text{for } 1 \le t \le 4 \\ 3L(t-1) + L(t-3) + 3L(t-4) & \text{for } t \ge 5. \end{cases}$$

We show now how to construct a cycle of length $L(t)$ in $G_t(x_1, \ldots, x_{2t+1})$ by joining together three cycles of length $L(t-1)$, one cycle of length $L(t-3)$, and three cycles of length $L(t-4)$.

Theorem 3 *For $t \ge 1$, and $L(t)$ as defined above, $G_t(x_1, \ldots, x_{2t+1})$ has a cycle C of length $L(t)$, with all edge labels distinct. Furthermore, C contains each of the following edges:*

UV edge: $(\{x_1, x_3, \ldots, x_{2t-1}\}, \{x_3, x_5, \ldots, x_{2t+1}\})$ (when $t \ge 1$),

WX edge: $(\{x_1, x_3, \ldots, x_{2t-3}, x_{2t}\}, \{x_3, x_5, \ldots, x_{2t-1}, x_{2t}\})$ (when $t \ge 1$), and

YZ edge: $(\{x_1, x_3, \ldots, x_{2t-5}, x_{2t-2}, x_{2t+1}\}, \{x_3, x_5, \ldots, x_{2t-3}, x_{2t-2}, x_{2t+1}\})$ (when $t \ge 1$).

Proof.

For $1 \le t \le 4$, the theorem follows from Theorem 1. Assume the theorem is true for $t = k-1 \ge 4$. Note that the following seven graphs are vertex-disjoint and edge-label disjoint subgraphs of $G_k(x_1, \ldots, x_{2k+1})$.

(1) $x_{2k+1} \circ G_{k-1}(x_1, \ldots, x_{2k-1})$.

(2) $x_{2k} \circ G_{k-1}(x_1, \ldots, x_{2k-2}, x_{2k+1})$.

(3) $x_{2k-1} \circ G_{k-1}(x_1, \ldots, x_{2k-2}, x_{2k})$.

(4) $x_{2k+1} \circ x_{2k} \circ x_{2k-1} \circ x_{2k-2} \circ G_{k-4}(x_1, \ldots, x_{2k-7})$.

(5) $x_{2k+1} \circ x_{2k} \circ x_{2k-1} \circ x_{2k-3} \circ G_{k-4}(x_1, \ldots, x_{2k-7})$.

(6) $x_{2k+1} \circ x_{2k} \circ x_{2k-1} \circ x_{2k-4} \circ G_{k-4}(x_1, \ldots, x_{2k-7})$.

(7) $x_{2k+1} \circ x_{2k} \circ x_{2k-1} \circ G_{k-3}(x_1, \ldots, x_{2k-5})$.

Graphs (1), (2), and (3) are each isomorphic to $G_{k-1}(x_1, \ldots, x_{2k-1})$; similarly, graphs (4), (5), and (6) are each isomorphic to $G_{k-4}(x_1, \ldots, x_{2k-4})$ and graph (7) is isomorphic to $G_{k-3}(x_1, \ldots, x_{2k-5})$.

Thus, by induction, each of these subgraphs has a cycle, in which no edge label occurs more than once, of length $L(t-1)$ in graphs (1), (2), and (3), of length $L(t-4)$ in graphs (4), (5), and (6), and of length $L(t-3)$ in graph (7). In addition, certain edges will appear on these cycles, as guaranteed by the induction hypothesis.

Our strategy will be to link these seven cycles into a single cycle in which no edge label is repeated. This cycle will have length $3L(t-1)+L(t-3)+3L(t-4)$, as claimed in the theorem.

It remains to show how to construct one large cycle from the seven small ones and to establish that the new cycle contains the required edges.

For graphs (1) through (7), let $C^{(i)}$ denote the cycle in graph G_i guaranteed by the induction hypothesis. For $1 \le i \le 7$, let (U_i, V_i) be the UV-edge in $C^{(i)}$, let (W_1, X_1) be the WX-edge in $C^{(1)}$, let (W_7, X_7) be the WX-edge in $C^{(7)}$, and let (Y_2, Z_2) be the YZ-edge in $C^{(2)}$.

Step 1. We first link together the cycles $C^{(1)}$, $C^{(2)}$, and $C^{(3)}$, exactly as in the proof of Theorem 1, by deleting edges (U_i, V_i) for $i = 1, 2, 3$ and adding edges $(U_1, V_2), (U_2, V_3)$, and (U_3, V_1).

Step 2. Next, link together the four cycles $C^{(4)}$ through $C^{(7)}$ as follows: delete edges (U_i, V_i) for $i = 4, 5, 6$ and edge (W_7, X_7). Add edges (U_4, U_5), (V_5, V_6), (U_6, W_7), and (V_4, X_7). (See Figure 5 for an example when $k = 5$.)

Step 3. The cycles formed in Steps 1 and 2 are joined with cycle $C^{(7)}$ as follows. Delete edges (U_7, V_7) and (Y_2, Z_2). Add edges (U_7, Y_2) and (V_7, Z_2).

The resulting cycle C has length $L(t)$ and we now show that all edge labels are distinct. Figure 6 shows the new pairs of vertices which have been joined, together with their corresponding edge label. Note first that all of the new edge labels are distinct from each other. The label of the new edge (U_7, Y_2) is the same as the label of the edge (U_6, V_6) which was deleted from $C^{(6)}$ in Step 2. It suffices now to establish that none of the other new edge labels could have occurred in any of the original seven graphs.

All new edge labels contain $\{x_{2k+1}, x_{2k}, x_{2k-1}\}$ as a subset and therefore cannot occur in graphs (1) - (3). In addition, each new edge label contains an element of $\{x_{2k-4}, x_{2k-3}, x_{2k-2}\}$ and therefore cannot occur in graph (7). New edge labels containing x_{2k-2} also contain x_{2k-3} or x_{2k-6} and therefore cannot be in graph (4). The new edge labels containing x_{2k-3} also contain x_{2k-2} or x_{2k-4}, so cannot be in graph (5). The new edge labels containing x_{2k-4} (excluding (U_7, Y_2)) also contain x_{2k-3}, x_{2k-5}, or x_{2k-6}, and therefore cannot occur in graph (6).

Finally, the cycle C contains the three special edges required by the theorem: the UV-edge for C is (U_3, V_1), the WX-edge for C is (U_2, V_3), and the YZ-edge for C is (W_1, X_1).

Graph 4:
$\underline{\{11,10,9,8\} \circ G_1(1,2,3)}$

- $U_4 = \{11,10,9,8,1\}$
- $V_4 = \{11,10,9,8,3\}$

Graph 5:
$\underline{\{11,10,9,7\} \circ G_1(1,2,3)}$

- $U_5 = \{11,10,9,7,1\}$
- $V_5 = \{11,10,9,7,3\}$

Graph 6:
$\underline{\{11,10,9,6\} \circ G_1(1,2,3)}$

- $U_6 = \{11,10,9,6,1\}$
- $V_6 = \{11,10,9,6,3\}$

Graph 7:
$\underline{\{11,10,9\} \circ G_2(1,2,3,4,5)}$

- $U_7 = \{11,10,9,1,3\}$
- $V_7 = \{11,10,9,3,5\}$

- $W_7 = \{11,10,9,1,4\}$
- $X_7 = \{11,10,9,3,4\}$

Graph 1:
$\underline{\{11\} \circ G_4(1,2,3,\ldots 8,9)}$

- $U_1 = \{11,1,3,5,7\}$
- $V_1 = \{11,3,5,7,9\}$

- $W_1 = \{11,1,3,5,7\}$
- $X_1 = \{11,3,5,7,8\}$

Graph 2:
$\underline{\{10\} \circ G_4(1,2,3,\ldots 8,11)}$

- $U_2 = \{10,1,3,5,7\}$
- $V_2 = \{10,3,5,7,11\}$

- $Y_2 = \{10,1,3,6,11\}$
- $Z_2 = \{10,3,5,6,11\}$

Graph 3:
$\underline{\{9\} \circ G_4(1,2,3,\ldots 8,10)}$

- $U_3 = \{9,1,3,5,7\}$
- $V_3 = \{9,3,5,7,10\}$

Figure 5: Linking together the seven cycles when $k = 5$ in the proof of Theorem 2.

STEP	JOIN VERTEX:	TO VERTEX:	GIVING NEW EDGE LABEL:
Step 2	$U_4 =$ $\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-2}\}$ $\cup \{x_1, x_3, \ldots x_{2k-9}\}$	$U_5 =$ $\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-3}\}$ $\cup \{x_1, x_3, \ldots x_{2k-9}\}$	$\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-2}, x_{2k-3}\}$ $\cup \{x_1, x_3, \ldots x_{2k-9}\}$
Step 2	$V_5 =$ $\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-3}\}$ $\cup \{x_3, x_5, \ldots x_{2k-7}\}$	$V_6 =$ $\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-4}\}$ $\cup \{x_3, x_5, \ldots x_{2k-7}\}$	$\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-3}, x_{2k-4}\}$ $\cup \{x_3, x_5, \ldots x_{2k-7}\}$
Step 2	$U_6 =$ $\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-4}\}$ $\cup \{x_1, x_3, \ldots x_{2k-9}\}$	$W_7 =$ $\{x_{2k+1}, x_{2k}, x_{2k-1}\}$ $\cup \{x_1, x_3, \ldots x_{2k-9}, x_{2k-6}\}$	$\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-4}, x_{2k-6}\}$ $\cup \{x_1, x_3, \ldots x_{2k-9}\}$
Step 2	$V_4 =$ $\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-2}\}$ $\cup \{x_3, x_5, \ldots x_{2k-7}\}$	$X_7 =$ $\{x_{2k+1}, x_{2k}, x_{2k-1}\}$ $\cup \{x_3, x_5, \ldots x_{2k-7}, x_{2k-6}\}$	$\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-2}, x_{2k-6}\}$ $\cup \{x_3, x_5, \ldots x_{2k-7}\}$
Step 3	$U_7 =$ $\{x_{2k+1}, x_{2k}, x_{2k-1}\}$ $\cup \{x_1, x_3, \ldots x_{2k-7}\}$	$Y_2 =$ $\{x_{2k}\}$ $\cup \{x_1, x_3, \ldots x_{2k-7},$ $x_{2k-4}, x_{2k+1}\}$	$\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-4}\}$ $\cup \{x_1, x_3, \ldots x_{2k-7}\}$
Step 3	$V_7 =$ $\{x_{2k+1}, x_{2k}, x_{2k-1}\}$ $\cup \{x_3, x_5, \ldots x_{2k-5}\}$	$Z_2 =$ $\{x_{2k}\}$ $\cup \{x_3, x_5, \ldots x_{2k-5},$ $x_{2k-4}, x_{2k+1}\}$	$\{x_{2k+1}, x_{2k}, x_{2k-1}, x_{2k-4}\}$ $\cup \{x_3, x_5, \ldots x_{2k-5}\}$

Figure 6: New edge labels formed in the construction of Theorem 2.

□

Corollary 1 *M_k has a cycle of length at least $|M_k|^{.836}$.*

Proof. By Theorem 2, $G_k(1,\ldots,2k+1)$ has a cycle of length $L(k)$ satisfying

$$L(t) = \begin{cases} 3^t & \text{for } 1 \le t \le 4 \\ 3L(t-1) + L(t-3) + 3L(t-4) & \text{for } t \ge 5. \end{cases}$$

It can be shown by induction that $L(k) \ge \frac{3}{4}(3.19)^k$. This cycle in $G_k(1,\ldots,2k+1)$ corresponds to a cycle C_k in M_k of length

$$|C_k| = 2 * L(k) \ge \frac{3}{2}(3.19)^k.$$

Using Lemma 1 for $|M_k|$,

$$\frac{\ln|C_k|}{\ln|M_k|} \ge \frac{\ln(3/2) + k\ln(3.19)}{\ln(3/2) + k\ln 4} \ge \frac{\ln(3.19)}{\ln 4}.$$

So,

$$|C_k| \ge |M_k|^{\ln(3.19)/\ln 4} \ge |M_k|^{.836}.$$

□

3 Concluding Remarks

It remains open whether the graph formed by the middle two levels of B_{2k+1} has a Hamilton cycle, or even whether there is a cycle of length at least $c * N$, where N is the number of vertices in the graph. In this paper we have constructed a cycle of length at least $N^{.836}$. How much better than this can one do? P. Winkler asks whether it might be possible to construct a cycle of length $\Omega(N^{1-\epsilon})$, that is, a family of cycles $\{C_k\}_{k\ge1}$ in $\{M_k\}_{k\ge1}$ such that for any $\epsilon > 0$, there is a $k_0 \ge 1$ such that $|C_t| \ge |M_t|^{1-\epsilon}$ for infinitely mant $t \ge k_0$.

A variation on this problem is the *antipodal layers problem:* for which values of k, n is there a Hamilton path among the k-subsets and $(n-k)$ - subsets of $\{1,\ldots,n\}$, where two sets are joined by an edge if and only if one is a subset of the other? Hurlbert shows there is a Hamilton cycle for $n > ck^2 + k$ and for $k \le 5, n > 2k$ [Hu]. Other limited results are reported in [Si]. Another variation is the *central layers problem*: in B_n, for which values of k is there a Hamilton path among layers k through $n-k$? (Edges are defined by the covering relation in the lattice.) The graph in this variation is no longer vertex transitive. For both the antipodal and the central layers problems, it would be interesting to find good lower bounds on the length of the longest cycle.

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