Durfee Polynomials
Extended Abstract

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Summary (English)

Let $\mathbf{F}(n)$ be a family of partitions of n and let $\mathbf{F}(n,d)$ denote the set of partitions in $\mathbf{F}(n)$ with Durfee square of size d. We define the Durfee polynomial of $\mathbf{F}(n)$ to be the polynomial $P_{\mathbf{F},n} = \sum |\mathbf{F}(n,d)| y^d$, where $0 \le d \le \lfloor \sqrt{n} \rfloor$. This paper describes ongoing efforts to compute statistics associated with the Durfee polynomial for various families $\mathbf{F}$.

We give empirical evidence which leads to the conjecture that for several families $\mathbf{F}$, including unrestricted partitions and partitions into distinct parts, all roots of the Durfee polynomial are real (and negative). It would follow then that for these families the sequence of coefficients $\{|\mathbf{F}(n,d)|\}$ is log-concave and unimodal and, by a result of Darroch, that the mean and the mode of the sequence differ by less than 1. The mean and the mode are, respectively, the average size and the most likely size of the Durfee square of a partition in $\mathbf{F}(n)$. Experimentally we have observed that for several families $\mathbf{F}(n)$ the mean and mode grow as $cn^{1/2}$, for an appropriate constant $c = c_{\mathbf{F}}$. An empirical method from asymptotic analysis yields analytical expressions for the constants $c_{\mathbf{F}}$ which agree numerically with the observed values.

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For the family of ordinary partitions, $\mathbf{P}(n)$, we find an asymptotic formula for $|\mathbf{P}(n,d)|$, determine the average, most likely, and asymptotic distributions of the Durfee square size, and prove some unimodality results.

Summary (French)

Soient $\mathbf{F}(n)$ l'ensemble des partitions de n de la famille $\mathbf{F}$ et $\mathbf{F}(n,d)$ l'ensemble des partitions dans $\mathbf{F}(n)$ dont la taille du carré de Durfee est d. Nous définissons le polynôme de Durfee relatif à $\mathbf{F}(n)$ comme étant $P_{\mathbf{F},n} = \sum_{d=0}^{\lfloor\sqrt{n}\rfloor} |\mathbf{F}(n,d)| y^d$. Nous nous intéressons, dans cet article, aux statistiques associées au polynôme de Durfee pour différentes familles $\mathbf{F}$.

Nous présentons des résultats expérimentaux qui conduisent à la conjecture que, pour plusieurs familles $\mathbf{F}$, y compris les partitions sans restrictions et les partitions en parts inégales, toutes les racines du polynôme de Durfee sont réelles (et négatives). Il suivrait alors que, pour ces familles et pour tout n, la suite $\{|\mathbf{F}(n,d)|\}$, $0 \leq \mathrm{d} \leq \lfloor\sqrt{n}\rfloor$, est log-concave et unimodale. De plus, par un résultat de Darroch, la différence entre le mode et la moyenne serait inférieure à 1. La moyenne et le mode sont respectivement la taille moyenne et la taille la plus probable du carré de Durfee des partitions de $\mathbf{F}(n)$. Nous avons observé expérimentalement qu'il existe une constante $c_{\mathbf{F}}$ propre a chaque famille, telle que le mode et la moyenne semblent être asymptotiquement $c_{\mathbf{F}} n^{1/2}$. Une méthode empirique de l'analyse asymptotique donne une expression analytique pour les constantes $c_{\mathbf{F}}$, qui est en accord avec les valeurs observées. Pour la famille des partitions sans restrictions, $\mathbf{P}$, nous donnons le comportement asymptotique de $|\mathbf{P}(n,d)|$, déterminons la moyenne, le mode et la distribution asymptotique de la taille du carré de Durfee et démontrons des résultats concernant l'unimodalité.

1 Introduction

A *partition* λ of an integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_\ell)$ of positive integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. The Ferrers diagram of λ is a 2-dimensional array of dots in which row i has λ_i dots and rows are left justified. The *Durfee square* of λ is the largest square array of dots contained in its Ferrers diagram and $d(\lambda)$ denotes the length of a side of this square. We let $\mathbf{P}(n)$ be the set of all partitions of n and let $\mathbf{P}(n,d)$ be the set of all partitions of n with Durfee square of size d. To simplify notation, we use the same symbol to denote a set and its size when the meaning is clear from context.

For a finite sequence of positive integers $\mathbf{s} = \{a_d\}$, $0 \leq d \leq N$, the *mean* of $\mathbf{s}$ is the ratio $\sum(d a_d)/\sum a_d$ and the *mode* of $\mathbf{s}$ is that index i such that $a_i = \max\{a_d\}$. It is well-known that if all roots of the polynomial $\sum a_d x^d, 0 \leq d \leq N$ are real (and hence negative), then $\{a_d\}$ is strictly log-concave in d and therefore unimodal with a peak or a plateau of two points. (See [4, 13], for example). What is perhaps less well-known is that this condition on the roots guarantees that the mode and the mean of $\{a_d\}$ differ by at most 1 [6, 1].

For a family of partitions $\mathbf{F}$, let $\mathbf{F}(n,d)$ be the set of partitions in $\mathbf{F}(n)$ with Durfee square of size d. We investigate the sequences $\{\mathbf{F}(n,d)\}$ for fixed n. The Durfee polynomial is their generating function $P_{\mathbf{F},n}(y) = \sum_d \mathbf{F}(n,d)y^d$, $0 \le d \le \lfloor\sqrt{n}\rfloor$. The mode and the mean of $\{\mathbf{F}(n,d)\}$ are, respectively, the most likely and the average size of the Durfee square of a partition in $\mathbf{F}(n)$.

In Section 2, we present the different families and show that there is a common form for the generating functions $\mathbf{F}_d(x) = \sum_n \mathbf{F}(n,d)x^n$. For each of these families $\mathbf{F}$, we give evidence that there is a constant $c_\mathbf{F}$ for which $mode\{\mathbf{F}(n,d)\} \sim c_\mathbf{F}\sqrt{n}$ and present the theoretical value of $c_\mathbf{F}$ computed by using asymptotic methods. We also present experimental evidence which leads us to the conjecture that the Durfee polynomials have all roots real.

In Section 3, for the family of ordinary partitions, $\mathbf{P}$, we find an asymptotic formula for $\mathbf{P}(n,d)$, determine the average, most likely Durfee square size, and show that the numbers $\{\mathbf{P}(n,d)\}$ are asymptotically normal. The results show that for n sufficiently large, $|mode\{\mathbf{P}(n,d)\} - mean\{\mathbf{P}(n,d)\}| \le 1/2$ and that $\{\mathbf{P}(n,d)\}$ is log-concave, but leave open the question whether the Durfee polynomial has all roots real.

In Section 4 we explain how the theoretical values for the constants $c_\mathbf{F}$ given in Table 1 of Section 2 were found.

2 Statistics of the Durfee Polynomial

We consider the Durfee polynomial for several families of partitions $\mathbf{F}$.

(1) $\mathbf{P}$: unrestricted partitions
(2) $\mathbf{B}$: basis partitions [8, 12]
(3) $\mathbf{D}$: partitions into distinct parts
(4) $\bar{\mathbf{D}}$: partitions λ into distinct parts with $\lambda_{d(\lambda)} > d(\lambda)$
(5) $\tilde{\mathbf{D}}$: partitions λ into distinct parts with $\lambda_{d(\lambda)+1} < d(\lambda)$
(6) $\mathbf{SC}$: self conjugate partitions
(7) $\mathbf{O}$: partitions into odd parts
(8) $\mathbf{E}$: partitions into even parts
(9) $\mathbf{Z}$: partitions λ in which the number of parts is $d(\lambda)$

The families $\bar{\mathbf{D}}$ and $\tilde{\mathbf{D}}$ were included because of the form of their generating functions. $\mathbf{Z}$ was included because its defining recurrence and generating function are similar to the other families and the mean and mode of $\{\mathbf{Z}(n,d)\}$ differ by less than 1 (for $n \le 5000$), but the Durfee polynomial fails to have all roots real. Note that $\mathbf{Z}(n,d)$ is equal to the number of partitions into d distinct parts that differ at least by 2 (counted by one side of the first Rogers-Ramanujan identity).

Observe that for self-conjugate partitions (6), $\mathbf{SC}(n,d) = 0$ if n and d have opposite parity, so the sequence $\{\mathbf{SC}(n,d)\}$ cannot be unimodal and consequently the Durfee polynomial cannot have all roots real. We consider instead the subsequence consisting of nonzero entries. Similarly, the sequences $\{\mathbf{E}(n,d)\}$ (for even n) and $\{\mathbf{O}(n,d)\}$ are not log-concave, but we can consider the subsequences corresponding to even d or odd d. The Durfee polynomial is then $\sum_d \mathbf{F}(n,2d)y^d$ or $\sum_d \mathbf{F}(n,2d+1)y^d$.

In Section 2.1, we present the generating functions $\mathbf{F}_d(x) = \mathbf{F}(n,d)x^n$ which permit us to derive recurrences and compute $\mathbf{F}(n,d)$. In Section 2.2, we give evidence of the existence of a constant $c_{\mathbf{F}}$ such that *mode* $\{\mathbf{F}(n,d)\} \sim c_{\mathbf{F}}\sqrt{n}$ and estimate its value. In Section 2.3, we present the results of our experiments to test whether all roots of the Durfee polynomial are real, to check the difference between the mode and the mean and to test for log-concavity.

2.1 Generating Functions

The generating function for (1) is straightforward; (2) is from [12]; (3) - (9) are explained in detail in [5]. The basic idea is that a partition in $\mathbf{F}(n,d)$ can be decomposed as the Durfee square together with a partition below the square and a partition to the right of the square, each having largest part at most d.

(1) $\mathbf{P}_d(x) = x^{d^2}\prod_{i=1}^{d}(1-x^i)^{-2}$

(2) $\mathbf{B}_d(x) = x^{d^2}\prod_{i=1}^{d}\frac{(1+x^i)}{(1-x^i)}$

(3) $\mathbf{D}_d(x) = \frac{x^{3d^2/2-d/2}(1+x^{2d})}{1+x}\prod_{i=1}^{d}\frac{(1+x^i)}{(1-x^i)}$

(4) $\bar{\mathbf{D}}_d(x) = x^{3d^2/2+d/2}\prod_{i=1}^{d}\frac{(1+x^i)}{(1-x^i)}$

(5) $\tilde{\mathbf{D}}_d(x) = \frac{x^{3d^2/2-d/2}}{1+x}\prod_{i=1}^{d}\frac{(1+x^i)}{(1-x^i)}$

(6) $\mathbf{SC}_d(x) = x^{d^2}\prod_{i=1}^{d}(1-x^{2i})^{-1}$

(7) $\mathbf{O}_d(x) = x^{d(2\lfloor d/2\rfloor+1)}\prod_{i=1}^{d}(1-x^{2i})^{-1}\prod_{i=1}^{\lceil d/2\rceil}(1-x^{2i-1})^{-1}$

(8) $\mathbf{E}_d(x) = x^{2d\lceil d/2\rceil}\prod_{i=1}^{d}(1-x^{2i})^{-1}\prod_{i=1}^{\lfloor d/2\rfloor}(1-x^{2i})^{-1}$

(9) $\mathbf{Z}_d(x) = x^{d^2}\prod_{i=1}^{d}(1-x^i)^{-1}$

2.2 The Mode of $\{\mathbf{F}(n,d)\}$

For a family $\mathbf{F}$ of partitions and an integer n, let $m(i) = \min\{n \mid mode\{\mathbf{F}(n,d)\} = i\}$. ¿From our experiments, it appears that for all of the families (1) - (9), the second difference of $m(i)$, $\triangle^2 m(i)$, is essentially constant. If true, then $m(i) \sim bi^2/2$ and thus

Family $\mathbf{F}$	$mode\{\mathbf{F}(n,d)\} \sim c_{\mathbf{F}}\sqrt{n}$	
	Experimental value	Theoretical value
(1) $\mathbf{P}$	0.54	$\sqrt{6}\ln 2/\pi \approx 0.54044$
(2) $\mathbf{B}$	0.62	$^{(*)}$ 0.6192194165...
(3) $\mathbf{D}$	0.53	$2\sqrt{3}\ln((1+\sqrt{5})/2)/\pi \approx 0.530611$
(4) $\bar{\mathbf{D}}$	0.53	$2\sqrt{3}\ln((1+\sqrt{5})/2)/\pi$
(5) $\tilde{\mathbf{D}}$	0.53	$2\sqrt{3}\ln((1+\sqrt{5})/2)/\pi$
(6) **SC**, odd d, n	0.54	$\sqrt{6}\ln 2/\pi$
(6) **SC**, even d, n	0.54	$\sqrt{6}\ln 2/\pi$
(7) **O**, odd d	0.53	$2\sqrt{3}\ln((1+\sqrt{5})/2)/\pi$
(7) **O**, even d	0.53	$2\sqrt{3}\ln((1+\sqrt{5})/2)/\pi$
(8) **E**, odd d, even n	0.53	$2\sqrt{3}\ln((1+\sqrt{5})/2)/\pi$
(8) **E**, even d, n	0.53	$2\sqrt{3}\ln((1+\sqrt{5})/2)/\pi$
(9) $\mathbf{Z}$	0.60	$\sqrt{15}\ln((1+\sqrt{5})/2)/\pi \approx 0.59324$

$^{(*)}$ See Section 4 in text.

Table 1: Most likely size Durfee square: tested for $0 \le n \le 5000$.

$i \sim \sqrt{(2m(i)/b)}$. This means that mode $\{\mathbf{F}(n,d)\} \sim \sqrt{2n/b}$. A slight modification of this calculation is required for the families in which we consider the sequence $\{\mathbf{F}(n,d)\}$ only for odd d or even d.

The results of our experiments are displayed in Table 1. Each of the families of partitions $\mathbf{F}(n)$ in column 1 was checked for $n = 0, \ldots, 5000$. Column 2 gives the numerical value of $c_{\mathbf{F}}$ based on computation, and column 3 gives the conjectured analytical expression for $c_{\mathbf{F}}$. For the family $\mathbf{P}(n,d)$, the analytical expression is proven in Section 3.

2.3 Roots of the Durfee Polynomial

First notice that the recurrences associated with the genrating functions presented in Section 2.1 will have the common form

$$\mathbf{F}(n,d) = \sum_{i=1}^{m_0} C_i \mathbf{F}(n - e_i d + f_i, d) + \sum_{i=1}^{m_1} D_i \mathbf{F}(n - g_i d + h_i, d-1). \tag{2.1}$$

(See [5] for details).

In this equation $e_i, f_i, g_i, h_i, m_0, m_1$ are all nonnegative integers. Because the recursion for the numbers on row n involves numbers appearing on rows a constant times d earlier, we see no way to use the often successful technique of [9] to prove the Durfee polynomials have all their roots negative.

We tested the Durfee polynomials of all the families (1) - (9) and, except for the family $\mathbf{Z}$ (first complex root when $n = 75$), found that all roots are real and negative for $n \leq 1000$. It was also confirmed by our experiments for $n \leq 5000$ that for all of the families $\mathbf{F}$ (1) - (9), the mean and the mode of $\{\mathbf{F}(n,d)\}$ differ by less than 1 and that the sequences $\{\mathbf{F}(n,d)\}$ are strictly log-concave. These results help to support the conjecture that the Durfee polynomials have all roots real, as they are necessary conditions. Moreover if $|mean - mode| < 1$ is true, then we also have $mean\{\mathbf{F}(n,d)\} \sim c_{\mathbf{F}}\sqrt{n}$.

3 The asymptotics of $\mathbf{P}(n,d)$

In this section, we study $\mathbf{P}(n,d)$, the number of partitions of the integer n having Durfee square size d. We find an asymptotic formula for $\mathbf{P}(n,d)$; determine the average, most likely, and asymptotic distribution of the Durfee square size; and prove some unimodality results. We denote by $\mathbf{p}(n)$ the number of partitions of n and by $\mathbf{p}(n,k)$ the number of partitions of n with at most k parts. As is well known, $\mathbf{p}(n,k)$ also counts partitions of n into parts all less than or equal to k. We have found the following asymptotic formula for $\mathbf{P}(n,d)$.

Theorem 1 *Uniformly for $\epsilon \leq x \leq 1-\epsilon$ we have*

$$\mathbf{P}(n, xn^{1/2}) \;=\; \frac{F(x)}{n^{5/4}}\exp\Big\{n^{1/2}G(x) + O(n^{-1/2})\Big\}. \tag{3.1}$$

Here, the functions $F(x)$ and $G(x)$ are given by:

$$F(x) = 2\pi^{1/2}f(u)^2(2+u^2)^{5/4}(g(u) - ug'(u) - u^2g''(u))^{-1/2} \tag{3.2}$$

and

$$G(x) = 2g(u)(2+u^2)^{-1/2}, \tag{3.3}$$

where $u = (2x^2/(1-x^2))^{1/2}$ and the functions $f(u)$, $g(u)$, and $v = v(u)$ are:

$$f(u) = \frac{1}{2\pi\sqrt{2}}\frac{v}{u}(1 - e^{-v} - \frac{u^2e^{-v}}{2})^{-1/2} \tag{3.4}$$

$$g(u) = \frac{2v}{u} - u\log(1-e^{-v}), \tag{3.5}$$

and, an implicit definition for v,

$$u^2 = v^2 \Big/ \int_0^v \frac{t}{e^t-1}dt. \tag{3.6}$$

Proof. (outline) The Ferrers diagram of a partition counted by $\mathbf{P}(n,d)$ consists of a $d\times d$ square with two independent partitions of n_1 and n_2, $n_1+n_2 = n-d^2$, attached to the east and south; the one to the east has at most d parts, and the one to the south has no parts exceeding d. Thus

$$\mathbf{P}(n, xn^{1/2}) = \sum_{n_1+n_2=(1-x^2)n} \mathbf{p}(n_1, xn^{1/2})\mathbf{p}(n_2, xn^{1/2}). \tag{3.7}$$

In this summation, we can estimate the factors in the terms by using the following asymptotic formula of Szekeres [16]:

$$\mathbf{p}(n, un^{1/2}) = \frac{f(u)}{n}\exp\Big\{n^{1/2}g(u) + O(n^{-1/2})\Big\}, \text{ uniformly for } u \geq \epsilon > 0. \tag{3.8}$$

Introduce the variable t by

$$n_1, n_2 \;=\; \frac{1-x^2}{2}n \pm tn^{1/2}, \tag{3.9}$$

the plus sign being used for n_1, the minus for n_2. View the summation as extending over a discrete set of real t, with stepsize $n^{-1/2}$:

$$\mathbf{P}(n, xn^{1/2}) = \sum_t \text{term}(t); \tag{3.10}$$

we find, uniformly for $|t| \leq n^{1/3}$:

$$\text{term}(t) \;=\; \frac{4f(u)^2}{(1-x^2)^2n^2}\exp\Big\{\frac{2g(u)}{\sqrt{2+u^2}}n^{1/2} + \frac{\beta t^2}{n^{1/2}} + O(\frac{t^4}{n^{3/2}} + \frac{t^2}{n} + \frac{1}{n^{1/2}})\Big\},$$

where

$$\beta = (2+u^2)^{3/2}(-g(u) + ug'(u) + u^2g''(u))/4.$$

We sum over t by approximation with an integral; bounding the error committed, and justifying the replacement of a finite integral with an infinite one, are standard arguments in asymptotic analysis (see, for example, [7]). Algebraic simplification leads to the functions $F(x)$ and $G(x)$ given in the statement of the theorem; it remains for us only to bound the tails by showing:

$$\sum_{|t|>n^{1/3}} \text{term}(t) = \frac{O(n^{-1/2})}{n}\exp\{\frac{2g(u)}{\sqrt{2+u^2}}n^{1/2}\}. \tag{3.11}$$

This is accomplished by using the following convenient upper-bound. For all $n \geq k \geq 1$,

$$\mathbf{p}(n,k) \;<\; \exp\Big\{n^{1/2}g(\frac{k}{n^{1/2}})\Big\}. \tag{3.12}$$

The upper-bound is proven by double induction on k and n, the well known recursion for $\mathbf{p}(n,k)$, and the convexity of both $g(u)$ and $ug(K/u)$ (K a constant). This completes the sketch of the proof of the theorem. □

Corollary 1 *Let* $x_0 = \sqrt{6}\log 2/\pi$, $d = x_0n^{1/2} + tn^{1/4}$, $c_1 = \pi(2/3)^{1/2}$, $c_2 = \frac{\pi}{2(6)^{5/4}}(\frac{\pi^2}{6} - (\log 2)^2)^{-1/2}$, *and* $c_3 = \frac{-32\pi^3}{(24)^{3/2}}(\frac{\pi^2}{6} - (\log 2)^2)^{-1}$. *Then, uniformly for* $t = o(n^{1/4})$,

$$\mathbf{P}(n,d) = \frac{c_1}{n^{5/4}}\exp\{c_2n^{1/2} + \tfrac{1}{2}c_3t^2 + o(1)\}.$$

Thus the numbers $\mathbf{P}(n,d)$ *are asymptotically normal as* $n \to \infty$.

Remark. The constants c_i are the values of F, G, and G'' at $x_0 = \frac{\log 2\times\sqrt{6}}{\pi}$. (The function $G(x)$ is maximized at $x = x_0$).

Corollary 2 *For any positive ϵ there is an integer n_0 $(= n_0(\epsilon))$ such that for all $n \geq n_0$ and d satisfying $\epsilon n^{1/2} \leq d \leq (1-\epsilon)n^{1/2}$ we have*

$$\mathbf{P}(n,d)^2 > \mathbf{P}(n,d-1)\mathbf{P}(n,d+1).$$

Proof. We know from Szekeres' work that there is a function $H(x)$ such that

$$\mathbf{P}(n, xn^{1/2}) = \frac{F(x)}{n^{5/4}}\exp\{n^{1/2}G(x) + H(x)/n^{1/2} + O(n^{-1})\}.$$

It is not necessary to know, and we have not computed, the explicit form of $H(x)$. Letting $x = d/n^{1/2}$, we evaluate the functions F, G, H at the arguments $x, x \pm n^{-1/2}$ and obtain

$$\frac{\mathbf{P}(n,d)^2}{\mathbf{P}(n,d-1)\cdot\mathbf{P}(n,d+1)} = \exp\{-G''(x)/n^{1/2} + O(n^{-1})\}.$$

The proof is completed by the calculation

$$G''(x) = (2+u^2)^{3/2}\frac{-ve^v}{u(e^v - 1 - \frac{1}{2}u^2)}.$$

$G''(x)$ is indeed negative, as it is easy to show that $e^v - 1 - \frac{1}{2}u^2$ is positive. This proves Corollary 2. □

Corollary 3 *Let $m(n)$ and $d(n)$ be the mean and mode of $d(\lambda)$, λ varying uniformly over all partitions of n. We have:*

$$m(n) = x_0 n^{1/2} + \frac{F'/F}{-G''} + O(n^{-1/2}).$$

In particular, for n sufficiently large,

$$m(n) - \tfrac{1}{2} \leq d(n) \leq m(n) + \tfrac{1}{2},$$

so $|m(n) - d(n)| \leq 1/2$.

Proof. Knowing the existence of $H(x)$ gives us

$$\mathbf{P}(n, x_0 n^{1/2} + t) = \frac{F e^{G\sqrt{n}+H/\sqrt{n}}}{n^{5/4}}\exp\{\frac{\frac{1}{2}t^2 G'' + tF'/F}{n^{1/2}} + O(n^{-1})\}. \qquad (3.13)$$

Here we follow the convention that $F, F'G, G''$ denote the latter functions evaluated at the special value x_0. Let t_0 be such that the expression $\frac{1}{2}t^2G'' + tF'/F$ is the same at both t_0 and $t_0 + 1$:

$$t_0 = \frac{F'/F}{-G''} - \tfrac{1}{2}.$$

It need not be the case that $x_0 n^{1/2} + t_0$ is an integer. It is not hard to check that

$$d < x_0 n^{1/2} + t_0 \Rightarrow \mathbf{P}(n,d) > \mathbf{P}(n,d-1),$$

and

$$d > x_0 n^{1/2} + t_0 + 1 \Rightarrow \mathbf{P}(n,d) > \mathbf{P}(n,d+1).$$

Hence,

$$x_0 n^{1/2} + t_0 \leq d(n) \leq x_0 n^{1/2} + t_0 + 1.$$

Multiplying both sides of (3.13) by $x_0 n^{1/2} + t$, summing over t, and dividing by $p(n)$ gives

$$D(n) = x_0 n^{1/2} + \frac{F'F}{-G''},$$

and this completes the proof. □

Note that the assertions of the three corollaries are consequences of the conjecture that the Durfee polynomial $P_{\mathbf{P},n}(y)$ has only real roots; as such, they may be taken as evidence of this conjecture.

4 Analytical Expressions for the Constants

Assume that the counting function $\mathbf{F}(n,d)$ is given, uniformly in d, by an asymptotic formula

$$\mathbf{F}(n,d) \;=\; \frac{f(u)}{n^a} \exp\{n^{1/2} g(u) + \mathrm{o}(1)\}, \quad \mathrm{u} = \mathrm{d}/\mathrm{n}^{1/2} \tag{4.1}$$

for suitable functions $f(u)$, $g(u)$ and exponent a. Associate with the recursion (2.1) the two polynomials

$$Q(w) = \sum_{i=1}^{m_0} C_i w^{e_i}, \quad R(w) = \sum_{i=1}^{m_1} D_i w^{g_i}. \tag{4.2}$$

Substitution of (4.1) into the recursion (2.1) produces differential equations for $f(u)$ and $g(u)$. This formal method requires the use of Taylor's series, as illustrated here:

$$\begin{aligned}
(n - e_i d + f_i)^{1/2} &= n^{1/2} - \tfrac{1}{2} e_i u + \cdots \\
d(n - e_i d + f_i)^{-1/2} &= u + \frac{\tfrac{1}{2} e_i u^2}{n^{1/2}} + \cdots \\
(n - e_i d + f_i)^{1/2} g(d(n - e_i d + f_i)^{-1/2}) &= n^{1/2} g(u) - e_i v + \cdots
\end{aligned}$$

in which the frequently appearing function $v = v(u)$ is defined by

$$v \;=\; \tfrac{1}{2} u g(u) - \tfrac{1}{2} u^2 g'(u). \tag{4.3}$$

In the above Taylor series, the ellipsis "$\cdots$" denotes terms of lower order. Further terms are needed to determine the function $f(u)$, (here we need only $g(u)$). See [3] for a detailed example. When all terms on the right of (2.1) are expanded similarly, substitution of (4.1) into (2.1) yields, after division by common factors,

$$1 \;=\; Q(e^{-v}) + e^{-g'}R(e^{-v}), \tag{4.4}$$

with $Q(w)$, $R(w)$ given in (4.2). Differentiating with respect to u and multiplying by -1:

$$0 \;=\; (Q'(e^{-v}) + e^{-g'}R'(e^{-v}))e^{-v}\frac{dv}{du} \;+\; e^{-g'}g''R(e^{-v}). \tag{4.5}$$

We can eliminate g'' and $e^{-g'}$ from the previous by, first, differentiating the definition (4.3) of v with respect to u, and rearranging:

$$g'' \;=\; \frac{-1}{v}\cdot(\frac{v^2}{u^2})',$$

and, second, solving (4.5) for $e^{-g'}$. Isolating the term $(v^2/u^2)'$ in the result, we obtain v implicitly as a function of u (and u explicitly as a function of v):

$$\frac{v^2}{u^2} \;=\; \int_0^v H(t)\,dt, \tag{4.6}$$

the function $H(t)$ being the integrand which appears below in equation (4.7). The mode of $\mathbf{F}(n,d)$ occurs at $d = u_0 n^{1/2}$, where $g'(u_0) = 0$. The value of u_0 satisfying the latter condition is obtained by first solving the following polynomial in e^{-v_0}

$$1 \;=\; Q(e^{-v_0}) \;+\; R(e^{-v_0}),$$

(thus v_0 is the logarithm of a certain algebraic number), and then using

$$u_0 \;=\; v_0 \;\times\; \left[\int_0^{v_0} t\Big(\frac{Q'(e^{-t})e^{-t}}{1-Q(e^{-t})} + \frac{R'(e^{-t})e^{-t}}{R(e^{-t})}\Big)dt\right]^{-1/2} \tag{4.7}$$

for u_0. For all the families of partitions which we have considered, the preceding integral can be evaluated in closed form using the dilogarithm function $\mathrm{Li}(X)$. For instance, in the case of the basis partitions $\mathbf{B}(n,d)$, we find that e^{-v_0} is the positive root of the cubic

$$X^3 + X^2 + X \;=\; 1,$$

and $u_0 = 0.6192194165\cdots$ is given by

$$u_0^2 \;=\; \frac{v_0^2}{-v_0^2 + \pi^2/4 - 2\,\mathrm{Li}(X) + \frac{1}{2}\,\mathrm{Li}(X^2)}.$$

The possibility of evaluating a dilogarithm by asymptotic partition counting was suggested by Andrews and carried out by Richmond and Szekeres [14]. See also [10].

5 Conclusion

All of the partition families (1) - (9) of Section 2 appear to have in common that the sequences $\{\mathbf{F}(n,d)\}$ are strictly log-concave, that $|mean\{\mathbf{F}(n,d)\} - mode\{\mathbf{F}(n,d)\}| < 1$ and, further, for (1) - (8), that the Durfee polynomial has all roots real. The common form of their generating functions may suffice to guarantee that all roots of the Durfee polynomial are real. Note that the family $\mathbf{Z}$ that has complex roots is the only one that does not have anything below the Durfee square.

These properties have been studied for many combinatorial sequences [15, 2] and in particular for the sequences $\{\mathbf{f}(n,k)\}$ for fixed n, where $\mathbf{f}(n,k)$ is the number of partitions of n in $\mathbf{F}(n)$ and k is the size of a chosen parameter. For example, if $\mathbf{p}(n,k)$ is the number of partitions of n with exactly k parts, the polynomial $\sum \mathbf{p}(n,k)y^k$, $0 \le k \le \lfloor n \rfloor$, does not, in general, have all roots real and the sequences $\{\mathbf{p}(n,k)\}$ are not log-concave, but are unimodal. Also, the difference between the mean and the mode is unbounded [11]. If $\mathbf{d}(n,k)$ is the number of partitions of n with exactly k distinct parts, the polynomial $\sum \mathbf{d}(n,k)y^k$, $0 \le k \le \lfloor n \rfloor$, does not, in general, have all roots real and the sequences $\{\mathbf{d}(n,k)\}$ seem to be log-concave. Also, the difference between the mean and the mode is less than one [11]. However for these sequences derived from partitions, combinatorial techniques seem difficult to apply. In fact, Szekeres' analytic proof [16] is the only proof that $\{\mathbf{p}(n,k)\}$ and $\{\mathbf{d}(n,k)\}$ are unimodal for n sufficiently large. No combinatorial proof of this unimodality exists.

Because of similar difficulties, we have only been able to show that our results on log-concavity hold only for n sufficiently large. We want to remark that $\mathbf{F}(n,d)$ can be expressed as a convolution of two sequences $\{\mathbf{g}(n,d)\}$, the number of partitions of n into (at most) d parts in some family $\mathbf{G}$. As log-concavity is closed under convolution, combinatorial proofs of the log-concavity of sequences $\{\mathbf{g}(n,d)\}$, would generalize the results for any n.

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