ANTI-LECTURE HALL COMPOSITIONS AND ANDREWS' GENERALIZATION OF THE WATSON-WHIPPLE TRANSFORMATION

SYLVIE CORTEEL, JEREMY LOVEJOY AND CARLA SAVAGE

ABSTRACT. For fixed n and k, we find a three-variable generating function for the set of sequences $(\lambda_1, \ldots, \lambda_n)$ satisfying

$$k \geq \frac{\lambda_1}{a_1} \geq \frac{\lambda_2}{a_2} \geq \ldots \geq \frac{\lambda_n}{a_n} \geq 0,$$

where $a := (a_1, \ldots, a_n) = (1, 2, \ldots, n)$ or $(n, n-1, \ldots, 1)$. When $k \to \infty$ we recover the refined anti-lecture hall and lecture hall theorems. When $a = (1, 2, \ldots, n)$ and $n \to \infty$, we obtain a refinement of a recent result of Chen, Sang and Shi. The main tools are elementary combinatorics and Andrews' generalization of the Watson-Whipple transformation.

1. Introduction and main result

A *lecture hall partition* of length n is a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_n}{1} \geq 0.$$

Let L_n denote the set of such partitions. These were introduced and extensively studied in a series of three papers by Bousquet-Mélou and Eriksson [3, 4, 5]. In their third paper they established the three-variable generating function

$$\mathcal{L}_n(u, v, q) := \sum_{\lambda \in L_n} u^{|\lceil \lambda \rceil|} v^{\mathrm{o}(\lceil \lambda \rceil)} q^{|\lambda|} = \frac{(-uvq)_n}{(u^2 q^{n+1})_n}, \tag{1.1}$$

where

$$\lceil \lambda \rceil = (\lceil \lambda_1/n \rceil, \ldots, \lceil \lambda_n/1 \rceil),$$
$$|\lambda| = \lambda_1 + \cdots + \lambda_n,$$
$$(a)_n := (a;q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \tag{1.2}$$

and $\mathrm{o}(w)$ is the number of odd parts in a sequence $w = (w_1, \ldots, w_n)$.

Subsequently the first and last author introduced the set A_n of *anti-lecture hall compositions* [7]. These are compositions $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \geq \frac{\lambda_n}{n} \geq 0.$$

Date: March 6, 2015.

The first two authors are partially funded by ANR-08-JCJC-0011. The third author is partially funded by Simons Foundation Grant 244963.

The analogue of (1.1) for anti-lecture hall compositions is [7]

$$\mathcal{A}_n(u,v,q) := \sum_{\lambda \in A_n} u^{|\lfloor \lambda \rfloor|} v^{o(\lfloor \lambda \rfloor)} q^{|\lambda|} = \frac{(-uvq)_n}{(u^2q^2)_n}, \tag{1.3}$$

where

$$\lfloor \lambda \rfloor = (\lfloor \lambda_1/1 \rfloor, \ldots, \lfloor \lambda_n/n \rfloor).$$

In a recent paper, Chen, Sang and Shi [6] studied anti-lecture hall compositions in A_n satisfying $\lambda_1 \leq k$. Let $A_{n,k}$ denote the set of these compositions, and let $\mathcal{A}_{n,k}(u,v,q)$ denote the generating function

$$\mathcal{A}_{n,k}(u,v,q) = \sum_{\lambda \in A_{n,k}} u^{|\lfloor \lambda \rfloor|} v^{o(\lfloor \lambda \rfloor)} q^{|\lambda|}.$$

The main result in [6] is the identity

$$\mathcal{A}_{\infty,k}(1,1,q) = \frac{(-q)_\infty (q, q^{k+1}, q^{k+2}; q^{k+2})_\infty}{(q)_\infty}, \tag{1.4}$$

where the notation in (1.2) is extended to

$$(a_1, \ldots, a_j)_n = (a_1;q)_n \cdots (a_j;q)_n = \prod_{i=0}^{n-1} (1 - a_1 q^i) \cdots (1 - a_j q^i). \tag{1.5}$$

To prove (1.4), Chen, Sang and Shi first used two long and involved combinatorial arguments, motivated by constructions of Bousquet-Mélou and Eriksson [5] and depending on the parity of k, to express $\mathcal{A}_{\infty,k}(1,1,q)$ as a q-hypergeometric multisum. Then they applied Andrews' generalization of the Watson-Whipple transformation (see (2.3)) to convert the multisum to an infinite product. For example, in the even case their combinatorial argument gives

$$\mathcal{A}_{\infty,2k-2}(1,1,q) = \sum_{n_1 \geq n_2 \geq \ldots n_{k-1} \geq 0} \frac{q^{\binom{n_1+1}{2} + 2\binom{n_2+1}{2} + \ldots + 2\binom{n_{k-1}+1}{2}} (-q)_{n_1}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-2}-n_{k-1}} (q)_{n_{k-1}}}, \tag{1.6}$$

and then an appropriate specialization of (2.3) turns (1.6) into (1.4). (For other combinatorial interpretations of the multisum in (1.6), see [11].)

Here we observe that an elementary recurrence combined with a different application of Andrews' transformation leads swiftly and neatly to the following generating function for $\mathcal{A}_{n,k}(u,v,q)$, and as a consequence, equations (1.3) and (1.4). We employ the q-binomial coefficient,

$$\begin{bmatrix} n \\ m \end{bmatrix} := \frac{(q)_n}{(q)_{n-m}(q)_m}, \tag{1.7}$$

which we note (for later use) is the generating function for partitions into at most $n-m$ parts, each less than or equal to m.

Theorem 1.1. *If k is odd then*

$$\mathcal{A}_{n,k}(u,v,q) = \frac{(-uvq)_n}{(u^2q^2)_n} \sum_{m=0}^{n} \frac{(1-u^2q^{2m+1})}{(1-u^2q)} \frac{(u^2q)_m (-1)^m q^{k\binom{m+1}{2} + m^2} u^{(k+1)m}}{(u^2q^{n+2})_m} \begin{bmatrix} n \\ m \end{bmatrix}, \tag{1.8}$$

and if k is even then

$$\mathcal{A}_{n,k}(u,v,q) = \frac{(-uvq)_n}{(u^2q^2)_n} \sum_{m=0}^{n} \frac{(1-u^2q^{2m+1})}{(1-u^2q)} \frac{(u^2q)_m(-uq/v)_m(-1)^m q^{k\binom{m+1}{2}+m^2} u^{(k+1)m} v^m}{(u^2q^{n+2})_m(-uvq)_m} \begin{bmatrix} n \\ m \end{bmatrix}. \tag{1.9}$$

Note that if $v=1$ then equations (1.8) and (1.9) are identical. Namely, for all k,

$$\mathcal{A}_{n,k}(u,1,q) = \frac{(-uq)_n}{(u^2q^2)_n} \sum_{m=0}^{n} \frac{(1-u^2q^{2m+1})}{(1-u^2q)} \frac{(u^2q)_m(-1)^m q^{k\binom{m+1}{2}+m^2} u^{(k+1)m}}{(u^2q^{n+2})_m} \begin{bmatrix} n \\ m \end{bmatrix}. \tag{1.10}$$

Also note that if $k \to \infty$ in (1.8) or (1.9), only the $m=0$ term survives and we obtain (1.3). If $n \to \infty$ and $u=v=1$, then the sums on the right-hand sides reduce to

$$\sum_{m \geq 0} \frac{(-1)^m(1-q^{2m+1})q^{k\binom{m+1}{2}+m^2}}{1-q} = \sum_{m \in \mathbb{Z}} \frac{(-1)^m q^{k\binom{m+1}{2}+m^2}}{1-q} = \frac{(q,q^{k+1},q^{k+2};q^{k+2})_\infty}{1-q}$$

by the triple product identity,

$$\sum_{n \in \mathbb{Z}} z^n q^{\binom{n+1}{2}} = (-1/z, -zq, q; q)_\infty, \tag{1.11}$$

and this gives (1.4). In fact, the sum on the right-hand side of (1.8) is independent of v, so we have the following refinement of (1.4) for odd k:

Corollary 1.2. *If $k \geq 1$ then*

$$\mathcal{A}_{\infty,2k-1}(1,v,q) = \frac{(-vq)_\infty(q,q^{2k},q^{2k+1};q^{2k+1})_\infty}{(q)_\infty}.$$

We present the proof of Theorem 1.1 in Section 2, and in Section 3 we show how Theorem 1.1 implies a similar result for lecture hall partitions. In Section 4 we also show how to extend the results to the truncated anti-lecture hall compositions and truncated lecture hall partitions. We conclude with some remarks.

2. Proof of Theorem 1.1

We use a decomposition of anti-lecture hall compositions given in [9, Proposition 7], namely:

Lemma 2.1. *For $k > 0$, we have*

$$\mathcal{A}_{n,k}(u,v,q) = \sum_{m=0}^{n} \mathcal{A}_{m,k-1}(u,1/v,q) u^m v^m q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}. \tag{2.1}$$

Moreover $\mathcal{A}_{n,0}(u,v,q) = 1$.

Proof. The proof is straightforward. Given an anti-lecture hall composition λ in $A_{n,k}$, let m be the largest index such that $\lambda_i \geq i$. Then $(\lambda_1 - 1, \lambda_2 - 2, \dots, \lambda_m - m)$ is in $A_{m,k-1}$ and $(\lambda_{m+1}, \lambda_{m+2}, \dots, \lambda_n)$ is a partition into $n-m$ non negative parts which are at most m. □

Iterating Lemma 2.1 gives the following generating function.

Proposition 2.2. *We have*

$$\mathcal{A}_{n,k}(u,v,q) = \sum_{n\geq n_k\geq n_{k-1}\geq\ldots\geq n_1\geq 0} \frac{u^{\sum_{i=1}^k n_i} v^{\sum_{i=1}^k (-1)^{k-i}n_i} q^{\sum_{i=1}^k \binom{n_i+1}{2}} (q)_n}{(q)_{n-n_k}\cdots(q)_{n_2-n_1}(q)_{n_1}}. \tag{2.2}$$

To finish the proof of Theorem 1.1, we need Andrews' transformation [2],

$$\sum_{m=0}^{N} \frac{(1-aq^{2m})}{(1-a)} \frac{(a,b_1,c_1,\ldots,b_k,c_k,q^{-N})_m}{(q,aq/b_1,aq/c_1,\ldots,aq/b_k,aq/c_k,aq^{N+1})_m} \left(\frac{a^k q^{k+N}}{b_1c_1\cdots b_kc_k}\right)^m$$
$$= \frac{(aq,aq/b_kc_k)_N}{(aq/b_k,aq/c_k)_N} \sum_{N\geq n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{(b_k,c_k)_{n_{k-1}}\cdots(b_2,c_2)_{n_1}(aq)^{n_{k-2}+\cdots+n_1}q^{n_{k-1}}}{(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}(b_{k-1}c_{k-1})^{n_{k-2}}\cdots(b_2c_2)^{n_1}}$$
$$\times \frac{(q^{-N})_{n_{k-1}}(aq/b_{k-1}c_{k-1})_{n_{k-1}-n_{k-2}}\cdots(aq/b_2c_2)_{n_2-n_1}(aq/b_1c_1)_{n_1}}{(b_kc_kq^{-N}/a)_{n_{k-1}}(aq/b_{k-1},aq/c_{k-1})_{n_{k-1}}\cdots(aq/b_1,aq/c_1)_{n_1}}. \tag{2.3}$$

In this transformation we replace k by $k+1$ and N by n, set $a=u^2q$, $b_i = -uqv^{(-1)^{i+k}}$, and let $c_i\to\infty$. Simplifying using standard q-series limits and the identity

$$(q^{-n})_j = \frac{(q)_n}{(q)_{n-j}}(-1)^j q^{\binom{j}{2}-nj},$$

we obtain equations (1.8) and (1.9).

3. Application to Lecture Hall partitions

We can use Theorem 1.1 to compute similar generating functions for lecture hall partitions with largest part less than or equal to nk, i.e., sequences $\lambda=(\lambda_1,\ldots,\lambda_n)$ such that

$$k\geq\frac{\lambda_1}{n}\geq\frac{\lambda_2}{n-1}\geq\ldots\geq\frac{\lambda_n}{1}\geq 0.$$

Let $L_{n,k}$ be the set of such partitions and write

$$\mathcal{L}_{n,k}(u,v,q) = \sum_{\lambda\in L_{n,k}} u^{|\lceil\lambda\rceil|}u^{\mathrm{o}(\lceil\lambda\rceil)}q^{|\lambda|}.$$

From [8, Corollary 3], we know that if k is odd, then

$$\mathcal{L}_{n,k}(u,v,q) = u^{kn}v^nq^{k\binom{n+1}{2}}\mathcal{A}_{n,k}(1/u,1/v,1/q), \tag{3.1}$$

and if k is even, then

$$\mathcal{L}_{n,k}(u,v,q) = u^{kn}q^{k\binom{n+1}{2}}\mathcal{A}_{n,k}(1/u,v,1/q). \tag{3.2}$$

Using Theorem 1.1 together with (3.1) and (3.2), we obtain the following:

Theorem 3.1. *If k is odd then*

$$\mathcal{L}_{n,k}(u,v,q) = \frac{(-uvq)_n}{(u^2q^2)_n}\sum_{m=0}^{n}\frac{(1-u^2q^{2m+1})}{(1-u^2q)}\frac{(u^2q)_m(-1)^{n+m}q^{k\left(\binom{n+1}{2}-\binom{m+1}{2}\right)+n-m}u^{(k+1)(n-m)}}{(u^2q^{n+2})_m}\begin{bmatrix}n\\m\end{bmatrix}, \tag{3.3}$$

and if k is even then

$$\mathcal{L}_{n,k}(u,v,q) = \frac{(-uq/v)_n}{(u^2q^2)_n} \times \sum_{m=0}^{n} \frac{(1-u^2q^{2m+1})}{(1-u^2q)} \frac{(u^2q)_m(-uvq)_m(-1)^{n+m}q^{k\left(\binom{n+1}{2}-\binom{m+1}{2}\right)+n-m}u^{(k+1)(n-m)}v^{n-m}}{(u^2q^{n+2})_m(-uq/v)_m} \begin{bmatrix} n \\ m \end{bmatrix}. \tag{3.4}$$

When $k \to \infty$, only the term $m = n$ in the sum survives and we recover (1.1). When $q \to 1^-$, applying the classical binomial theorem gives

$$\mathcal{L}_{n,k}(u,v,1) = \left(\frac{(1+uv)(1-u^{k+1})}{1-u^2}\right)^n \tag{3.5}$$

for k odd and

$$\mathcal{L}_{n,k}(u,v,1) = \left(\frac{1+uv-u^{k+1}v-u^{k+2}}{1-u^2}\right)^n \tag{3.6}$$

for k even. Setting $v = 1$ in either, we recover the "Mahonian" generating function from [12, Corollary 1]:

$$\mathcal{L}_{n,k}(u,1,1) = \left(\frac{1-u^{k+1}}{1-u}\right)^n. \tag{3.7}$$

4. Truncated objects

Let $A_{n,j,k}$ be the set of sequences $(\lambda_1, \dots, \lambda_j)$ such that

$$k \geq \frac{\lambda_1}{n-j+1} \geq \frac{\lambda_2}{n-j+2} \geq \dots \geq \frac{\lambda_j}{n} \geq 0.$$

These are called *truncated anti-lecture hall compositions* [9]. Let

$$\mathcal{A}_{n,j,k}(u,v,q) = \sum_{\lambda \in A_{n,j,k}} u^{|\lfloor\lambda\rfloor|} v^{o(\lfloor\lambda\rfloor)} q^{|\lambda|}$$

where $\lfloor\lambda\rfloor = (\lfloor\lambda_1/(n-j+1)\rfloor, \dots, \lfloor\lambda_j/n\rfloor)$.

Arguing in the spirit of Lemma 2.1 (see also Proposition 8 of [9]), one obtains the following recurrence :

$$\mathcal{A}_{n,j,k}(u,v,q) = \begin{bmatrix} n \\ j \end{bmatrix} \mathcal{A}_{j,k-1}(uq^{n-j}, v, q) + v^{\mathrm{odd}(k)} u^k q^{k(n-j+1)} \mathcal{A}_{n,j-1,k}(u,v,q) \tag{4.1}$$

if $j > 0$ and $k > 0$, and $\mathcal{A}_{n,0,k}(u,v,q) = \mathcal{A}_{n,j,0}(u,v,q) = 1$. Here $\mathrm{odd}(k) = 1$ if k is odd and 0 otherwise. This gives

$$\mathcal{A}_{n,j,k}(u,v,q) = \sum_{m=0}^{j} \begin{bmatrix} n \\ m \end{bmatrix} \mathcal{A}_{m,k-1}(uq^{n-m}, v, q) v^{\mathrm{odd}(k)(j-m)} u^{k(j-m)} q^{k(j-m)(n-j+1)+\binom{j-m}{2}}, \tag{4.2}$$

and an application of Theorem 1.1 then gives a double sum formula for $\mathcal{A}_{n,j,k}(u,v,q)$. When $k \to \infty$, only the term $j = m$ in (4.2) survives and we recover Theorem 2 of [9],

$$\mathcal{A}_{n,j,\infty} = \begin{bmatrix} n \\ j \end{bmatrix} \frac{(-uvq^{n-j+1})_j}{(u^2q^{2(n-j+1)})_j}. \tag{4.3}$$

Next let $L_{n,j,k}$ be the set of sequences $(\lambda_1,\ldots,\lambda_j)$ such that

$$k \geq \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_j}{n-j+1} \geq 0.$$

These are called *truncated lecture hall partitions* [9]. Let

$$\mathcal{L}_{n,j,k}(u,v,q) = \sum_{\lambda \in L_{n,j,k}} u^{|\lceil\lambda\rceil|} v^{o(\lceil\lambda\rceil)} q^{|\lambda|},$$

where $\lceil\lambda\rceil = (\lceil\lambda_1/n\rceil,\ldots,\lceil\lambda_j/(n-j+1)\rceil)$.

As usual, we can treat the lecture hall case by setting $q = 1/q$ in the anti-lecture hall case. Namely, as with equations (3.1) and (3.2), we have

$$\mathcal{L}_{n,j,k}(u,v,q) = u^{kj} v^j q^{k\binom{j+1}{2}+k(n-j)j} \mathcal{A}_{n,j,k}(1/u,1/v,1/q) \tag{4.4}$$

if k is odd and

$$\mathcal{L}_{n,j,k}(u,v,q) = u^{kj} q^{k\binom{j+1}{2}+k(n-j)j} \mathcal{A}_{n,j,k}(1/u,v,1/q) \tag{4.5}$$

if k is even. This gives a double sum formula for $\mathcal{L}_{n,j,k}(u,v,q)$, and when $k \to \infty$ we recover Theorem 1 of [9]. We leave the details to the reader.

5. Concluding remarks

The results in this paper raise a number of interesting combinatorial prospects. For example, if we set $v = 0$ in Corollary 1.2, we obtain a relation between anti-lecture hall compositions λ with $\lfloor\lambda\rfloor$ containing only even parts and certain of the Andrews-Gordon identities (for $k = 2$ this is the second Rogers Ramanujan identity).

Another promising line of research would be to investigate the relationship between q-series identities like (2.3) and Ehrhart theory, following up on the connections between lecture hall objects and Ehrhart theory made by the third author [10, 12].

6. Acknowledgements

The authors are indebted to the referees for their careful reading of the paper and suggestions for its improvement. The first author would like to thank the Simons Foundation, who funded her trip to NCSU where part of this work was done.

References

[1] George E. Andrews, The theory of partitions. Reprint of the 1976 original. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. xvi+255 pp.

[2] George E. Andrews, Problems and prospects for basic hypergeometric series, in: R. Askey, Theory and Application of Special Functions, Academic Press, New York 1975, 191–224.

[3] Mireille Bousquet-Mélou and Kimmo Eriksson, Lecture hall partitions, Ramanujan J. **1** No.1 (1997) 101–111.

[4] Mireille Bousquet-Mélou and Kimmo Eriksson, Lecture hall partitions 2, Ramanujan J. **1** No.2 (1997) 165–185.

[5] Mireille Bousquet-Mélou and Kimmo Eriksson, A refinement of the lecture hall theorem. J. Combin. Theory Ser. A 86 (1999), no. 1, 63–84.

[6] William Y. C. Chen, Doris D.M. Sang and Diane Y. H. Shi, Anti-lecture hall compositions and overpartitions. J. Combin. Theory Ser. A 118 (2011), no. 4, 1451–1464.

[7] Sylvie Corteel and Carla D. Savage, Anti-lecture hall compositions. Discrete Math. 263 (2003), no. 1-3, 275–280.

[8] Sylvie Corteel, Sunyoung Lee, and Carla D. Savage, Enumeration of sequences constrained by the ratio of consecutive parts. Sém. Lothar. Combin. 54A (2005/07), Art. B54Aa, 12 pp.
[9] Sylvie Corteel and Carla D. Savage, Lecture hall theorems, q-series and truncated objects. J. Combin. Theory Ser. A 108 (2004), no. 2, 217–245.
[10] Fu Liu and Richard Stanley, The lecture hall parallelepiped, Ann. Comb., 18 (2014), no. 3, 473-488.
[11] Jeremy Lovejoy and Olivier Mallet, Overpartition pairs and two classes of basic hypergeometric series, Adv. Math. **217** (2008), 386–418.
[12] Carla D. Savage and Michael Schuster, Ehrhart series of lecture hall polytopes and Eulerian polynomials for inversion sequences. J. Combin. Theory Ser. A 119 (2012), no. 4, 850–870.

LIAFA, CNRS et Université Paris Diderot, Paris, France
E-mail address: `corteel@liafa.univ-paris-diderot.fr`

LIAFA, CNRS et Université Paris Diderot, Paris, France
E-mail address: `lovejoy@math.cnrs.fr`

Department of Computer Science, NCSU, Raleigh NC, USA
E-mail address: `savage@ncsu.edu`