Hamilton Cycles which Extend Transposition Matchings in Cayley Graphs of S_n

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Abstract

Let B be a basis of transpositions for S_n and let $Cay(B:S_n)$ be the Cayley graph of S_n with respect to B. It was shown by Kompel'makher and Liskovets that $Cay(B:S_n)$ is hamiltonian. We extend this result as follows. Note that every transposition b in B induces a perfect matching M_b in $Cay(B:S_n)$. We show here when $n > 4$ that for any $b \in B$, there is a Hamilton cycle in $Cay(B:S_n)$ which includes every edge of M_b. That is, for $n > 4$, for any basis B of transpositions of S_n, and for any $b \in B$, it is possible to generate all permutations of $1, 2, \ldots, n$ by transpositions in B so that every other transposition is b.

Keywords. Cayley graph, perfect matching, hamiltonian graph, transposition.

AMS(MOS) subject classifications. 05C25, 05C45.

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1 Introduction

For a finite group G with generating set X, the *Cayley graph of* G *with respect to the generating set* X is the graph $Cay(X:G)$ with vertex set G, in which g and gx are joined by an undirected edge for every $g \in G$ and $x \in X$. We will think of the edge $\{g, gx\}$ as being labeled x. A compelling question in graph theory is whether every undirected Cayley graph is hamiltonian. Although there are results such as [CuWi] and [KeWi] which show that the answer is yes for certain subclasses of Cayley graphs, the general question remains open. If we require only a Hamilton path, the question is still open and is, in fact, a special case of the more general conjecture of Lovász that every connected, undirected, vertex transitive graph has a Hamilton path [Lo].

If we restrict our attention to the case when $G = S_n$, the symmetric group of all permutations of $[n] = \{1, 2, \ldots, n\}$, it is still an open problem whether every Cayley graph of S_n is hamiltonian. The question remains open even when we require that every generator $x \in X$ satisfy $x^2 = id$. What is known is that for every generating set X of *transpositions*, the Cayley graph of S_n is hamiltonian. This was first shown by Kompel'makher and Liskovets [KoLi]. Slater showed in [Sl] that one could always find a Hamilton path in $Cay(X:S_n)$ which ended at a permutation with a j in position k for any $j, k \in [n]$. Tchuente generalized both of these results by showing that any two permutations of different parity are joined by a Hamilton path in $Cay(X:S_n)$ [Tc]. As an example, the well-known algorithm of Steinhaus [St], Johnson [Jo], and Trotter [Tr], for generating permutations by adjacent transpositions, gives a Hamilton cycle through the Cayley graph of S_n with generating set $\{(12), (23), (34), \ldots, (n-1\ n)\}$.

However, an element of S_n of order two need not be a transposition, so it remains open whether the Cayley graph of S_n on a set of generators, each of order two, is hamiltonian. Recently it has been shown that the Cayley graphs of Coxeter groups, generated by order two elements which are *geometric reflections*, are hamiltonian [CoSlWi]. A related result is that for A_n generated by the set of 3-cycles $\{(12n), (13n), \ldots, (1\ n-1\ n)\}$, the Cayley graph is hamiltonian [GoRo].

In this paper, we consider S_n with any generating set of transpositions, X. Note that each $x \in X$ defines a *perfect matching* in $Cay(X:S_n)$, that is, a set M_x of edges of the graph with the property that each vertex of $Cay(X:S_n)$ is the end of exactly one edge in M_x:

$$M_x = \{\{g,\ gx\} \mid g \in S_n\}.$$

Knowing that $Cay(X:S_n)$ is hamiltonian by [KoLi], we can ask if M_x extends to a Hamilton cycle. Such a cycle corresponds to a listing of all permutations of $[n]$, in which successive permutations differ by a transposition in X, so that alternate transpositions correspond to the element x.

The graph $C = Cay(\{(12),(23),(34)\}:S_4)$ is shown in Figure 1. The tripled lines of the figure indicate edges in the matching $M_{(23)}$, and the list of permutations of Figure 2 is a Hamilton cycle in C that contains every edge of $M_{(23)}$.

A specific instance of this problem arose initially in the work of Pruesse and Ruskey on listing the linear extensions of certain posets by transpositions [PrRu]. Let $\mathcal{R}$ be the class of ranked posets in which every non-maximal element has at least two upper covers. Examples of posets in $\mathcal{R}$ include the odd fences, crowns, the Boolean algebra lattices, the lattices of subspaces of a finite-dimensional vector spaces over $GF(q)$, and partition lattices. In [PrRu] it is proven that the linear extensions of any poset in $\mathcal{R}$ can be listed so that every extension differs by a transposition from its predecessor in the list.

Their proof required a cyclic listing of all permutations of $[n]$ by transpositions so that every other transposition was an exchange of the elements in positions 1 and 2. Although they were able to show such a listing was always possible, in some cases the transpositions were not of elements in adjacent positions; these transpositions were the only ones in the proof that were nonadjacent. In [RuSa] we showed that it is possible to list permutations of $[n]$ by *adjacent* transpositions so that every other transposition exchanges the elements in positions 1 and 2. See Figure 3 for an example when $n = 5$. This result is equivalent to showing that in the Cayley

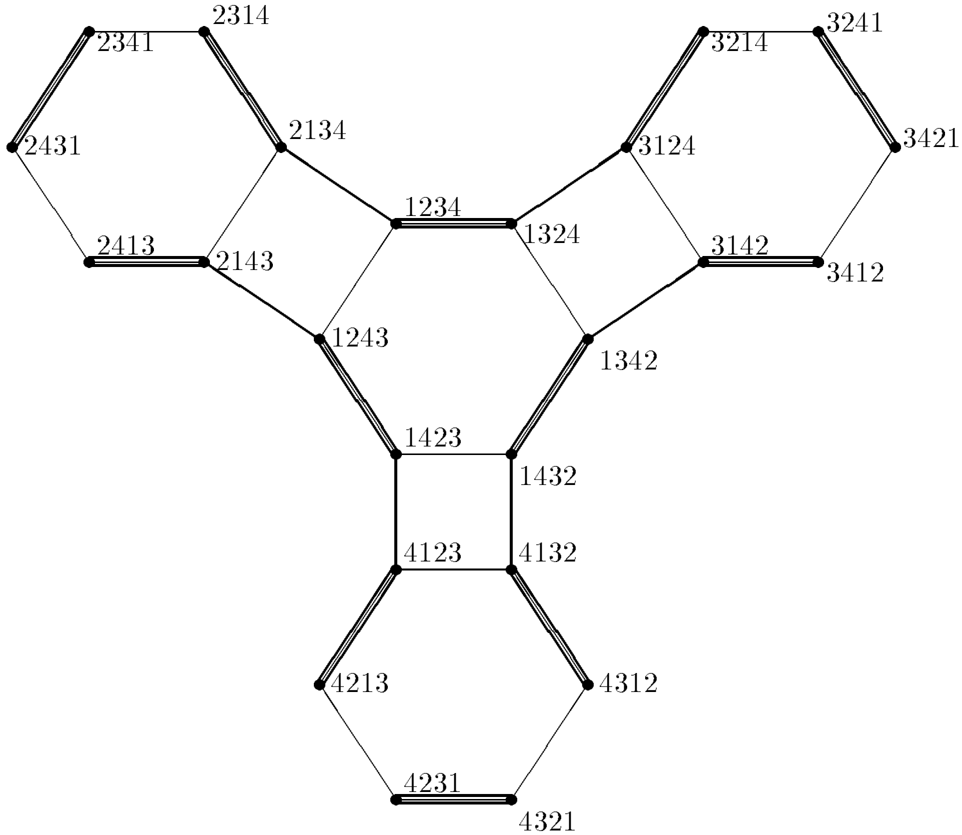


Figure 1: The graph $Cay(\{(12),(23),(34)\} : S_4)$ with $M_{(23)}$.

1234	1324	3124	3214	2314	2134	2143	2413
2431	2341	3241	3421	3412	3142	1342	1432
4132	4312	4321	4231	4213	4123	1423	1243

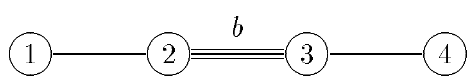


Figure 2: $B = \{(12),(23),(34)\}$ and $b = (23)$ (read across.)

12345	21345	23145	32145	31245	13245	13425	31425
34125	43125	43152	34152	31452	13452	13542	31542
31524	13524	13254	31254	32154	23154	23514	32514
32541	23541	25341	52341	53241	35241	35421	53421
54321	45321	45231	54231	52431	25431	25413	52413
54213	45213	42513	24513	24531	42531	42351	24351
23451	32451	32415	23415	24315	42315	43215	34215
34251	43251	43521	34521	34512	43512	45312	54312
53412	35412	35142	53142	53124	35124	35214	53214
52314	25314	25134	52134	51234	15234	15324	51324
51342	15342	15432	51432	54132	45132	41532	14532
14352	41352	41325	14325	14235	41235	42135	24135
21435	12435	12453	21453	24153	42153	41253	14253
14523	41523	45123	54123	51423	15423	15243	51243
52143	25143	21543	12543	12534	21534	21354	12354

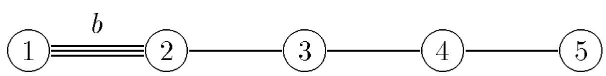


Figure 3: $B = \{(12),(23),(34),(45)\}$ and $b = (12)$ (read across.)

graph $Cay(X\!:\!S_n)$ where $X = \{(12),(23),\ldots,(n{-}1\ n)\}$, the perfect matching $M_{(12)}$ extends to a Hamilton cycle. For $n = 4$ there is a Hamilton path including every edge of $M_{(12)}$, but no Hamilton cycle. A consequence of this result, which is a special case of our main theorem below, is that the linear extensions of the posets in $\mathcal{R}$ can, in fact, be listed by *adjacent* transpositions.

Our major result in this paper is the following theorem.

Main Theorem. Let X be a generating set of transpositions for S_n, where $n > 4$. Then for any $x \in X$, M_x extends to a Hamilton cycle in $Cay(X\!:\!S_n)$.

A *basis* for S_n is a minimal set of generators for S_n. Without loss of generality, we may assume that our generating set of transpositions for S_n is a basis, call it B, so that the transpositions can be described as a tree T_B: the vertices of T_B are the positions $1, 2, \ldots, n$, where i and j are joined by an edge if and only if (ij) is a

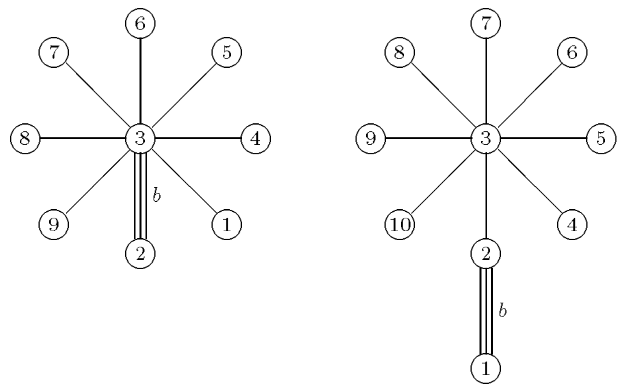


Figure 4: Exceptional combinations: star (left) and flare (right) with b as indicated.

transposition in B. For $b \in B$ we refer to the ordered pair $\langle T_B, b\rangle$ as a *combination.* A combination $\langle T_B, b\rangle$ is said to be *ordinary* if there are two edges e_1, e_2 in T_B such that (a) $e_1 \neq b$, $e_2 \neq b$, and (b) the edges e_1 and e_2 are not adjacent. A combination that is not ordinary is *exceptional.*

The reason for distinguishing between ordinary and exceptional combinations is that our basic proof technique is to splice together Hamilton cycles in certain induced subgraphs. This splicing is based on small cycles that don't contain any edges labeled b. If $\langle T_B, b\rangle$ is ordinary then every vertex of $Cay(B\colon S_n)$ is on a 4-cycle with no edge labeled b. Specifically, if $c, d \neq b$ are nonadjacent edges of T_B then $(cd)^2 = id$, so for any vertex π of $Cay(B\colon S_n)$, the sequence

$$\pi, \pi c, \pi cd, \pi cdc, \pi cdcd = \pi$$

is a 4-cycle. However, if $\langle T_B, b\rangle$ is exceptional, any edges $c, d, \neq b$ of T_B are adjacent, so generators c and d do not commute. In this case there will be no 4-cycles in $Cay(B\colon S_n)$ not containing b. Instead, $(cd)^3 = id$, which gives rise to 6-cycles not containing b.

A *star* is a tree of n vertices in which one vertex has degree $n-1$ and a *flare* is a tree of n vertices in which one vertex has degree $n-2$ and one vertex has degree

2. We refer to a vertex of degree one in a tree as a *leaf* of the tree. See Figure 4. Exceptional combinations are characterized in the following lemma, which we state without proof.

Lemma 1 *An exceptional combination $\langle T, (ij)\rangle$ for $n > 4$ must either be a star or be a flare in which i is a leaf and j is a vertex of degree 2 (or vice-versa).*

The proof of the Main Theorem uses a different construction for each of the following three families of combinations:

1. Ordinary combinations.

2. Exceptional combinations in which $n > 4$ and T is a flare.

3. Exceptional combinations in which $n > 4$ and T is a star.

Within each family, the construction relies inductively only on members of the same family, so the three cases can be handled independently. Section 2 concerns ordinary combinations. Exceptional combinations are handled in Section 3. Section 4 contains extensions and open problems.

2 Ordinary Combinations

An ordinary combination $\langle T, (ij)\rangle$ is *minimal* if for every leaf $k \neq i, j$ the combination $\langle T - k, (ij)\rangle$ is exceptional. The following lemma is easily proven.

Lemma 2 *There are three non-isomorphic minimal ordinary combinations. They are shown in Figures 2, 3, 5.*

For $b \in B$, define a *b-alternating path (cycle)* to be a path (cycle) in $Cay(B\colon S_n)$ in which alternate edges are labeled b. Furthermore, in the case of a b-alternating path, the first and last edge of the path must be labeled b. For example, the cycle in Figure 3 is a (12)-alternating Hamilton cycle in $Cay(B\colon S_n)$ where $B = \{(12), (23), \ldots, (n{-}1\ n)\}$.

12345	21345	21354	12354	13254	31254	31245	13245
14235	41235	41253	14253	12453	21453	21435	12435
13425	31425	31452	13452	14352	41352	43152	34152
34125	43125	42135	24135	23145	32145	32154	23154
25134	52134	52143	25143	24153	42153	45123	54123
54132	45132	41532	14532	14523	41523	42513	24513
24531	42531	43521	34521	34512	43512	45312	54312
51342	15342	13542	31542	35142	53142	53124	35124
31524	13524	15324	51324	52314	25314	25341	52341
53241	35241	32541	23541	23514	32514	35214	53214
51234	15234	12534	21534	21543	12543	15243	51243
54213	45213	45231	54231	52431	25431	25413	52413
51423	15423	15432	51432	53412	35412	35421	53421
54321	45321	42351	24351	23451	32451	34251	43251
43215	34215	32415	23415	24315	42315	41325	14325

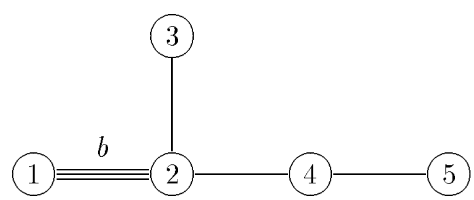


Figure 5: $B = \{(12),(23),(24),(45)\}$ and $b = (12)$ (read across.)

In this section we will show that when B is a basis of transpositions for S_n, with $b \in B$ and $\langle T_B, b\rangle$ is an ordinary combination, then $Cay(B\colon S_n)$ has a b-alternating Hamilton cycle. The proof is by an inductive construction and will require a somewhat stronger hypothesis.

If Q is any b-alternating cycle in $Cay(B\colon S_n)$, an (i,j)-*insertion pair* for Q is a pair of consecutive vertices, α, β on Q satisfying (1) $\alpha(i) = \beta(i) = j$ and (2) the edge joining α and β is not labeled b (i.e., $\{\alpha,\beta\} \notin M_b$).

Theorem 1 *Let B be a basis of transpositions for S_n, and let $b \in B$ be such that $\langle T_B, b\rangle$ is an ordinary combination. Then* $\mathrm{Cay}(B\colon S_n)$ *has a b-alternating Hamilton cycle Q. Further, Q can be chosen so that for every $i,j \in [n]$, Q has an (i,j)-insertion pair, and for each $i \in [n]$ there is some $j \in [n]$ for which Q has two distinct*

(i,j)-insertion pairs.

Proof. If the ordinary combination $\langle T_B, b\rangle$ is minimal, then by Lemma 2 it must be isomorphic to one of the combinations in Figures 2, 3, or 5, each shown with a cycle Q satisfying the conditions of the theorem.

Otherwise, assume inductively that the theorem is true for all ordinary combinations with fewer vertices than T_B. Since $\langle T_B, b\rangle$ is not minimal, T_B contains a leaf, v, not incident with the edge labeled b, such that $\langle T_B - v, b\rangle$ is an ordinary combination. Let z be the unique vertex of T_B adjacent to v.

The Cayley graph of S_n on the set $B \setminus \{(zv)\}$ has n components $G_1, G_2, \ldots, G_n$, where G_k is the subgraph of $Cay(B\!:\!S_n)$ induced by all permutations π with $\pi(v) = k$. Let G' denote the Cayley graph of permutations of $[n] \setminus \{v\}$, generated by the set $B \setminus \{(zv)\}$. Then the induction hypothesis holds for G'. Each G_k is isomorphic to G', so by induction, G_v, in particular, has a b-alternating Hamilton cycle Q_v. Further, for each i, j satisfying $i \neq v$, $j \neq v$, Q_v has an (i,j)-insertion pair and for each $i \neq v$ there is some $j \neq v$ for which Q_v has two (i,j)-insertion pairs.

For $k \neq v$, interchanging k and v in every permutation on Q_v gives a b-alternating Hamilton cycle, Q_k, in G_k.

Now, to obtain the desired cycle Q for $Cay(B\!:\!S_n)$, each of the cycles Q_k, where $k \neq v$, is spliced into the cycle Q_v at a (z,k)-insertion pair of Q_v (z is the unique vertex of T_B adjacent to v.) This is done as follows (see Figure 6). Let α, β be the (z,k) insertion pair of Q_v. Let α', β' be the corresponding pair on the cycle Q_k. That is, α' is obtained from α by interchanging v and k, and similarly for β and β'. Then simply delete edges $\alpha\beta$ and $\alpha'\beta'$ and add edges $\alpha\alpha'$ and $\beta\beta'$ corresponding to the generator (zv) in B.

It remains to show that after all Q_k are spliced into Q_v to form Q, that there is still an (i,j)-insertion pair for every $i, j \in [n]$ and that for each i, there is some j for which Q has two (i,j)-insertion pairs.

First consider $i \neq z, v$ and $j \neq v$. The cycle Q_v has an (i,j)-insertion pair and for some $t \neq v$ there are two (i,t)-insertion pairs. These pairs are still in the final cycle

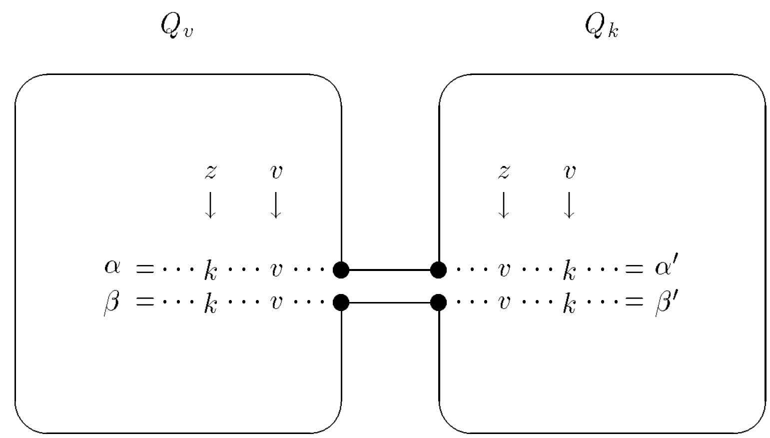


Figure 6: Splicing cycle Q_k into cycle Q_v at a (z, k) insertion pair in proof of Theorem 1.

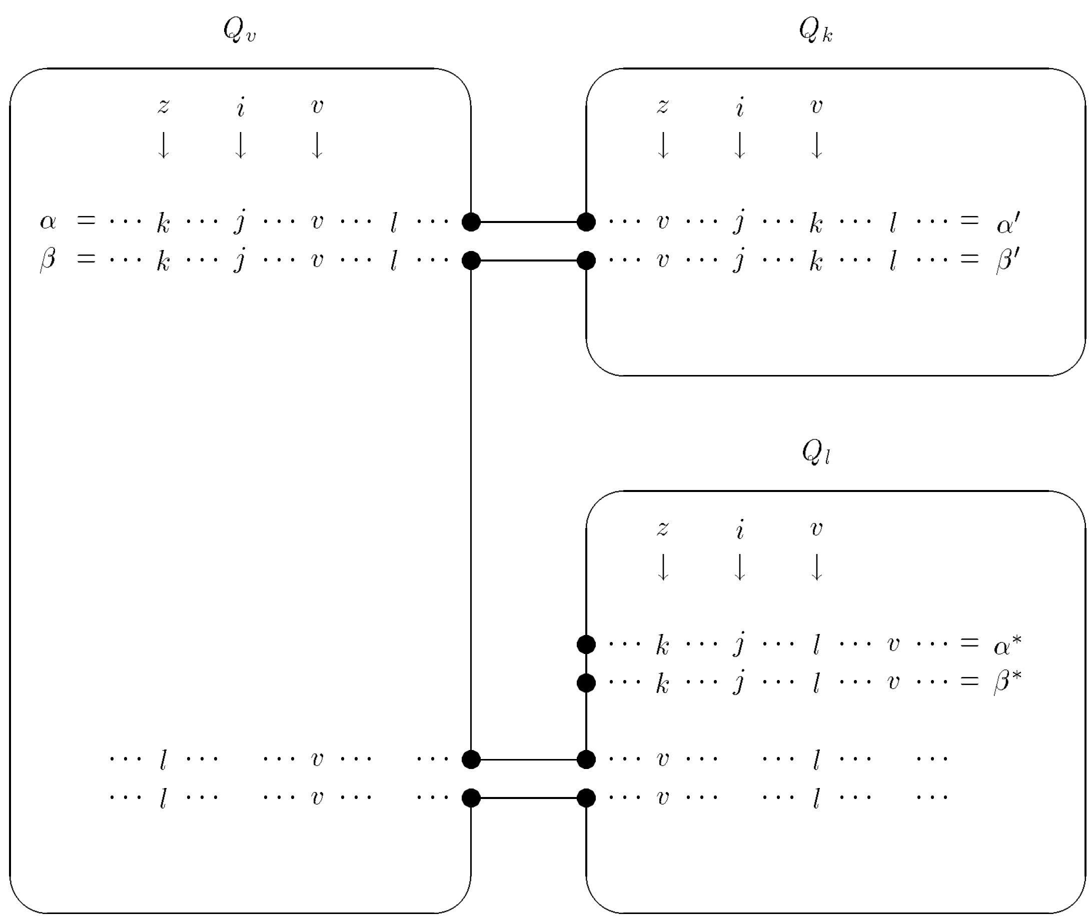


Figure 7: Conservation of (i, j)-insertion pairs when $i \neq z, v$ and $j \neq v$.

Q unless some Q_k was spliced into Q_v at a (z,k)-insertion pair α, β which was also an (i,j)-insertion pair. But then, for any $l \neq j, k$ consider the consecutive pair α^*, β^* on Q_l obtained by swapping elements v and l in each of α, β (see Figure 7). Then α^*, β^* is an (i,j)-insertion pair on Q_l. Note that $\alpha^*(z) = \beta^*(z) = k$. But, in splicing Q_l into Q_v, Q_l is split only at a pair with element v in position z, so α^*, β^* is still an (i,j)-insertion pair in Q. Thus, after splicing, there is no net loss in insertion pairs for $i \neq z, v$ and $j \neq v$.

For $i \neq z, v$ and $j = v$, choose $k \neq v$. In Q_v there was an (i,k)-insertion pair α, β. Interchanging elements v and k in each of α, β gives an (i,v)-insertion pair on Q_k. Since $i \neq z$, this is not the pair in Q_k which was split when Q_z was spliced into Q_v. Thus each Q_k, $k \neq v$ contributes an (i,v)-insertion pair to Q.

If $i = v$, the number of (v,k)-insertion pairs on Q_k is $(n-1)!/2$. During the splicing, only one pair is split for $k \neq v$ and only $n-1$ pairs for $k = v$. So, Q contains a (v,k)-insertion pair for every k, as well as two (v,v)-insertion pairs.

In the case that $i = z$, splicing Q_k into Q_v for $k \neq v$ can only split Q_k at a (z,j)-insertion pair for $j = v$. So, even after splicing, Q_k will contain (z,j)-insertion pairs for every $j \neq v, k$. Choose any l, m, distinct from k and v. Then each of Q_l and Q_m contains a (z,k)-insertion pair, even after splicing.

Finally, we must check for a (z,v)-insertion pair. The cycle Q_v has none and each Q_k, $k \neq v$ gets split at a (z,v)-insertion pair during splicing. However, by induction, there is some $j \neq v$ for which Q_v contains two distinct (z,j)-insertion pairs. Corresponding to these, Q_j contains two (z,v)-insertion pairs. Thus, even after splicing, Q_j contains a (z,v)-insertion pair. □

3 Stars and Flares: Exceptional Combinations

Let B be a basis of transpositions for S_n and $b \in B$. We consider here the cases where T_B is a star or a flare in which b joins the vertex of degree 2 with a leaf (see Figure 4.) In both cases, any two edges in $B \setminus \{b\}$ are adjacent, so the technique used for

12345	21345	21435	12435	12534	21534	21354	12354
12453	21453	21543	12543	15243	51243	51423	15423
15324	51324	51234	15234	15432	51432	51342	15342
13542	31542	31452	13452	13254	31254	31524	13524
13425	31425	34125	43125	43215	**34215**	**34512**	43512
43152	34152	34251	43251	43521	34521	35421	53421
53124	35124	35214	53214	53412	35412	35142	53142
53241	35241	32541	23541	25341	52341	52143	25143
25413	**52413**	**52314**	25314	25134	52134	52431	25431
24531	42531	45231	54231	54321	45321	45123	54123
54213	45213	45312	54312	54132	45132	41532	14532
14352	41352	41253	14253	14523	41523	41325	14325
14235	41235	42135	24135	24315	42315	42513	24513
24153	42153	42351	24351	23451	32451	32154	23154
23514	32514	32415	23415	23145	32145	31245	13245

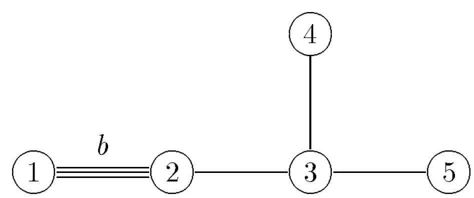


Figure 8: Basis case for flares (read across.)

ordinary combinations will not work. We focus attention on flares, and then show that stars can be handled similarly.

If T_B is a flare, we can assume that $B = F_n = \{(12), (23), (34), (35), ..., (3n)\}$. For $n \geq 5$ flares are isomorphic to ordinary combinations, unless $b = (12)$. We show now that even in this case, $Cay(F_n : S_n)$ has a (12)-alternating Hamilton cycle.

Theorem 2 *For $n \geq 5$, $\mathrm{Cay}(F_n : S_n)$ has a (12)-alternating Hamilton cycle H satisfying*

1. *For n odd, there are consecutive permutations σ_n, τ_n on H satisfying*

$$\sigma_n(3) = 2, \quad \sigma_n(n-1) = 1, \quad \sigma_n(n) = n$$

$$\tau_n(3) = n, \quad \tau_n(n-1) = 1, \quad \tau_n(n) = 2$$

2. For $0 \le k < (n-1)/2$ when n is even and for $1 \le k < (n-1)/2$ when n is odd, there are consecutive permutations $\alpha_n^{(k)}$ and $\beta_n^{(k)}$ on H satisfying

$$\alpha_n^{(k)}(3) = 2k+1, \quad \alpha_n^{(k)}(n) = 2k+2$$

$$\beta_n^{(k)}(3) = 2k+2, \quad \beta_n^{(k)}(n) = 2k+1$$

Proof. The theorem is true when $n = 5$, as demonstrated in Figure 8. Note that on the cycle of Figure 8, the required consecutive permutations σ_5 and τ_5 are 34215 and 34512. The required consecutive permutations $\alpha_5^{(1)}$ and $\beta_5^{(1)}$ are 52314 and 52413. (Note that the order does not matter as long as the permutations appear consecutively.)

Assume that for some $n \ge 5$, $Cay(F_n : S_n)$ has a (12)-alternating Hamilton cycle H satisfying conditions (1) and (2) of the theorem. If we append $n+1$ to every permutation on H, we have a b-alternating cycle in $Cay(F_{n+1} : S_{n+1})$, call it H_{n+1}, still satisfying (1) and (2).

For $1 \le i \le n$, the subgraph of $Cay(F_{n+1} : S_{n+1})$, induced by the elements of S_{n+1} with i in position $n+1$, is isomorphic to $Cay(F_n : S_n)$, and therefore it contains a (12)-alternating Hamilton cycle H_i. Note that given any permutation π with $\pi(n+1) = i$, and any transposition of the form $(3k) \in F_n$, we may assume that π is followed by an edge labeled $(3k)$ on H_i. (Some edge labeled $(3k)$ must appear on H_i since F_n is a basis for S_n. Simply arrange the cyclic list of generators corresponding to the edges along H_i to begin with $(3k)$ and apply them, starting with permutation π. This yields a new H_i with the required property.)

The idea of the construction is to splice $H_1, \ldots, H_n$ into H_{n+1} in such a way to obtain a (12)-alternating Hamilton cycle H^* in $Cay(F_{n+1} : S_{n+1})$ and preserve properties (1) and (2) of the theorem.

For n odd, we first splice H_1, H_2, and H_n into H_{n+1} at the pair σ_n', τ_n' on H_{n+1} corresponding to σ_n, τ_n on H (see Figure 9.) To do this we use the fact that the following composition of transpositions is the identity:

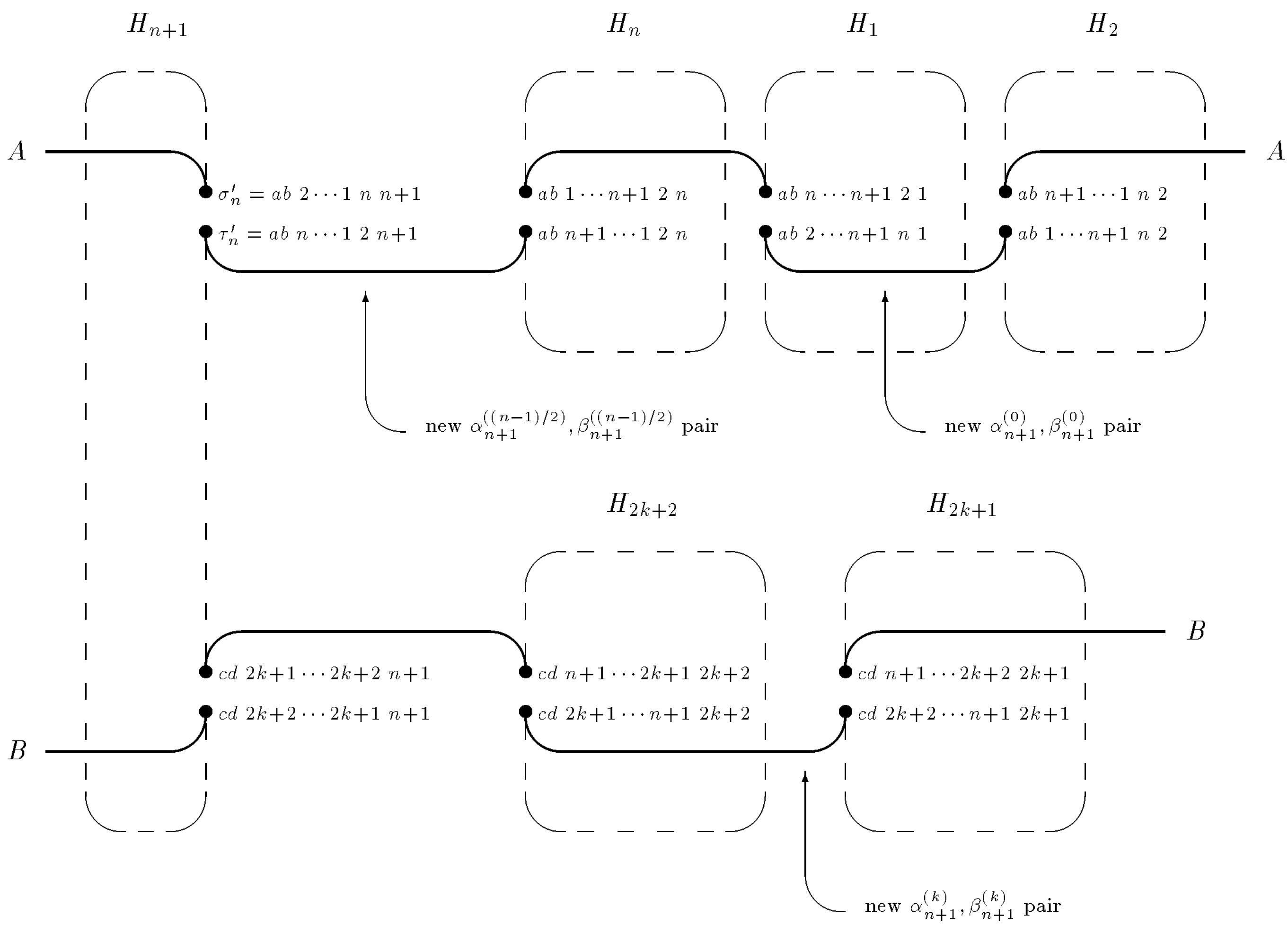


Figure 9: For n odd, splicing the cycles H_i into H_{n+1}.

$$(3\ n)(3\ n{+}1)(3\ n{-}1)(3\ n{+}1)(3\ n)(3\ n{+}1)(3\ n{-}1)(3\ n{+}1) = id$$

We know that σ'_n and τ'_n appear consecutively on H_{n+1} and, as discussed above, we may assume without loss of generality that:

- $[\tau'_n(3\ n{+}1)]$ and $[\tau'_n(3\ n{+}1)](3\ n{-}1)$ appear consecutively on H_n,
- $[\tau'_n(3\ n{+}1)(3\ n{-}1)(3\ n{+}1)]$ and $[\tau'_n(3\ n{+}1)(3\ n{-}1)(3\ n{+}1)](3\ n)$ appear consecutively on H_1, and
- $[\tau'_n(3\ n{+}1)(3\ n{-}1)(3\ n{+}1)(3\ n)(3\ n{+}1)]$ and $[\tau'_n(3\ n{+}1)(3\ n{-}1)(3\ n{+}1)(3\ n)(3\ n{+}1)](3\ n{-}1)$ appear consecutively on H_2.

In each pair above, as well as for the pair σ'_n, τ'_n, delete the edges joining the two elements of the pair in their respective cycles. Then use edges corresponding to the generator $(3\ n{+}1)$ to join together the cycles as shown in Figure 9. Note from Figure 9 that this construction provides us with the required pairs $\alpha^{(0)}_{n+1}, \beta^{(0)}_{n+1}$ and $\alpha^{((n-1)/2)}_{n+1}, \beta^{((n-1)/2)}_{n+1}$ for the (12)-alternating cycle H^* being constructed in $Cay(F_{n+1} : S_{n+1})$.

For $0 \leq k < (n-1)/2$ when n is even and $1 \leq k < (n-1)/2$ when n is odd, we splice H_{2k+1} and H_{2k+2} into H_{n+1} at the consecutive pair on H_{n+1} corresponding to $\alpha^{(k)}_n, \beta^{(k)}_n$ on H, similar to the method above, but using the identity

$$(3\ n)(3\ n{+}1)(3\ n)(3\ n{+}1)(3\ n)(3\ n{+}1) = id$$

(see Figures 9 and 10.) Note from Figure 9 that this provides for the cycle H^* the pairs $\alpha^{(k)}_{n+1}, \beta^{(k)}_{n+1}$ for $1 \leq k < (n-1)/2$ and, from Figure 10, when n is even, gives σ_{n+1}, τ_{n+1}. □

The case of stars can be handled similarly. For a basis B of transpositions of S_n, if T_B is a star, we may assume that $B = R_n = \{(31), (32), (34), (35), \ldots (3n)\}$, and that the distinguished edge b of B is (32) (see Figure 4.) In this case we have the following theorem.

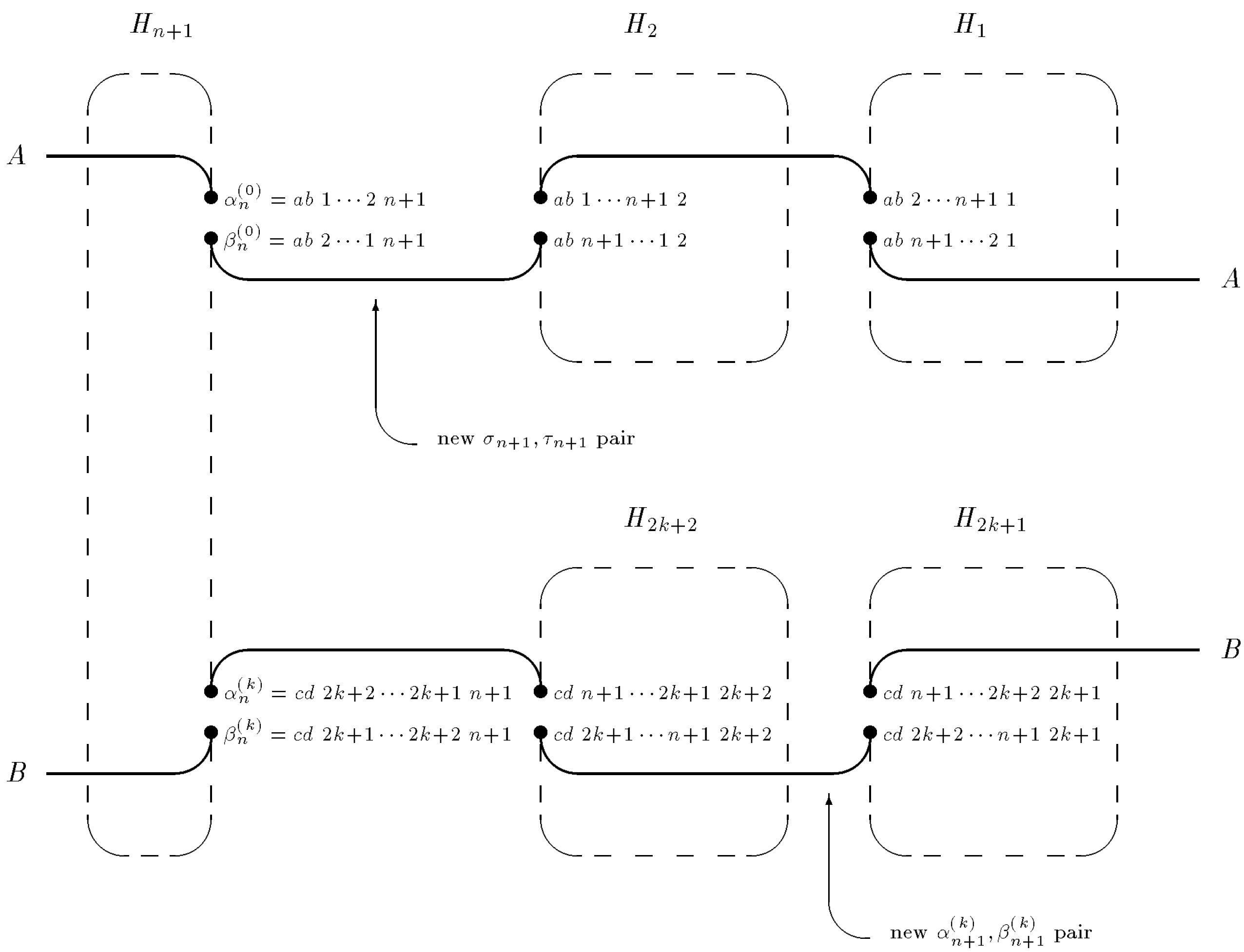


Figure 10: For n even, splicing the H_i into H_{n+1}

Theorem 3 *For $n \geq 5$,* $\text{Cay}(R_n : S_n)$ *has a (32)-alternating Hamilton cycle H satisfying*

1. *For n odd, there are consecutive permutations σ_n, τ_n on H satisfying*

$$\sigma_n(3) = 2, \quad \sigma_n(n-1) = 1, \quad \sigma_n(n) = n$$

$$\tau_n(3) = n, \quad \tau_n(n-1) = 1, \quad \tau_n(n) = 2$$

2. *For $0 \leq k < (n-1)/2$ when n is even and for $1 \leq k < (n-1)/2$ when n is odd, there are consecutive permutations $\alpha_n^{(k)}$ and $\beta_n^{(k)}$ on H satisfying*

$$\alpha_n^{(k)}(3) = 2k+1, \quad \alpha_n^{(k)}(n) = 2k+2$$

$$\beta_n^{(k)}(3) = 2k+2, \quad \beta_n^{(k)}(n) = 2k+1$$

Proof. The theorem is true when $n = 5$, as demonstrated in Figure 11. Note that on the cycle of Figure 11, the required consecutive permutations σ_5 and τ_5 are 34215 and 34512. The required consecutive permutations $\alpha_5^{(1)}$ and $\beta_5^{(1)}$ are 25314 and 25413.

The remainder of the proof is identical to the proof of Theorem 2. □

4 Final Remarks

There have been some other papers written about finding Hamilton cycles through specified matchings in graphs, but not in connection with Cayley graphs ([Ha],[Wo]). For example, Häggkvist [Ha] has shown that if $d(u) + d(v) \geq |V(G)| + 1$ for all nonadjacent vertices u and v of G, then G has a Hamilton path through any given perfect matching.

By deleting all odd permutations from our lists we obtain listings of the alternating group A_n. In the case of a star, where $B = \{(1\ n), (2\ n), \ldots, (n-1\ n)\}$ and $b = (1\ n)$, note that since $(1\ n)(j\ n) = (1\ j\ n)$, our results provide another proof of the result of Gould and Roth [GoRo] that the *digraph Cay*$(X : A_n)$ is hamiltonian for $n \geq 5$, where $X = \{(1\ j\ n) \mid 1 < j < n\}$.

12345	13245	13542	15342	15432	14532	54132	51432
51234	52134	52314	53214	23514	**25314**	**25413**	24513
24153	21453	41253	42153	42351	43251	43521	45321
35421	34521	34125	31425	31245	32145	32415	**34215**
34512	35412	35142	31542	51342	53142	53241	52341
52431	54231	54321	53421	53124	51324	31524	35124
35214	32514	32154	31254	31452	34152	34251	32451
32541	35241	25341	23541	23145	21345	21543	25143
15243	12543	12453	14253	14352	13452	43152	41352
41532	45132	45312	43512	53412	54312	54213	52413
52143	51243	51423	54123	14523	15423	15324	13524
13254	12354	12534	15234	25134	21534	21354	23154
23451	24351	24531	25431	45231	42531	42135	41235
21435	24135	24315	23415	43215	42315	42513	45213
45123	41523	41325	43125	13425	14325	14235	12435

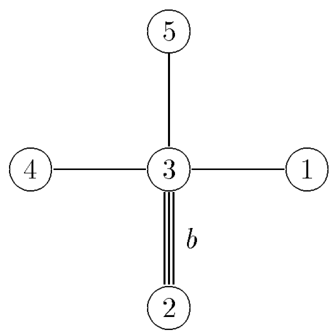


Figure 11: Basis case for stars (read across.)

Tchuente [Tc] showed that there is a Hamilton path between any two permutations of opposite parity in $Cay(B:S_n)$ for any basis of transpositions B. The next lemma shows that it is not in general the case that there is a b-alternating path containing M_b between any two permutations of opposite parity.

Let $\langle T_B, b\rangle$ be a combination. If the edge b is removed from T_B then two trees remain; these trees induce a partition of $[n]$ into two sets, say X and Y.

Lemma 3 *Let X, Y be the partition of $[n]$ induced by $\langle T_B, b\rangle$. Any b-alternating Hamilton path in* $\mathrm{Cay}(B:S_n)$ *that starts at the permutation π and ends at the permu-*

tation π' must satisfy the following condition.

$$\bigcup_{i \in X} \pi(i) \;=\; \bigcup_{i \in X} \pi'(i)$$

Proof. Consider the multigraph $\mathcal{M}$ formed from $Cay(B\!:\!S_n)$ by condensing into a single vertex, for each k-subset S of $[n]$, those permutations π for which $\{\pi(i) \mid i \in X\} = S$. Thus $\mathcal{M}$ has $\binom{n}{k}$ vertices and each vertex is regular of degree $k!(n-k)!$. Every edge of $\mathcal{M}$ is labeled b since every transposition other than b either swaps two elements with positions in X or swaps two elements with positions in Y. A b-alternating path in in $Cay(B\!:\!S_n)$ that contains every edge of M_b becomes an Euler tour in $\mathcal{M}$. Clearly, this tour has to start and end at the same condensed vertex. □

If $n = 4$ then there are two non-isomorphic exceptional combinations $\langle T_B, b\rangle$, namely the star $B = \{(12), (13), (14)\}$ with $b = (12)$, and the path $B = \{(12), (23), (34)\}$ again with $b = (12)$. In these cases it is not too difficult to show that there is no b-alternating Hamilton cycle. However, there are b-alternating Hamilton paths containing M_b as shown in Figure 12.

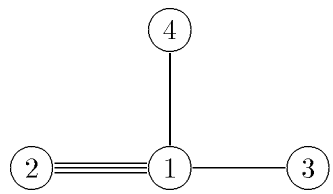
Below we list some questions for further investigation.

1. Is there an efficient algorithm to generate the permutations on a b-alternating Hamilton cycle? We would like an algorithm whose total storage requirement is $O(n)$ and whose total running time is $O(n!)$. A straightforward implementation of our proofs leads to algorithms that require $\Theta(n \cdot n!)$ time and $\Theta(n \cdot n!)$ space.

2. Is the necessary condition of Lemma 3, together with the condition that π and π' have opposite parity, also a sufficient condition for the existence of a Hamilton path from π to π'? We conjecture that the condition is sufficient.

3. Given a matching M in the n-cube $\mathcal{Q}_n$, is there a Hamilton cycle in $\mathcal{Q}_n$ that includes every edge of M?

1234	2134	3124	1324	2314	3214
4213	2413	3412	4312	1342	3142
4132	1432	2431	4231	3241	2341
4321	3421	1423	4123	2143	1243



1234	2134	2314	3214	3124	1324
1342	3142	3412	4312	4321	3421
3241	2341	2431	4231	4213	2413
2143	1243	1423	4123	4132	1432

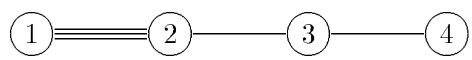


Figure 12: (12)-alternating Hamilton paths.

4. If X is a set of generators for a group G, and $x \in X$ is an involution (i.e., $x^2 = id$), then x induces a perfect matching, M_x, in $Cay(X:G)$. A natural question is whether there is a x-alternating path in $Cay(X:G)$.

   In general, there is no x-alternating Hamilton path in $Cay(X:G)$. For example, if $X = \{(1\ 2), (1\ 2\ \cdots\ n)\}$, where $n \geq 3$ is odd, then the following little argument, similar to the proof of Lemma 3, shows that $Cay(X:S_n)$ has no (1 2)-alternating path. Condense into single super-vertices all those permutations inequivalent under the rotation $(1\ 2\ \cdots\ n)$. The resulting multigraph has $(n-1)!$ vertices, each of degree n, and any Hamilton path in $Cay(X:S_n)$ becomes a Eulerian cycle in the multigraph. Clearly, there is no Eulerian cycle if n is odd.

   On the other hand, there does appear to be a x-alternating path in $Cay(X:G)$

if G is a reflection group and x is one of its generators, but we have not proven this fully. In [CoSlWi], the Cayley graphs of reflection groups are shown to be Hamiltonian.

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