A Pentagonal Number Sieve

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Abstract

We prove a general "pentagonal sieve" theorem that has corollaries such as the following. First, the number of pairs of partitions of n that have no parts in common is

$$p(n)^2 - p(n-1)^2 - p(n-2)^2 + p(n-5)^2 + p(n-7)^2 - \ldots.$$

Second, if two unlabeled rooted forests of the same number of vertices are chosen i.u.a.r., then the probability that they have no common tree is .8705.... Third, if f, g are two monic polynomials of the same degree over the field $GF(q)$, then the probability that f, g are relatively prime is $1 - 1/q$. We give explicit involutions for the pentagonal sieve theorem, generalizing earlier mappings found by Bressoud and Zeilberger.

1 The Main Theorem

The natural context in which our results lie is that of *prefabs*. A prefab ([1, 3, 4]) $\mathcal{P}$ is a combinatorial structure in which each object ω is uniquely representable as a product ('synthesis') of powers of prime objects, and in which there is an *order* function $\omega \to |\omega| \in \mathbf{Z}^+$ which satisfies $|\omega\omega'| = |\omega| + |\omega'|$. We denote the primes of $\mathcal{P}$ by $p_1, p_2, \ldots$. Examples of prefabs are integer partitions, rooted unlabeled forests, plane partitions, etc.

Let $\mathcal{P}$ be a prefab in which the number of objects of order n is $f(n)$, for $n = 0, 1, 2, \ldots$, and the number of "prime" objects of order n is b_n, for $n \geq 1$. The unique factorization of all objects in $\mathcal{P}$ into products of

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powers of prime objects is expressed by the formula

$$\sum_{n\geq 0} f(n)x^n = \prod_{i\geq 1} \frac{1}{(1-x^i)^{b_i}}. \qquad (1)$$

For a fixed positive integer m, we are interested here in the number $f_m(n)$, of m-tuples of objects of order n in $\mathcal{P}$, such that no prime object is a factor of every member of the m-tuple. We will call such a tuple *coprime*. As special cases we mention the number of pairs of partitions of n with no common part, the number of pairs of rooted forests with no common tree, and the number of relatively prime pairs of monic polynomials over a finite field.

To find $f_m(n)$ we note that we can uniquely factor an m-tuple $(\omega_1, \ldots, \omega_m)$ of objects of order n into a product of their "gcd" α and an m-tuple $(\omega'_1, \ldots, \omega'_m)$ of coprime objects of orders $n - |\alpha|$. Thus

$$\sum_{n\geq 0} f(n)^m x^n = \frac{1}{\prod_{i\geq 1}(1-x^i)^{b_i}} \sum_{n\geq 0} f_m(n)x^n,$$

which yields

$$\sum_{n\geq 0} f_m(n)x^n = \left(\sum_{n\geq 0} f(n)^m x^n\right)\left(\prod_{i\geq 1}(1-x^i)^{b_i}\right). \qquad (2)$$

This is the general form of the pentagonal number sieve. The effect of multiplying by the product on the right is to sieve out of the generating function for *all* m-tuples of objects of order n, the gf for just the coprime tuples.

Some consequences of the sieve (2) are as follows.

(A) In the prefab of integer partitions, (2) yields the following.

Proposition 1 *The number of m-tuples of partitions of n that have no part in common is*

$$p(n)^m - p(n-1)^m - p(n-2)^m + p(n-5)^m + p(n-7)^m - p(n-12)^m - p(n-15)^m + \ldots, \qquad (3)$$

in which the decrements are the pentagonal numbers $\{j(3j\pm 1)/2\}_{j\geq 0}$.

(B) Let $\mathcal{P}$ be the prefab of rooted, unlabeled forests. For fixed n, the probability that if we choose two forests of n vertices i.u.a.r. then they will have no tree in common, is, according to (2) with $m = 2$,

$$1 + c_1\left(\frac{f(n-1)}{f(n)}\right)^2 + c_2\left(\frac{f(n-2)}{f(n)}\right)^2 + \ldots,$$

in which $\prod_{i\geq 1}(1-x^i)^{b_i} = \sum_i c_i x^i$ defines the c's. Now it is well known that the number of rooted forests of n vertices is $f(n) \sim KC^n/n^{1.5}$, where $C = 2.95576...$ Hence each $(f(n-k)/f(n))^2$ above approaches C^{-2k}, and in the limit as $n \to \infty$ we obtain the following.

Proposition 2 *The probability that two rooted forests of n vertices have no tree in common approaches*

$$1 + \frac{c_1}{C^2} + \frac{c_2}{C^4} + \ldots = \prod_{i\geq 1}\left(1 - \frac{1}{C^2}\right)^{b_i} = 0.8705...$$

as $n \to \infty$.

(C) Now let $\mathcal{P}$ be the prefab of monic polynomials over a finite field $GF(q)$. There are q^n such polynomials of order (degree) n, so (1) reads as

$$\frac{1}{1-qx} = \prod_{i\geq 1} \frac{1}{(1-x^i)^{b_i}},$$

where b_i is the number of irreducible monic polynomials of degree i. Now from (2) we find that

$$\sum_{n\geq 0} f_m(n)x^n = \left(\sum_{n\geq 0} q^{nm}x^n\right)\left(\prod_{i\geq 1}(1-x^i)^{b_i}\right) = \frac{1-qx}{1-q^m x}.$$

If we compare the coefficients of like powers of x on both sides, we find the following.

Proposition 3 *The number of coprime m-tuples of monic polynomials of degree n over $GF(q)$ is $q^{nm} - q^{(n-1)m+1}$. Alternatively, if m monic polynomials of degree n over $GF(q)$ are chosen i.u.a.r., then the probability that their gcd is 1 is $1 - 1/q^{m-1}$.*

(D) What is the *average* number of different parts that m randomly chosen partitions of the integer n have in common? We use a well known property of the sieve method: the average number of properties that objects have is $\sum N(\supseteq i)/N$, where i runs over all single properties, $N(\supseteq \{i\})$ is the number of objects that have at least the ith property, and N is the total number of objects. In the present case, the average number of common parts is

$$\frac{1}{p(n)}(p(n-1)^m + p(n-2)^m + \ldots + p(1)^m + 1).$$

If we now use the classical asymptotic formula for $p(n)$ it is easy to see that this last expression is $\sim \sqrt{6n}/(m\pi)$. It is well known that the average number of distinct parts in a single random partition of n is $\sim \frac{\sqrt{6n}}{\pi}$. It follows that *the average number of different parts that are common to all members of an m-tuple of partitions of n is $1/m$-th of the average number of distinct parts in a single partition.* For instance, the average number of different common parts in a random pair of partitions of n is one-half of the average number of distinct parts in a single partition of n.

A question. A special case of Proposition 3 is this: Among the ordered pairs of monic polynomials of degree n over $GF(2)$ there are as many relatively prime pairs as non-relatively prime pairs. What is a nice simple bijection that proves this result?

2 Combinatorial proofs

We give combinatorial proofs of (2) and (3) from Section 1.

Rewrite (2) as

$$f_m(n) = \sum_{k\geq 0} f(n-k)^m (q_e(k) - q_o(k)), \tag{4}$$

where $q_e(k)$ (resp. $q_o(k)$) is the number of objects of order k which consist of an even (resp. odd) number of distinct primes. We claim that *any* parity-changing involution which establishes this equation in the $m = 1$ case,

$$\delta_{n,0} = \sum_{k\geq 0} f(n-k)(q_e(k) - q_o(k)), \tag{5}$$

will generalize to an involution for the $m > 1$ case.

To see this, Let $F(n)^m$, $F_m(n)$ be the sets counted by $f(n)^m$, $f_m(n)$, respectively. For an object α and for $\Omega = (\omega_1, \omega_2, \cdots, \omega_m) \in F(n)^m$, let $\alpha\Omega$ denote the m-tuple $(\alpha\omega_1, \alpha\omega_2, \cdots, \alpha\omega_m) \in F(n+|\alpha|)^m$. Then any $\Omega \in F(n)^m$ can be decomposed uniquely as $\alpha\Omega'$ for some object α and some $\Omega' \in F_m(n-|\alpha|)$. Thus

$$f(n)^m = \sum_{l \geq 0} f_m(n-l) \cdot f(l). \tag{6}$$

Then using (6) followed by (5) we find

$$\begin{aligned} \sum_{k\geq 0} f(n-k)^m (q_e(k) - q_o(k)) &= \sum_{k,l\geq 0} f_m(n-k-l) \cdot f(l) \cdot (q_e(k) - q_o(k)) \\ &= \sum_{j\geq 0} f_m(n-j) \sum_{k\geq 0} f(j-k) \cdot (q_e(k) - q_o(k)) \qquad (7) \\ &= \sum_{j\geq 0} f_m(n-j) \cdot \delta_{j,0} = f_m(n). \end{aligned}$$

Let $Q_e(k)$, $Q_o(k)$ be the sets of objects counted by $q_e(k)$, $q_o(k)$, respectively. Now, suppose we have an involution proof of (5). Specifically, let

$$\psi_1 : \bigcup_{k\geq 0} F(n-k) \times F(k) \quad \longrightarrow \quad \bigcup_{k\geq 0} F(n-k) \times F(k)$$

be an involution satisfying (i) $\psi_1(\alpha, \beta) = (\alpha, \beta)$ if and only if $\alpha\beta = \lambda$, the empty object (i.e. $n = 0$) and otherwise (ii) if $\psi_1(\alpha, \beta) = (\gamma, \delta)$ then $\beta \in Q_e(k)$ if and only if $\delta \in Q_o(k)$.

Then it follows from (7) that for any $m > 1$, ψ_1 extends to the following parity-changing involution ψ_m on $\bigcup_{k\geq 0}(F(n-k)^m \times F(k))$, in which the fixed points are $F_m(n) \times \{\lambda\}$. For $\Omega \in F(n-k)^m$ and $\beta \in F(k)$, decompose Ω as $\alpha\Omega'$, where $\Omega' \in F_m(n-k-|\alpha|)$. Then ψ_m is defined by

$$\psi_m((\Omega, \beta)) = \psi_m((\alpha\Omega', \beta)) = (\gamma\Omega', \delta),$$

where $(\gamma, \delta) = \psi_1((\alpha, \beta))$, thus establishing (4).

Some examples follow.

- The involution of [5] for the inclusion-exclusion principle, adapted for (5) gives the following involution, ψ_m, to prove (2). Let $(\Omega, \beta) \in F(n-k)^m \times F(k)$. Decompose Ω as $\alpha\Omega'$, where $\Omega' \in F_m(n-k-|\alpha|)$ and let p be the prime factor of largest index, in some fixed list $p_1, p_2, \ldots$ of all primes in the prefab, occurring in $\alpha\beta$. Then ψ_m is defined by

$$\psi_m((\Omega, \beta)) = \psi_m((\alpha\Omega', \beta)) = \begin{cases} (\Omega, \beta) & \text{if } \alpha\beta = \lambda \\ (\alpha p\Omega', \beta - p)) & \text{if } p \in \beta \\ ((\alpha - p)\Omega', \beta p) & \text{otherwise,} \end{cases}$$

  where $\alpha - p$ denotes the object obtained from α by removing one copy of p, and similarly for $\beta - p$.

- In the prefab of integer partitions, we will write a partition of n as a nonincreasing sequence of positive integers $\pi(1) \geq \pi(2) \geq \cdots \geq \pi(t) > 0$ such that $|\pi| = \pi(1)+\pi(2)+\cdots+\pi(t) = n$. The set of partitions of n is denoted by $\mathrm{P}(n)$, its cardinality by $p(n)$. Euler's identity for $p(n)$, $n \geq 1$,

$$\sum_{j \text{ even}} p(n-a(j)) = \sum_{j \text{ odd}} p(n-a(j)),$$

where the $a(j) = (3j^2+j)/2$ are the pentagonal numbers, and j ranges over all integers, was proved in [2] by exhibiting a bijection between the sets $S_o = \bigcup_{j \text{ odd}} \mathrm{P}(n-a(j))$ and $S_e = \bigcup_{j \text{ even}} \mathrm{P}(n-a(j))$ for $n > 0$. The bijection can be interpreted as a parity-changing involution Φ_1 on $S_e \cup S_o$, where when $n = 0$, $\Phi_1(\lambda) = \lambda$. This gives a proof of (5), where first Euler's pentagonal number theorem is applied in (5) to replace $q_e(k) - q_o(k)$ by $(-1)^j$ if $k = (3j^2 \pm j)/2$ and by 0 otherwise. Thus, Φ_1 extends to a parity-changing involution, Φ_m on

$$\bigcup_{j \text{ even}} P(n-a(j))^m \;\cup\; \bigcup_{j \text{ odd}} P(n-a(j))^m,$$

to prove (3). The involution Φ_m is defined as follows. For $\Pi = \alpha\Pi' \in \mathrm{P}(n-a(j))^m$, where $\alpha = (\alpha(1), \cdots, \alpha(t))$ and $\Pi' \in \mathrm{P}_m(n-|\alpha|)$,

$$\Phi_m(\Pi) = \Phi_m(\alpha\Pi') = \begin{cases} \Pi, & \text{if } j = 0 \text{ and } |\alpha| = 0 \\ (t+3j-1, \alpha(1)-1, \cdots, \alpha(t)-1)\Pi', & \text{if } t+3j \geq \alpha(1), \\ (\alpha(2)+1, \cdots, \alpha(t)+1, 1, \cdots, 1)\Pi', \text{where} & \\ \quad \text{there are } (\pi(1)-3\mathrm{j}-\mathrm{t}-1) \text{ ones at the end,} & \text{otherwise.} \end{cases}$$

As a further check, note that $\Phi_m(\Pi) = \Pi$ if and only if $\Pi \in \mathrm{P}_m(n)$. Otherwise, $\Pi \in P_e(n)$ if and only if $\Phi_m(\Pi) \in P_o(n)$, where $P_e(n) = \bigcup_{j \text{ even}} P(n-a(j))^m$ and $P_o(n) = \bigcup_{j \text{ odd}} P(n-a(j))^m$. It can be checked that Φ_m is its own inverse.

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