

# On an identity of Gessel and Stanton and the new little Göllnitz identities

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*Dedicated to Dennis Stanton on the occasion of his 60th birthday.*

## Abstract

We show that an identity of Gessel and Stanton (*Trans. Amer. Math Soc.* 277 (1983), p. 197, Eq. (7.24)) can be viewed as a symmetric version of a recent analytic variation of the little Göllnitz identities. This is significant, since the Göllnitz-Gordon identities are considered the usual symmetric counterpart to little Göllnitz theorems. Is it possible, then, that the Gessel-Stanton identity is part of an infinite family of identities like those of Göllnitz-Gordon?

Toward this end, we derive partners and generalizations of the Gessel-Stanton identity. We show that the new little Göllnitz identities enumerate partitions into distinct parts in which even-indexed (resp. odd-indexed) parts are even, and derive a refinement of the Gessel-Stanton identity that suggests a similar interpretation is possible. We study an associated system of  $q$ -difference equations to show that the Gessel-Stanton identity and its partner are actually two members of a three-element family.

**Keywords:** integer partitions,  $q$ -series identities,  $q$ -Gauss summation, little Göllnitz partition theorems, Göllnitz-Gordon partition theorem, Lebesgue identity

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# 1 Introduction

In 1983, Gessel and Stanton presented the following Rogers-Ramanujan type identity [11, p. 197, Eq. (7.24)]:

$$\sum_{n=0}^{\infty} \frac{(-q; q^2)_{2n} q^{2n^2}}{(q^8; q^8)_n (q^2; q^4)_n} = (-q^3, -q^5; q^8)_{\infty} (-q^2; q^2)_{\infty}, \quad (1.1)$$

where

$$(a; q)_{\infty} := \prod_{j=0}^{\infty} (1 - aq^j),$$

$$(a_1, a_2, \dots, a_r; q)_{\infty} := (a_1; q)_{\infty} (a_2; q)_{\infty} \cdots (a_r; q)_{\infty},$$

and

$$(a; q)_n := (a)_{\infty} / (aq^n)_{\infty}.$$

If the base in a rising  $q$ -factorial is omitted, it is assumed to be  $q$ , i.e.

$$(a)_{\infty} := (a; q)_{\infty} \text{ and } (a)_n := (a; q)_n.$$

Rogers-Ramanujan type identities rarely occur in isolation; where there is one, there is often one or more “partners” and these identities generalize in a number of ways. We will use a variety of techniques to derive partners and generalizations.

Identity (1.1) was one of many derived by Gessel and Stanton in their 1983 paper on  $q$ -Lagrange inversion [11]. This identity came to our attention as a candidate for a symmetric version of a recently discovered variation of the little Göllnitz identities [9] in the same way that the Göllnitz-Gordon theorem is a symmetric version of the familiar little Göllnitz identities.

As such, there were natural questions to ask about the Gessel-Stanton identity (1.1), such as: what would its partner(s) be? Does it have a combinatorial interpretation that relates it to the little Göllnitz identities as do the Göllnitz -Gordon identities? Does it have a multiparameter generalization? And what do those parameters count? Does the identity generalize to an infinite family as the Göllnitz-Gordon identities do?

In this paper we answer some of these questions and suggest approaches to others.

Section 2 supplies the background on the the Göllnitz-Gordon theorem and its relationship to the little Göllnitz identities. We describe the “new” little Göllnitz identities, how they are related to the Gessel-Stanton identity (1.1), and why one would expect a partner for (1.1).

In Section 3, we derive a 3-parameter generalization of (1.1) and a partner for (1.1) from Andrews’  $q$ -analog of Bailey’s sum [3, p. 526, Eq. (1.9)]. The new little Göllnitz identities have a similar generalization and there is a combinatorial interpretation of the generalized infinite products that encompasses both pairs.

In Section 4, the focus is on the infinite sums. We prove an interpretation of the new little Göllnitz identities, which in itself is interesting, but which we conjecture can extend to the Gessel-Stanton pair.

In Section 5 we describe the generalized Göllnitz-Gordon identities and consider whether there is an analogous generalization of the Gessel-Stanton identities to an infinite family. We study associated  $q$ -difference equations via the methods Andrews used to derive infinite families for the Rogers-Ramanujan identities and the Göllnitz-Gordon identities [5, Chapter 7]. Although we are not successful at finding the combinatorial counterpart of the analogous infinite family, we show via this method that the Gessel-Stanton identity and its partner are actually two members of a three-element family of identities.

In Section 6 we suggest directions for further inquiry.

## 2 Background

The little Göllnitz identities [12, Eq. (2.22, 2.24)] have the analytic form:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^{-1}; q^2)_n}{(q^2; q^2)_n} = \prod_{\substack{k \geq 1 \\ k \equiv 1,5,6 \pmod{8}}} \frac{1}{1-q^k} \quad (2.1)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q; q^2)_n}{(q^2; q^2)_n} = \prod_{\substack{k \geq 1 \\ k \equiv 2,3,7 \pmod{8}}} \frac{1}{1-q^k}. \quad (2.2)$$

In contrast, the following Göllnitz-Gordon identities (Göllnitz [12, pp. 162–163, Satz 2.1 and 2.2], Gordon [13, p. 741, Thms. 2 and 3]; cf. Slater [16, p. 155, Eq. (36) and (34)] and Ramanujan [8, p. 37, Entries 1.7.12 and 1.7.13]) involve *symmetric* residue classes modulo 8:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q^2)_n}{(q^2; q^2)_n} = \prod_{\substack{k \geq 1 \\ k \equiv 1,4,7 \pmod{8}}} \frac{1}{1-q^k} \quad (2.3)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q; q^2)_n}{(q^2; q^2)_n} = \prod_{\substack{k \geq 1 \\ k \equiv 3,4,5 \pmod{8}}} \frac{1}{1-q^k}. \quad (2.4)$$

A set,  $R = \{r_1, r_2, \dots, r_k\}$ , of residue classes modulo  $m$ , is *symmetric* if  $R = \{m - r_1, m - r_2, \dots, m - r_k\}$ . It is immediately clear that identities (2.1–2.4) are “close relatives”, with (2.1) and (2.2) as asymmetric variants of (2.3) and (2.4). In the same way, we will

show that (2.1) and (2.2) are close relatives of *another* pair of asymmetric variants, one of which is the Gessel-Stanton identity (1.1).

The similarity of identities (2.1–2.4) is reinforced when their combinatorial versions are considered. The combinatorial counterparts to (2.1) and (2.2) are known as “Göllnitz’s little partition theorems” [12, pp. 166–167, Satz 2.3 and 2.4]:

**Theorem 2.1** (Göllnitz). *The number of partitions of  $N$  into parts differing by at least 2 and no consecutive odd parts equals the number of partitions of  $N$  into parts congruent to 1, 5 or 6 (mod 8).*

**Theorem 2.2** (Göllnitz). *The number of partitions of  $N$  into parts differing by at least 2, no consecutive odd parts, and no ones equals the number of partitions of  $N$  into parts congruent to 2, 3 or 7 (mod 8).*

Similarly, combinatorial counterparts to (2.3) and (2.4) are the following:

**Theorem 2.3** (Göllnitz-Gordon). *The number of partitions of  $N$  into parts differing by at least 2 and no consecutive even parts equals the number of partitions of  $N$  into parts congruent to 1, 4 or 7 (mod 8).*

**Theorem 2.4** (Göllnitz-Gordon). *The number of partitions of  $N$  into parts differing by at least 2, no consecutive even parts, and no ones equals the number of partitions of  $N$  into parts congruent to 3, 4 or 5 (mod 8).*

The identities (2.1) and (2.2) are special cases  $L(-q^{-1}; q^2)$  and  $L(-q; q^2)$ , respectively, of an identity due to V. A. Lebesgue ([14]; cf. [5, p. 21, Cor. 2.7]):

$$L(a; q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} (a; q)_n}{(q; q)_n} = (-q; q)_{\infty} (aq; q^2)_{\infty}. \quad (2.5)$$

We use here and throughout the fact that  $(q; q^2)_{\infty}^{-1} = (-q; q)_{\infty}$ .

Recall Heine’s  $q$ -Gauss summation [10, p. 354, Eq. (II.8)]:

$$H(a, b, c; q) := \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} \left( \frac{c}{ab} \right)^n = \frac{(c/a; q)_{\infty} (c/b; q)_{\infty}}{(c; q)_{\infty} (c/(ab); q)_{\infty}}. \quad (2.6)$$

As in [9], we can use the following specialization of  $q$ -Gauss

$$H(a, \infty, c; q) = \sum_{n=0}^{\infty} \frac{(a; q)_n (-c/a)^n q^{\binom{n}{2}}}{(c; q)_n (q; q)_n} = \frac{(c/a; q)_{\infty}}{(c; q)_{\infty}} \quad (2.7)$$

to derive the “new little Göllnitz identities”:

$$H(-q, \infty, q^2; q^4) = \sum_{n=0}^{\infty} \frac{q^{2n^2-n} (-q; q^4)_n}{(q^2; q^2)_{2n}} = \frac{1}{(q; q^4)_{\infty} (q^6; q^8)_{\infty}} \quad (2.8)$$

and

$$H(-q^{-1}, \infty, q^2; q^4) = \sum_{n=0}^{\infty} \frac{q^{2n^2+n}(-q^{-1}; q^4)_n}{(q^2; q^2)_{2n}} = \frac{1}{(q^2; q^8)_{\infty}(q^3; q^4)_{\infty}}. \quad (2.9)$$

It is surprising that the infinite products in (2.8) and (2.9), which arose as special cases of the  $q$ -Gauss sum, are identical to those in (2.1) and (2.2).

If we now write the new little Göllnitz identities (2.8) and (2.9) in the form

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-n}(-q; q^4)_n(-q^4; q^4)_n}{(q^2; q^4)_n(q^8; q^8)_n} = \prod_{\substack{k \geq 1 \\ k \not\equiv 3,7 \pmod{8}}} (1 + q^k) \quad (2.10)$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+n}(-q^{-1}; q^4)_n(-q^4; q^4)_n}{(q^2; q^4)_n(q^8; q^8)_n} = \prod_{\substack{k \geq 1 \\ k \not\equiv 1,5 \pmod{8}}} (1 + q^k), \quad (2.11)$$

then (2.10) and (2.11) are revealed as asymmetric variants of

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(-q; q^4)_n(-q^3; q^4)_n}{(q^2; q^4)_n(q^8; q^8)_n} = \prod_{\substack{k \geq 1 \\ k \not\equiv 1,7 \pmod{8}}} (1 + q^k), \quad (2.12)$$

which, in a different guise, is the Gessel-Stanton identity (1.1). In the next section, we derive the following ‘‘Gessel-Stanley partner’’ for (2.12):

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^{-1}; q^4)_n(-q^5; q^4)_n}{(q^2; q^4)_n(q^8; q^8)_n} = \prod_{\substack{k \geq 1 \\ k \not\equiv 3,5 \pmod{8}}} (1 + q^k). \quad (2.13)$$

### 3 Three-parameter generalizations and a partner

Recall George Andrews’  $q$ -analog of Bailey’s sum [3, p. 526, Eq. (1.9)] (cf. [10, p. 354, Eq. (II.10)]):

$$G(b, c; q) := \sum_{n=0}^{\infty} \frac{(b)_n(q/b)_n c^n q^{n(n-1)/2}}{(q^2; q^2)_n (c)_n} = \frac{(cq/b; q^2)_{\infty} (bc; q^2)_{\infty}}{(c)_{\infty}}, \quad (3.1)$$

which follows from an application of the Bailey-Daum  $q$ -Kummer sum [10, p. 354, Eq. (II.9)] to a special case of Jackson’s  ${}_2\phi_1$  to  ${}_2\phi_2$  transformation [10, p. 359, Eq. (III.4)].

Then

$$G(-tq, zq^2; q^4) := \sum_{n=0}^{\infty} \frac{(-tq; q^4)_n (-t^{-1}q^3; q^4)_n z^n q^{2n^2}}{(q^8; q^8)_n (zq^2; q^4)_n} = \frac{(-t^{-1}zq^5; q^8)_{\infty} (-tzq^3; q^8)_{\infty}}{(zq^2; q^4)_{\infty}}. \quad (3.2)$$

Setting  $t = z = 1$ , we recover (2.12) and, simplifying, the Gessel-Stanton identity (1.1):

$$\sum_{n=0}^{\infty} \frac{(-q; q^2)_{2n} q^{2n^2}}{(q^8; q^8)_n (q^2; q^4)_n} = (-q^5; q^8)_{\infty} (-q^3; q^8)_{\infty} (-q^2; q^2)_{\infty}. \quad (3.3)$$

Similarly, in (3.1),

$$G(-t/q, zq^2; q^4) = \sum_{n=0}^{\infty} \frac{(-t/q; q^4)_n (-t^{-1}q^5; q^4)_n z^n q^{2n^2}}{(q^8; q^8)_n (zq^2; q^4)_n} = \frac{(-t^{-1}zq^7; q^8)_{\infty} (-tzq; q^8)_{\infty}}{(zq^2; q^4)_{\infty}}. \quad (3.4)$$

Setting  $t = z = 1$ , we recover the Gessel-Stanton partner (2.13) which, simplified, becomes

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-1} (-q; q^2)_{2n-1} (1 + q^{4n+1})}{(q^2; q^4)_n (q^8; q^8)_n} = (-q, -q^7; q^8)_{\infty} (-q^2; q^2)_{\infty}. \quad (3.5)$$

The new little Göllnitz identities have similar three-parameter generalizations:

$$H(-q/t, \infty, zq^2, q^4) = \sum_{n=0}^{\infty} \frac{(-q/t; q^4)_n t^n z^n q^{2n^2-n}}{(zq^2; q^4)_n (q^4; q^4)_n} = \frac{(-tzq; q^4)_{\infty}}{(zq^2; q^4)_{\infty}} \quad (3.6)$$

$$H(-(tq)^{-1}, \infty, zq^2, q^4) = \sum_{n=0}^{\infty} \frac{(-(tq)^{-1}; q^4)_n t^n z^n q^{2n^2+n}}{(zq^2; q^4)_n (q^4; q^4)_n} = \frac{(-tzq^3; q^4)_{\infty}}{(zq^2; q^4)_{\infty}} \quad (3.7)$$

The infinite products (3.2), (3.4), and (3.6), (3.7) have the interpretations, for  $i = 1, -1$ :

$$G(-tq^i, zq^2; q^4) = \sum_{\lambda \in \mathcal{S}_i} t^{s(\lambda)} z^{\ell(\lambda)} q^{|\lambda|}; \quad H(-q^i/t, \infty, zq^2, q^4) = \sum_{\lambda \in \mathcal{T}_i} t^{r(\lambda)} z^{\ell(\lambda)} q^{|\lambda|},$$

where  $\ell(\lambda)$  is the total number of parts of  $\lambda$ ;  $\mathcal{S}_i$  denotes the set of all partitions into parts that are congruent to  $\pm 2, \pm(i+2) \pmod{8}$ , where no odd part may be repeated, and  $s(\lambda)$  denotes the difference between the number of odd parts congruent to  $i+2 \pmod{8}$  and  $-(i+2) \pmod{8}$  in  $\lambda$ ; and  $\mathcal{T}_i$  denotes the set of all partitions into parts that are congruent to  $\pm 2, i, i+4 \pmod{8}$  where no odd part may be repeated, and  $r(\lambda)$  is the total number of odd parts  $\lambda$ .

## 4 Interpreting the new little Göllnitz identities

Recall Euler's partition theorem which states that *the number of partitions of  $N$  into distinct parts is equal to the number of partitions of  $N$  into parts congruent to 1 (mod 2)*. In this section we show:

**Corollary 4.1** (new little Göllnitz). *The the number of partitions of  $N$  into distinct parts in which even-indexed parts are even is equal to the number of partitions of  $N$  into parts congruent to 1, 5, or 6 (mod 8).*

**Corollary 4.2** (new little Göllnitz). *The the number of partitions of  $N$  into distinct parts in which odd-indexed parts are even is equal to the number of partitions of  $N$  into parts congruent to 2, 3, or 6 (mod 8).*

Corollaries 4.1 and 4.2 follow from Theorems 4.1 and 4.2 below, which interpret and refine the new little Göllnitz identities (2.8) and (2.9). We will observe that the Gessel-Stanton pair has a similar refinement.

**Theorem 4.3.** *Let  $E$  be the set of partitions  $\lambda = (\lambda_1, \lambda_2, \dots)$  into distinct parts in which even-indexed parts,  $\lambda_{2i}$ , are even. Then*

$$\sum_{\lambda \in E} x^{|\lambda|_o} y^{|\lambda|_e} t^{\text{odd}(\lambda)} = \sum_{j \geq 0} \frac{t^j x^{j^2} y^{j(j-1)} (-x/t; x^2 y^2)_j}{(x^2; x^2 y^2)_j (x^2 y^2; x^2 y^2)_j} = \frac{(-tx; x^2 y^2)_\infty}{(x^2; x^2 y^2)_\infty} \quad (4.1)$$

where  $|\lambda|_o = \lambda_1 + \lambda_3 + \dots$ ;  $|\lambda|_e = \lambda_2 + \lambda_4 + \dots$ ; and  $\text{odd}(\lambda)$  is the number of odd parts of  $\lambda$ .

*Proof.* The second equality is  $H(-x/t, \infty, x^2; x^2 y^2)$ , where  $H$  is the  $q$ -Gauss specialization (2.7). To prove the first equality, let  $E_n$  be the set of partitions in  $E$  with  $n$  positive parts and let  $f_n(x, y, t)$  be its generating function. Partitions  $\lambda$  from  $E_n$  can be constructed as follows.

First, if  $n = 2j - 1$ , start with the staircase  $(n, n - 1, \dots, 1)$ , which contributes  $t^j x^{j^2} y^{j^2 - j}$  to the weight of  $\lambda$ . If  $n = 2j$ , start with the staircase  $(n + 1, n, \dots, 2)$ , which contributes  $t^j x^{j^2 + 2j} y^{j^2 + j}$  to the weight of  $\lambda$ .

Next, decide which of the odd-indexed parts  $\lambda_{2i+1}$  of  $\lambda$  should be even. For each such  $i$ , add 2 to  $\lambda_1, \dots, \lambda_{2i}$ ; add 1 to  $\lambda_{2i+1}$ . These possibilities are generated by  $(-x/t; x^2 y^2)_j$ , where  $n = 2j - 1$  or  $n = 2j$ .

Finally, any partition into at most  $n$  even parts can be added to  $\lambda$ . We view it this way: Add 2 to any  $\lambda_i$ , but then 2 must also be added to each of  $\lambda_1, \lambda_2, \dots, \lambda_{i-1}$ . This may be repeated an arbitrary number of times. These possibilities are generated by  $(x^2; x^2 y^2)_j^{-1} (x^2 y^2; x^2 y^2)_{j-1}^{-1}$  if  $n = 2j - 1$  and by  $(x^2; x^2 y^2)_j^{-1} (x^2 y^2; x^2 y^2)_j^{-1}$  if  $n = 2j$ .

Putting everything together:

$$\begin{aligned} \sum_{\lambda \in E} x^{|\lambda|_o} y^{|\lambda|_e} t^{\text{odd}(\lambda)} &= \sum_{n \geq 0} f_n(x, y, t) = 1 + \sum_{j \geq 1} (f_{2j-1}(x, y, t) + f_{2j}(x, y, t)) \\ &= 1 + \sum_{j \geq 1} \left( \frac{t^j x^{j^2} y^{j^2 - j} (-x/t; x^2 y^2)_j}{(x^2; x^2 y^2)_j (x^2 y^2; x^2 y^2)_{j-1}} + \frac{t^j x^{j^2 + 2j} y^{j^2 + j} (-x/t; x^2 y^2)_j}{(x^2; x^2 y^2)_j (x^2 y^2; x^2 y^2)_j} \right) \\ &= 1 + \sum_{j \geq 1} \left( \frac{t^j x^{j^2} y^{j^2 - j} (-x/t; x^2 y^2)_j}{(x^2; x^2 y^2)_j (x^2 y^2; x^2 y^2)_j} ((1 - x^2 y^2 (x^2 y^2)^{j-1}) + x^{2j} y^{2j}) \right) \\ &= \sum_{j \geq 0} \frac{t^j x^{j^2} y^{j^2 - j} (-x/t; x^2 y^2)_j}{(x^2; x^2 y^2)_j (x^2 y^2; x^2 y^2)_j}. \end{aligned}$$

□

**Theorem 4.4.** *Let  $O$  be the set of partitions  $\lambda = (\lambda_1, \lambda_2, \dots)$  into distinct parts in which odd-indexed parts,  $\lambda_{2i+1}$ , are even. Then*

$$\sum_{\lambda \in O} x^{|\lambda|_o} y^{|\lambda|_e} t^{\text{odd}(\lambda)} = \sum_{j \geq 0} \frac{t^j y^{j^2} x^{j(j+1)} (-ty)^{-1}; x^2 y^2)_j}{(x^2; x^2 y^2)_j (x^2 y^2; x^2 y^2)_j} = \frac{(-tx^2 y; x^2 y^2)_\infty}{(x^2; x^2 y^2)_\infty} \quad (4.2)$$

*Proof.* The second equality is  $H(-ty)^{-1}, \infty, x^2; x^2 y^2)$ . To prove the first equality, let  $O_n$  be the set of partitions in  $O$  with  $n$  positive parts and let  $g_n(x, y, t)$  be its generating function. Partitions  $\lambda$  from  $O_n$  can be constructed as follows.

First, if  $n = 2j$ , start with the staircase  $(n, n-1, \dots, 1)$ , which contributes  $t^j x^{j^2+j} y^{j^2}$  to the weight of  $\lambda$ . If  $n = 2j-1$ , start with the staircase  $(n+1, n, \dots, 2)$ , which contributes  $t^{j-1} x^{j^2+j} y^{j^2-1}$  to the weight of  $\lambda$ .

Next, decide which of the even-indexed parts  $\lambda_{2i}$  of  $\lambda$  should be even. For each such  $i$ , add 2 to  $\lambda_1, \dots, \lambda_{2i-1}$ ; add 1 to  $\lambda_{2i}$ . These possibilities are generated by  $(-x^2 y/t; x^2 y^2)_j$ , if  $n = 2j$  or  $(-x^2 y/t; x^2 y^2)_{j-1}$  if  $n = 2j-1$ .

Finally, any partition into at most  $n$  even parts can be added to  $\lambda$ . We view it this way: Add 2 to any  $\lambda_i$ , but then 2 must also be added to each of  $\lambda_1, \lambda_2, \dots, \lambda_{i-1}$ . This may be repeated an arbitrary number of times. These possibilities are generated by  $(x^2; x^2 y^2)_j^{-1} (x^2 y^2; x^2 y^2)_j^{-1}$  if  $n = 2j$  and by  $(x^2; x^2 y^2)_j^{-1} (x^2 y^2; x^2 y^2)_{j-1}^{-1}$  if  $n = 2j-1$ .

Putting everything together:

$$\begin{aligned} & \sum_{\lambda \in O} x^{|\lambda|_o} y^{|\lambda|_e} t^{\text{odd}(\lambda)} \\ &= \sum_{n \geq 0} g_n(x, y, t) = 1 + \sum_{j \geq 1} (g_{2j-1}(x, y, t) + g_{2j}(x, y, t)) \\ &= 1 + \sum_{j \geq 1} \left( \frac{t^{j-1} x^{j^2+j} y^{j^2-1} (-x^2 y/t; x^2 y^2)_{j-1}}{(x^2; x^2 y^2)_j (x^2 y^2; x^2 y^2)_{j-1}} + \frac{t^j x^{j^2+j} y^{j^2} (-x^2 y/t; x^2 y^2)_j}{(x^2; x^2 y^2)_j (x^2 y^2; x^2 y^2)_j} \right) \\ &= 1 + \sum_{j \geq 1} \left( \frac{t^j x^{j(j+1)} y^{j^2} (-x^2 y/t; x^2 y^2)_{j-1}}{(x^2; x^2 y^2)_j (x^2 y^2; x^2 y^2)_j} \left( \frac{1 - x^{2j} y^{2j}}{ty} + \frac{1 + x^{2j} y^{2j}}{t} \right) \right) \\ &= \sum_{j \geq 0} \left( \frac{t^j x^{j(j+1)} y^{j^2} (-ty)^{-1}; x^2 y^2)_j}{(x^2; x^2 y^2)_j (x^2 y^2; x^2 y^2)_j} \right) \end{aligned}$$

□

Finally, note that we can get a similar refinement of the Gessel-Stanton pair via (3.1),



with  $G(-tx, x^2; x^2y^2)$  and  $G(-t/y, x^2; x^2y^2)$ , e.g.:

$$\begin{aligned} G(-tx, x^2; x^2y^2) &= \sum_{j=0}^{\infty} \frac{(-tx; x^2y^2)_j (-xy^2/t; x^2y^2)_j x^{j(j+1)} y^{j(j-1)}}{(x^4y^4; x^4y^4)_j (x^2; x^2y^2)_j} \\ &= \frac{(-x^3y^2/t; x^4y^4)_{\infty} (-tx^3; x^4y^4)_{\infty}}{(x^2; x^2y^2)_{\infty}}. \end{aligned}$$

## 5 An alternate generalization via a Bailey pair and $q$ -difference equations

The following generalization of the Göllnitz-Gordon identities is due to Andrews ([1]; cf. [5, p. 114, Theorem 7.11]):

**Theorem 5.1.** *For a partition  $\lambda$ , and positive integer  $j$ , let  $f_j$  be the multiplicity of the part  $j$  in  $\lambda$ . Let  $i$  and  $k$  be integers with  $0 < i \leq k$ . Then the number of partitions of  $n$  into parts not congruent to  $2 \pmod{4}$  nor to  $0, \pm(2i-1) \pmod{4k}$  is equal to the following: the number of partitions  $\lambda$  of  $n$ , satisfying: for  $j \geq 1$ ,*

$$f_{2j} + f_{2j+1} + f_{2j+2} \leq k - 1;$$

and in which no odd part is repeated and at most  $i - 1$  parts are  $\leq 2$ .

(The original Göllnitz-Gordon identities are the special cases  $k = i = 2$  and  $k = i + 1 = 2$ .) There is an analytic counterpart to Theorem 5.1 (see [5, p. 116, Eq. (7.4.4)]) which is an immediate consequence of the Bailey chain ([6]; cf. [7, Chapter 3]).

Is the same true of the Gessel-Stanton identities? We study the associated  $q$ -difference equations as Andrews did to derive the infinite family generalizations of the combinatorial Rogers-Ramanujan identities and the Göllnitz-Gordon identities in [5, Chapter 7]. Although we are not successful at finding the combinatorial counterpart to the infinite family, we show via this method that the Gessel-Stanton identity and its partner are actually two members of a three-element family, where the first and third partner are essentially equivalent.

We shall require a series of definitions and results due to Andrews [2], cf. [4].

$$\begin{aligned} H_{k,j}(a_1, a_2, a_3; x; q) &:= \frac{(xq/a_1, xq/a_2, xq/a_3; q)_{\infty}}{(xq)_{\infty}} \\ &\times \sum_{n \geq 0} \frac{x^{kn} (a_1 a_2 a_3)^{-n} q^{(k-1)n^2 + (2-j)n} (1 - x^j q^{2nj}) (x)_n (a)_n (b)_n (c)_n}{(1-x)(q)_n (xq/a_1)_n (xq/a_2)_n (xq/a_3)_n} \quad (5.1) \end{aligned}$$

and

$$\begin{aligned}
J_{k,j}(a_1, a_2, a_3; x; q) &:= \frac{(xq/a_1, xq/a_2, xq/a_3; q)_\infty}{(xq)_\infty} \\
&\times \sum_{n \geq 0} \frac{x^{kn} (a_1 a_2 a_3)^{-n} q^{(k-1)n^2 + (2+k-j)n} (xq)_n (a_1)_n (a_2)_n (a_3)_n}{(q)_n (xq/a_1)_n (xq/a_2)_n (xq/a_3)_n} \\
&\times \left\{ 1 + \frac{x^j q^{(2n+1)j-3n} (1-a_1 q^n) (1-a_2 q^n) (1-a_3 q^n)}{(a_1 - xq^{n+1}) (a_2 - bxq^{n+1}) (a_3 - xq^{n+1})} \right\}. \quad (5.2)
\end{aligned}$$

The following is the fundamental  $q$ -difference equation satisfied by the  $J_{k,j}$  [4, p. 337, Theorem 4.1]:

$$\begin{aligned}
J_{k,j}(a_1, a_2, a_3; x; q) - J_{k,j-1}(a_1, a_2, a_3; x; q) \\
= (1 - a_1^{-1})(1 - a_2^{-1})(1 - a_3^{-1})(xq)^{j-1} H_{k,k-j+1}(a_1 q, a_2 q, a_3 q; xq^2; q). \quad (5.3)
\end{aligned}$$

It is also the case that [2, pp. 435–6, Thms. 1–2]

$$H_{k,-j}(a_1, a_2, a_3; x; q) = -x^{-j} H_{k,j}(a_1, a_2, a_3; x; q), \quad (5.4)$$

$$H_{k,0}(a_1, a_2, a_3; x; q) = 0, \text{ and} \quad (5.5)$$

$$H_{k,1}(a_1, a_2, a_3; x; q) = J_{k,k}(a_1, a_2, a_3; x; q) = J_{k,k+1}(a_1, a_2, a_3; x; q). \quad (5.6)$$

Also, Jacobi's triple product identity allows us to conclude

$$J_{k,j}(a_1, a_2, a_3; 1; q) = \frac{\left( (-1)^{r-1} q^{2k-j-\frac{1}{2}}, (-1)^{r-1} q^{j-\frac{3}{2}}, q^{2k-2}; q^{2k-2} \right)_\infty \left( \frac{xq}{a_1}, \frac{xq}{a_2}, \frac{xq}{a_3}; q \right)_\infty}{(xq)_\infty}, \quad (5.7)$$

provided  $a_1 a_2 a_3 = (-1)^r q^{3/2}$  and as sets  $\{a_1, a_2, a_3\} = \{q/a_1, q/a_2, q/a_3\}$ .

The standard two-variable generalization of the first Rogers-Ramanujan identity follows from inserting the ‘‘standard multiparameter’’ Bailey pair  $(\alpha_n^{(1,1,2)}(x, q), \beta_n^{(1,1,2)}(x, q))$  from [15, p. 297, Eq. (3.1); p. 299, Eq. (3.8)] into a certain limiting case of Bailey's lemma [15, p. 297, Eq. (2.3)]:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^n q^{n^2}}{(q; q)_n} &= \frac{1}{(xq; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{n(5n-1)/2} (1 - xq^{2n})(x; q)_n}{(1-x)(q; q)_n} \\
&= H_{2,1}(\infty, \infty, \infty; x; q) = J_{2,2}(\infty, \infty, \infty; x; q). \quad (5.8)
\end{aligned}$$

That of the second Rogers-Ramanujan identity then follows from (5.3) with  $k = j = 2$ ,  $a_1 = a_2 = a_3 = \infty$  together with (5.6):

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^n q^{n^2+n}}{(q; q)_n} &= \frac{1}{(xq; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{n(5n+3)/2} (1 - xq^{2n+1})(xq; q)_n}{(q; q)_n} \\
&= J_{2,1}(\infty, \infty, \infty; x; q). \quad (5.9)
\end{aligned}$$

Setting  $x = 1$  in (5.8) and (5.9), and applying the Jacobi triple product identity, we recover the two Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_{\infty}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_{\infty}}.$$

By inserting the ‘‘Jackson-Slater multiparameter’’ Bailey pair  $(\bar{\alpha}_n^{(1,2,2)}(x^2, q^4), \bar{\beta}_n^{(1,2,2)}(x^2, q^4))$  from [15, p. 298, Eq. (3.3); p. 300, Eq. (3.29)] into the appropriate limiting case of Bailey’s lemma [15, p. 297, Eq. (2.4)], we find that the analogous generalization of the Gessel-Stanton identity is

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{x^{2n} q^{2n^2} (-q^2; q^4)_n (-xq; q^2)_{2n}}{(q^4; q^4)_n (-xq^2; q^2)_{2n} (x^2 q^2; q^4)_n} \\ &= \frac{(-x^2 q^2; q^4)_{\infty}}{(x^2 q^4; q^4)_{\infty}} \sum_{n=0}^{\infty} \frac{x^{3n} q^{4n^2-n} (-q^2; q^4)_n (1-xq^{4n}) (q; q^2)_n (x; q^2)_n}{(-x^2 q^2; q^4)_n (1-x) (xq; q^2)_n (q^2; q^2)_n} \\ &= \frac{H_{3,1}(iq, q, -iq; x; q^2)}{(xq, -xq^2; q^2)_{\infty}} = \frac{J_{3,3}(iq, q, -iq; x; q^2)}{(xq, -xq^2; q^2)_{\infty}}, \end{aligned} \quad (5.10)$$

where  $i = \sqrt{-1}$ .

Rogers-Ramanujan type identities occur in families of size  $k$ , so observing that (5.10) is an infinite product times an instance of  $J_{3,j}$ , we might hope to find two other partners to (5.10), corresponding to  $J_{3,1}$  and  $J_{3,2}$ .

**Lemma 5.2.**

$$\begin{aligned} & J_{3,3}(a, b, -a; x; q^2) \\ &= (xq^2/b, -xq^2; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n a^{-2n} x^{2n} q^{2n^2+2n} (a^2; q^4)_n (-xq^2/b; q^2)_{2n}}{(q^4; q^4)_n (-xq^2; q^2)_{2n} (x^2 q^4/b^2; q^4)_n}. \end{aligned} \quad (5.11)$$

*Proof.*

$$\begin{aligned} & J_{3,3}(a, b, -a; x; q^2) \\ &= H_{3,1}(a, b, -a; x; q^2) \quad (\text{by (5.6)}) \\ &= \frac{(x^2 q^4/a^2; q^4)_{\infty} (xq^2/b; q^2)_{\infty}}{(xq^2; q^2)_{\infty}} \\ & \quad \times \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n} a^{-2n} b^{-n} q^{4n^2+2n} (1-xq^{2n}) (x; q^2)_n (a^2; q^4)_n (b; q^2)_n}{(1-x) (q^2; q^2)_n (x^2 q^4/a^2; q^4)_n (xq^2/b; q^2)_n} \\ &= (xq^2/b, -xq^2; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n a^{-2n} x^{2n} q^{2n^2+2n} (a^2; q^4)_n (-xq^2/b; q^2)_{2n}}{(q^4; q^4)_n (-xq^2; q^2)_{2n} (x^2 q^4/b^2; q^4)_n}, \end{aligned}$$

where the last equality follows from a nonterminating basic hypergeometric formula due Verma and Jain which transforms a very-well-poised  $_{10}\phi_9$  series to a weighted sum of two balanced  ${}_5\phi_4$  series [17, p. 12, Eq. (3.1)]. Specifically, in their formula, replace  $q$  by  $q^2$ , put  $x$  for  $a$  and  $a$  for  $x$ , and set both  $y$  and  $z$  equal to  $q/t$ . Then take the limit as  $t \rightarrow 0$ .  $\square$

**Lemma 5.3.**

$$J_{3,2}(a, b, -a; x; q^2) = (xq^2/b, -xq^2; q^2)_\infty \times \sum_{n=0}^{\infty} \frac{(-1)^n a^{-2n} x^{2n} q^{2n^2+2n} (a^2; q^4)_n (-xq^2/b; q^2)_{2n-1}}{(q^4; q^4)_n (-xq^2; q^2)_{2n} (x^2q^4/b^2; q^4)_n} \left( \frac{xq^{4n}}{b} + q^{4n} + \frac{1}{b} - \frac{q^{4n}}{b} \right) \quad (5.12)$$

*Proof.* Take (5.3) with  $j = k = 3$ ,  $a_1 = -a_3 = a$ ,  $a_2 = b$ , and  $q$  replaced by  $q^2$ , apply (5.6), and conclude

$$J_{3,2}(a, b, -a; x, q^2) = J_{3,3}(a, b, -a; x; q^2) - \left(1 - \frac{1}{a^2}\right) \left(1 - \frac{1}{b}\right) x^2 q^4 J_{3,3}(aq^2, bq^2, -aq^2; xq^4; q^2). \quad (5.13)$$

Then apply Lemma 5.2 to the preceding equation to deduce

$$J_{3,2}(a, b, -a; x, q^2) = (xq^2/b, -xq^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n a^{-2n} x^{2n} q^{2n^2+2n} (a^2; q^4)_n (-xq^2/b; q^2)_{2n}}{(q^4; q^4)_n (-xq^2; q^2)_{2n} (x^2q^4/b^2; q^4)_n} - (xq^4/b; q^2)_\infty (-xq^6; q^2)_\infty \left(1 - \frac{1}{a^2}\right) \left(1 - \frac{1}{b}\right) x^2 q^4 \times \sum_{n=0}^{\infty} \frac{(-1)^n a^{-2n} x^{2n} q^{2n^2+6n} (a^2q^4; q^4)_n (-xq^4/b; q^2)_{2n}}{(q^4; q^4)_n (-xq^6; q^2)_{2n} (x^2q^8/b^2; q^4)_n}. \quad (5.14)$$

After some routine manipulations, including the reindexing of the second sum on the right side of (5.14), we obtain the result.  $\square$

**Theorem 5.4.**

$$\frac{J_{3,2}(iq, q, -iq; x; q^2)}{(xq, -xq^2; q^2)_\infty} = \sum_{n=0}^{\infty} \frac{x^{2n} q^{2n^2} (-q^2; q^4)_n (-xq; q^2)_{2n-1}}{(q^4; q^4)_n (-xq^2; q^2)_{2n} (x^2q^2; q^4)_n} (xq^{4n-1} + q^{4n} + q^{-1} - q^{4n-1})$$

*Proof.* Set  $a = iq$  and  $b = q$  in Lemma 5.3, and divide both sides by the infinite product  $(xq, -xq^2; q^2)_\infty$ .  $\square$

We now are ready to present a partner to the original Gessel-Stanton identity (1.1) (compare to (3.5)).

**Corollary 5.5.**

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-1}(-q; q^2)_{2n-1}(1+q^{4n+1})}{(q^2; q^4)_n(q^8; q^8)_n} = (-q, -q^7; q^8)_{\infty}(-q^2; q^2)_{\infty}$$

*Proof.* Set  $x = 1$  in Theorem 5.4, and apply (5.7). □

Next, we set out to find the partner to (1.1) associated with  $J_{3,1}$ .

**Lemma 5.6.**

$$\begin{aligned} J_{3,1}(iq, q, -iq; x; q^2) \\ = (1-x)(1+x^2)J_{3,3}(iq, q, -iq; xq^2; q^2) + xq^{-1}J_{3,2}(iq, q, -iq; x; q^2). \end{aligned}$$

*Proof.* Let  $\mathbf{a} := (iq, q, -iq)$ .

$$\begin{aligned} J_{3,2}(\mathbf{a}) &= H_{3,2}(\mathbf{a}; xq^2; q^2) - xqH_{3,1}(\mathbf{a}; xq^2; q^2) + x^2q^2H_{3,0}(\mathbf{a}; xq^2; q^2) \\ &\quad - x^3q^3H_{3,-1}(\mathbf{a}; xq^2; q^2) \text{ (by (5.3))} \\ &= H_{3,2}(\mathbf{a}; xq^2; q^2) - xq(1-x)J_{3,3}(\mathbf{a}; xq^2; q^2) \text{ (by (5.5), (5.4), and (5.6)).} \end{aligned}$$

Thus,

$$H_{3,2}(\mathbf{a}; xq^2; q^2) = J_{3,2}(\mathbf{a}; x; q^2) + xq(1-x)J_{3,3}(\mathbf{a}; xq^2; q^2). \quad (5.15)$$

Next,

$$\begin{aligned} J_{3,1}(\mathbf{a}) &= H_{3,1}(\mathbf{a}; xq^2; q^2) - xqH_{3,0}(\mathbf{a}; xq^2; q^2) + x^2q^2H_{3,-1}(\mathbf{a}; xq^2; q^2) \\ &\quad - x^3q^3H_{3,-2}(\mathbf{a}; xq^2; q^2) \text{ (by (5.3))} \\ &= (1-x)J_{3,3}(\mathbf{a}; xq^2; q^2) + xq^{-1}H_{3,2}(\mathbf{a}; xq^2; q^2) \text{ (by (5.5), (5.4), and (5.6))} \\ &= (1-x)J_{3,3}(\mathbf{a}; xq^2; q^2) \\ &\quad + xq^{-1}(J_{3,2}(\mathbf{a}; x; q^2) + xq(1-x)J_{3,3}(\mathbf{a}; xq^2; q^2)) \text{ (by (5.15))} \\ &= (1-x)(1+x^2)J_{3,3}(\mathbf{a}; xq^2; q^2) + xq^{-1}J_{3,2}(\mathbf{a}; x; q^2). \end{aligned}$$

□

**Theorem 5.7.**

$$\begin{aligned} \frac{J_{3,1}(iq, q, -iq; x; q^2)}{(xq, -xq^2; q^2)_{\infty}} \\ = \sum_{n=0}^{\infty} \frac{x^{2n-2}q^{2n^2-2}(-q^2; q^4)_{n-1}(-xq; q^2)_{2n-1}}{(q^4; q^4)_n(-xq^2; q^2)_{2n}(x^2q^2; q^4)_n} \\ \times ((1-x)(1+x^2)(1-q^{4n})(1+xq^{4n}) \\ + x^3(1+q^{4n-2})(xq^{4n} + q^{4n+1} + 1 - q^{4n})) \end{aligned}$$

*Proof.* Apply Lemma 5.2 and Lemma 5.3 to Lemma 5.6. □

Setting  $x = 1$  in the preceding yields a result equivalent to Corollary 5.5.

**Remark 5.8.** Now that we have placed (1.1) in the context of Bailey pairs, an infinite family multisum-product generalization is immediate via the Bailey chain [6].

**Remark 5.9.** If we let  $\mathcal{R}$  denote the set of partitions with difference at least 2 between all parts, then we have

$$H_{2,1}(0, 0, 0; x; q) = J_{2,2}(0, 0, 0; x; q) = \sum_{n=0}^{\infty} \frac{x^n q^{n^2}}{(q; q)_n} = \sum_{\lambda \in \mathcal{R}} x^{\ell(\lambda)} q^{|\lambda|},$$

i.e. the third member of (5.8), the “ $x$ -generalization” of the first Rogers-Ramanujan identity, is a generating for partitions in  $\mathcal{R}$ , where the exponent of  $q$  counts the number being partitioned and the exponent on  $x$  records the number of parts in the partition.

In the case of (5.10), however, the exponent on  $x$  does *not* count the number of parts in the partition, as witnessed by the fact that the coefficients of powers of  $x$  may be negative:

$$1 + q^2 x^2 + q^3 x^3 + q^4 (x^4 - x^3 + x^2) + q^5 (x^5 - x^4 + 2x^3) + \dots,$$

thus there is no hope of getting a *simple* partition theoretic interpretation interpretation of (5.10) or the identity in Theorem (5.4) analogous to that of (5.8).

Furthermore, even if such an interpretation could be found, extending it to an infinite family analogous to Theorem 5.1 may prove difficult due to the complexity of the associated family of  $q$ -difference equations.

## 6 Further Directions

We suggest two possible directions for further research.

1. Is there a combinatorial interpretation for the Gessel-Stanton identity in the spirit of Theorems 4.3 and 4.2?
2. Despite the difficulties acknowledged in Remark 5.9, is there a natural combinatorial interpretation of (5.10) and its partner? Could such an interpretation be extended to the infinite family implied by the Bailey chain?

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## References

- [1] George E. Andrews. A generalization of the Göllnitz-Gordon partition theorems. *Proc. Amer. Math. Soc.*, 18:945–952, 1967.
- [2] George E. Andrews. On  $q$ -difference equations for certain well-poised basic hypergeometric series. *Quart. J. Math. Oxford Ser. (2)*, 19:433–447, 1968.
- [3] George E. Andrews. On the  $q$ -analog of Kummer’s theorem and applications. *Duke Math. J.*, 40:525–528, 1973.
- [4] George E. Andrews. On Rogers-Ramanujan type identities related to the modulus 11. *Proc. London Math. Soc. (3)*, 30:330–346, 1975.
- [5] George E. Andrews. *The Theory of Partitions*, volume 2 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley, Reading, MA, 1976.
- [6] George E. Andrews. Multiple series Rogers-Ramanujan type identities. *Pacific J. Math.*, 114:267–283, 1984.
- [7] George E. Andrews.  *$q$ -Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, volume 66 of *CBMS*. AMS, Providence, RI, 1986.
- [8] George E. Andrews and Bruce C. Berndt. *Ramanujan’s Lost Notebook, Part II*. Springer, 2009.
- [9] Sylvie Corteel, Carla D. Savage, and Andrew V. Sills. Lecture hall sequences,  $q$ -series, and asymmetric partition identities. *Developments in Mathematics*. Springer, to appear.
- [10] George Gasper and Mizan Rahman. *Basic hypergeometric series*, volume 96 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2004. With a foreword by Richard Askey.
- [11] Ira Gessel and Dennis Stanton. Applications of  $q$ -Lagrange inversion to basic hypergeometric series. *Trans. Amer. Math. Soc.*, 277(1):173–201, 1983.
- [12] H. Göllnitz. Partitionen mit differenzenbedingungen. *J. Reine Angew. Math.*, 225:154–190, 1967.
- [13] Basil Gordon. Some continued fractions of the Rogers-Ramanujan type. *Duke J. Math.*, 31:741–748, 1965.
- [14] V. A Lebesgue. Sommatation de quelques séries. *Journal Math. Pures Appl.*, 5:42–71, 1840.

- [15] Andrew V. Sills. Identities of the Rogers-Ramanujan-Slater type. *Int. J. Number Theory*, 3(2):293–323, 2007.
- [16] Lucy J. Slater. Further identities of the Rogers-Ramanujan type. *Proc. London Math. Soc. (2)*, 54:147–167, 1952.
- [17] A. Verma and V. K. Jain. Transformations of nonterminating basic hypergeometric series, their contour integrals and applications to Rogers-Ramanujan identities. *J. Math. Anal. Appl.*, 87(1):9–44, 1982.