A Note on Partitions and Compositions Defined by Inequalities

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1 Introduction

For a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of integers, define the *weight* of λ to be $|\lambda| = \lambda_1 + \cdots + \lambda_n$. If sequence λ of weight N has all parts nonnegative, we call it a *composition* of N; if, in addition, λ is a nonincreasing sequence, we call it a *partition* of N.

Given an $n\times n$ integer matrix $C = [c_{i,j}]$, we consider the set S_C of compositions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying the constraints

$$c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \ldots + c_{i,n}\lambda_n \geq 0, \quad 1 \leq i \leq n. \tag{1}$$

In [3], we gave sufficient conditions for the generating function of S_C to have a product form:

$$F_C(q) = \sum_{\lambda \in S_C} q^{|\lambda|} = \prod_{j=1}^{n} \frac{1}{(1 - q^{b_j})}, \tag{2}$$

for some positive integers $b_1, \ldots, b_n$. Specifically, it was shown that if C is an upper triangular matrix with all $c_{i,i} = 1$, and if $B = C^{-1}$ has nonnegative entries, then $F_C(q)$ is given by (2), where b_j is the sum of the entries in column j of $B = [b_{i,j}]$. Moreover, it was shown that this provides a natural bijection between the "partitions" of N into parts from the multiset $\{b_1, \ldots, b_n\}$ and the compositions of N in S_C given by:

$$b_1^{m_1} b_2^{m_2} \cdots b_n^{m_n} \leftrightarrow (\lambda_1, \lambda_2, \ldots, \lambda_n), \tag{3}$$

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where $\lambda_i = \sum_{j=i}^n b_{i,j} m_j$. Consequently, both the generating function for S_C and the bijection can be "read off" from C^{-1}.

This result automatically gives the product form generating functions, as well as the corresponding "natural bijections", for all of the following families:

- *Hickerson partitions* [4]

  Given a positive integer r, the partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying $\lambda_i \geq r\lambda_{i+1}$ have generating function
$$\prod_{j=1}^{n} \frac{1}{1 - q^{1+r+\cdots+r^{j-1}}}.$$
  This follows by inverting the constraint matrix:
$$C^{-1} = \begin{bmatrix} 1 & -r & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -r & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & -r & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & r & r^2 & r^3 & r^4 & \ldots & r^{n-1} \\ 0 & 1 & r & r^2 & r^3 & \ldots & r^{n-2} \\ 0 & 0 & 1 & r & r^2 & \ldots & r^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

- *Santos' interpretation of Euler* [6]

  The number of partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of N into n nonnegative parts satisfying the additional constraint $\lambda_1 \geq \sum_{i=2}^n \lambda_i$, is the same as the number of partitions of N into odd parts in $\{1, 3, \ldots, 2n-1\}$. (See next example.)

- *Sellers' generalization of Santos* [7, 8]:

  The set of partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ into n nonnegative parts satisfying the additional constraint $\lambda_1 \geq \sum_{i=2}^n k_i \lambda_i$, for a given sequence of nonnegative integers $(k_2, \ldots, k_n)$ with $k_2 > 0$ has generating function
$$\frac{1}{(1-q)(1-q^{k_2+1})(1-q^{k_2+k_3+2})(1-q^{k_2+k_3+k_4+3})\cdots(1-q^{k_2+k_3+\cdots+k_n+n-1})}. \tag{4}$$
  This follows because the constraint matrix has the form
$$C = \begin{bmatrix} 1 & -k_2 & -k_3 & -k_4 & -k_5 & \ldots & -k_n \\ 0 & 1 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix},$$
  and its inverse is
$$C^{-1} = \begin{bmatrix} 1 & k_2 & k_2+k_3 & k_2+k_3+k_4 & k_2+k_3+k_4+k_5 & \ldots & k_2+\cdots+k_n \\ 0 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 1 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$
  Setting $k_2 = \cdots = k_n = 1$ gives Santos' result. In fact, note that we can relax Sellers' requirements that $k_2 > 0$ and $k_i \geq 0$: as long as all partial sums $k_2 + \cdots + k_i$, $2 \leq i \leq n$, are nonnegative, (4) is still the generating function for the set of compositions of N (no longer necessarily partitions) satisfying $\lambda_1 \geq \sum_{i=2}^n k_i \lambda_i$ and $\lambda_i \geq \lambda_{i+1}$ for $2 \leq i \leq n-1$. For example, we could have $k_i = 1$ if i is even, $k_i = -1$ if i is odd.

- *Partitions with nonnegative second differences* [1], *super-concave partitions* [9]

  Partitions $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with the additional constraint that $\lambda_i \geq 2\lambda_{i+1} - \lambda_{i+2}$ for $1 \leq i \leq n-1$, with $\lambda_{n+2} = 0$ were first considered by Andrews in [1] as *partitions with nonnegative second differences*. He used partition analysis to show that they have the same generating function as partitions into the *triangular numbers* $\{\binom{i+1}{2} | 1 \leq i \leq n\}$ and also to give a bijection. We can get the same generating function and a bijection from (2) and (3) by inverting the constraint matrix.

  Snellman and Paulsen arrived at the same family via a different definition in [9]. Their interest was in partitions $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying $\lambda_i(k-j) + \lambda_j(i-k) + \lambda_k(j-i) \geq 0$ for all positive integers $i < j < k \leq n+1$, where $\lambda_{n+1} = 0$. They called these *super concave partitions* and proved that they are equivalent to the partitions with nonnegative second differences.

- *Partitions with r-th differences nonnegative* [1, 2, 10]

  Generalizing the case for $r = 2$, these are the partitions $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying

$$\sum_{j=0}^{r} (-1)^j \lambda_{i+j} \binom{r}{j} \geq 0$$

  for $1 \leq i \leq n-1$, with $\lambda_{n+1} = \cdots = \lambda_{n+r-1} = 0$, defined by Andrews in [1]. He used partition analysis to show that they have the same generating function as partitions into parts in $\{\binom{i+r}{2} | 0 \leq i \leq n-1\}$ and a bijection appears in [2, 10]. Again, here, we can get the generating function and a bijection by inverting the constraint matrix.

- *Examples (0-5) of Pak in* [5].

Many other examples are provided in [3], where the main result was extended to handle the case where any subset of the constraints (1) could require equality and also to handle certain inhomogenous constraints.

On the other hand, consider the following simple example where the results of [3] do not directly apply. Let S be the set of integer sequences $(\lambda_1, \lambda_2, \lambda_3)$ satisfying

$$\begin{aligned} \lambda_1 &\geq \lambda_2 + \lambda_3 \\ \lambda_2 &\geq \lambda_3 \\ 2\lambda_3 &\geq \lambda_1 - \lambda_2. \end{aligned} \tag{5}$$

The constraint matrix is

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}, \tag{6}$$

which is not upper triangular and its diagonal entries are not all 1.

The purpose of this note is to extend the result of [3] to more general systems, including (5). We show that for any constraint matrix C, as long as the matrix $B = C^{-1}$ has all nonnegative *integer* entries, then the generating function for the set S_C, of integer sequences satisfying (1), has the form (2), where b_j is the sum of the entries in column j of $B = [b_{i,j}]$. Again, this comes supplied with the natural bijection (3). As in [3], this extends to the case where some of the inequalities are equalities and some of the inequalities may be inhomogeneous.

Furthermore, the multivariable generating function

$$F_C(x_1, x_2, \ldots, x_n) = \sum_{\lambda \in S_C} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \tag{7}$$

can also be "read" from the B matrix: q^{b_j} in (2) just becomes: $x_1^{b_{1,j}} x_2^{b_{2,j}} \cdots x_n^{b_{n,j}}$ in (7). Note that the polynomial $F_C(x_1, x_2, \ldots, x_n)$ is an encapsulation of the set S_C: the coefficient of q^N in $F_C(qx_1, qx_2, \ldots, qx_n)$ is a listing (as the terms of a polynomial) of all integer solutions to (1) of weight N.

In the next section we state and prove the extended theorem and follow in Section 3 with some examples.

2 The Main Theorem

Suppose S is a multiset of positive integers. We want to define a partition of n into parts taken from S. Suppose i is some element of S which appears with multiplicity > 1. Then imagine that the different copies of i have different colors. We will count as different a partition of n that uses two red copies of i and six green copies of i, on the one hand, and a partition of n that uses five red copies of i and three green copies of i, on the other hand. It is in this sense that we will be counting the partitions of n into parts taken from S.

Theorem 1 *Let C be an $n \times n$ matrix of integers such that $C^{-1} = B = [b_{i,j}]$ exists and has all entries nonnegative integers. Let $e_1, \ldots, e_n$ be nonnegative integer constants. Let EQ be a subset of $\{1, \ldots, n\}$. For each $1 \le i \le n$, let c_i be the constraint*

$$\begin{cases} c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \ldots c_{i,n}\lambda_n \ge e_i & \text{if } i \notin EQ \\ c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \ldots c_{i,n}\lambda_n = e_i & \text{if } i \in EQ \end{cases}$$

Let S_C be the set of nonnegative integer sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying the constraints c_i for all i, $1 \le i \le n$. The generating function for S_C is:

$$F_C(x_1, x_2, \ldots, x_n) = \sum_{\lambda \in S_C} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} = \frac{\prod_{j=1}^{n} (x_1^{b_{1,j}} x_2^{b_{2,j}} \cdots x_n^{b_{n,j}})^{e_j}}{\prod_{j \in \{1,\ldots,n\} - EQ} (1 - x_1^{b_{1,j}} x_2^{b_{2,j}} \cdots x_n^{b_{n,j}})}.$$

Furthermore, let b_j be the sum of the entries in column j of $B = C^{-1}$. Then an explicit bijection between

(i) *partitions of N into parts from the multiset $\{b_1, \ldots, b_n\}$ with the restriction that part b_j occurs at least e_j times if $j \notin EQ$ and exactly e_j times if $j \in EQ$ and*

(ii) *sequences $\lambda \in S_C$ of weight N*

is given by

$$b_1^{m_1} b_2^{m_2} \cdots b_n^{m_n} \rightarrow (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where $\lambda_i = \sum_{j=i}^{n} b_{i,j} m_j$.

Proof. If $\lambda \in S_C$, then for $1 \le i \le n$,

$$s_i = \left(\sum_{j=1}^{n} c_{i,j} \lambda_j \right) - e_i \ge 0,$$

and $s_i = 0$ if $i \in EQ$. Conversely, if B has all nonnegative integer entries, then given nonnegative integers $s_1, s_2, \ldots, s_n$, with $s_i = 0$ if $i \in EQ$, define λ by

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = B \begin{bmatrix} s_1 + e_1 \\ \vdots \\ s_n + e_n \end{bmatrix}. \tag{8}$$

Then the λ_i are nonnegative integers and since $B = C^{-1}$,

$$C \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} s_1 + e_1 \\ \vdots \\ s_n + e_n \end{bmatrix}.,$$

So for $1 \le i \le n$, λ satisfies

$$c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \ldots c_{i,n}\lambda_n = s_i + e_i \begin{cases} \ge e_i & \text{if } i \notin EQ \\ = e_i & \text{if } i \in EQ \end{cases}$$

that is, $\lambda \in S_C$.

Thus,

$$\begin{aligned} F_C(x_1, \ldots, x_n) &= \sum_{\lambda \in S_C} \prod_{i=1}^{n} x_i^{\lambda_i} \\ &= \sum_{\substack{s_1, \ldots, s_n \ge 0 \\ k \in EQ \to (s_k = 0)}} \prod_{i=1}^{n} x_i^{b_{i,1}(s_1+e_1)+\cdots+b_{i,n}(s_n+e_n)} \\ &= \sum_{\substack{s_1, \ldots, s_n \ge 0 \\ k \in EQ \to (s_k = 0)}} \prod_{j=1}^{n} (x_1^{b_{1,j}} x_2^{b_{2,j}} \cdots x_n^{b_{n,j}})^{(s_j+e_j)} \\ &= \prod_{j=1}^{n} (x_1^{b_{1,j}} x_2^{b_{2,j}} \cdots x_n^{b_{n,j}})^{e_j} \sum_{\substack{s_1, \ldots, s_n \ge 0 \\ k \in EQ \to (s_k = 0)}} \prod_{j=1}^{n} (x_1^{b_{1,j}} x_2^{b_{2,j}} \cdots x_n^{b_{n,j}})^{s_j} \end{aligned}$$

and the result follows. □

3 Examples

Example 1 We revisit the example from the introduction. Let S be the set of integer sequences $(\lambda_1, \lambda_2, \lambda_3)$ satisfying the constraints (5). The constraint matrix C is (6) which is not upper triangular. However, C^{-1} has all nonnegative integer entries:

$$C^{-1} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

so by Theorem 1, the multivariable generating function is

$$\sum_{(\lambda_1,\lambda_2,\lambda_3)\in S} x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3} \quad = \quad \frac{1}{(1-x_1^3x_2x_3)(1-x_1x_2)(1-x_1^2x_2x_3)}.$$

Example 2. Consider the set T of nondegenerate incongruent integer-sided triangles which were treated by Andrews in [1]. These are partitions into 3 parts $(\lambda_1,\lambda_2,\lambda_3)$ satisfying $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ and $\lambda_1+\lambda_2 > \lambda_3$, $\lambda_2+\lambda_3 > \lambda_1$, $\lambda_1+\lambda_3 > \lambda_2$. Equivalently, T is the set of nonnegative integer sequences $(\lambda_1,\lambda_2,\lambda_3)$ satisfying

$$\begin{aligned} \lambda_1 &\geq \lambda_2 \\ \lambda_2 &\geq \lambda_3 \\ \lambda_3 &\geq \lambda_1-\lambda_2+1. \end{aligned} \tag{9}$$

Then in the hypotheses of Theorem 1, $EQ=\emptyset$, $e_1=e_2=0$, $e_3=1$, and the constraint matrix is

$$C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

(which is not upper triangular). The inverse is:

$$B = C^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

so by Theorem 1 the multivariable generating function is

$$\begin{aligned} \sum_{(\lambda_1,\lambda_2,\lambda_3)\in T} x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3} &= \frac{x_1^{b_{1,3}}x_2^{b_{2,3}}x_3^{b_{3,3}}}{(1-x_1^{b_{1,1}}x_2^{b_{2,1}}x_3^{b_{3,1}})(1-x_1^{b_{1,2}}x_2^{b_{2,2}}x_3^{b_{3,2}})(1-x_1^{b_{1,3}}x_2^{b_{2,3}}x_3^{b_{3,3}})} \\ &= \frac{x_1x_2x_3}{(1-x_1^2x_2x_3)(1-x_1x_2)(1-x_1x_2x_3)}. \end{aligned}$$

Setting $x_1=x_2=x_3=q$ gives

$$\sum_{(\lambda_1,\lambda_2,\lambda_3)\in T} q^{\lambda_1+\lambda_2+\lambda_3} = \frac{q^3}{(1-q^4)(1-q^2)(1-q^3)}.$$

Also, Theorem 1 gives an explicit bijection between the partitions of N into parts $2,3,4$ with at least one part 3 (i.e. at least $e_3=1$ copies of $b_3=1+1+1$) and the triangles $(\lambda_1,\lambda_2,\lambda_3)$ of perimeter N satisfying (9). Specifically: the partition $4^{m_1}2^{m_2}3^{m_3}$ (using multiplicity representation), with $m_3 \geq 1$, maps to the triangle $(2m_1+m_2+m_3, m_1+m_2+m_3, m_1+m_3)$, the same bijection obtained in [1].

Continuing with the integer-sided triangles, suppose we only wanted those in which the lengths of the three sides were distinct. Then $EQ=\emptyset$, $e_1=e_2=e_3=1$, and the multivariable generating function would be:

$$\frac{(x_1^2x_2x_3)(x_1x_2)(x_1x_2x_3)}{(1-x_1^2x_2x_3)(1-x_1x_2)(1-x_1x_2x_3)} = \frac{x_1^4x_2^3x_3^2}{(1-x_1^2x_2x_3)(1-x_1x_2)(1-x_1x_2x_3)}.$$

If we wanted the two smallest sides to be the same length and to differ from the largest by at least 2, then $EQ=2$, $e_1=2, e_2=0, e_3=1$, and the generating function is:

$$\frac{(x_1^2x_2x_3)^2(x_1x_2x_3)}{(1-x_1^2x_2x_3)(1-x_1x_2x_3)} = \frac{(x_1^5x_2^3x_3^3)}{1-x_1^2x_2x_3(1-x_1x_2x_3)}.$$

Example 3. Analogous to the super concave partitions of Snellman and Paulsen in [9], call a partition *super convex* if it satisfies

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0; \qquad 2\lambda_i \geq \lambda_{i-1} + \lambda_{i+1}, \quad i \geq 2,$$

assuming $\lambda_{n+1} = 0$. To our knowledge, these were first considered by Andrews in [1], where he called them *partitions with mixed difference conditions* and computed their generating function using partition analysis. The generating function is also easy to compute from Theorem 1. The constraint matix is (e.g., for $n = 6$)

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

and the inverse is

$$C^{-1} = \begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

so by Theorem 1, the multivariable generating function is

$$\prod_{j=1}^{n} \left(1 - \prod_{i=1}^{j} (x_i \cdots x_n)\right)^{-1}.$$

We can generalize this a bit: the generating function for the nonnegative integer sequences λ satisfying

$$\lambda_1 \geq t\lambda_2; \quad (t+1)\lambda_i \geq \lambda_{i-1} + t\lambda_{i+1}, \quad i \geq 2,$$

assuming $\lambda_{n+1} = 0$, is

$$\prod_{j=1}^{n} \left(1 - \prod_{i=1}^{j} (x_i^{t^0} x_{i+1}^{t^1} \cdots x_n^{t^{n-i}})\right)^{-1}.$$

Example 4. In [1], Andrews defines Δ_m as the number of partitions $\lambda_1, \lambda_2, \ldots, \lambda_{2m+1}$ satisfying the additional constraints

$$\lambda_{2i-1} - \lambda_{2i} - \lambda_{2i+1} + \lambda_{2i+2} \leq 0, \quad 1 \leq i \leq m-1; \quad \lambda_{2m-1} - \lambda_{2m} - \lambda_{2m+1} \leq 0, \tag{10}$$

and shows that the generating function of Δ_m is

$$\prod_{h=1}^{m} \frac{1}{1 - q^{2h}} \prod_{j=0}^{m} \frac{1}{1 - q^{(j+1)(2m+1-j)}}. \tag{11}$$

(Then Δ_1 is the set of incongruent *nonnegative* integer-sided triangles.)

The constraints (10), together with the constraints $\lambda_i \geq \lambda_{i+1}$, $1 \leq i \leq 2m$, give a total of $3m-1$ constraints in $2m+1$ variables. It is easy to check that the set of constraints $\{\lambda_{2i+1} \geq \lambda_{2i+2} | 1 \leq i <$

$m\}$ is redundant. Deleting these gives a system of $2m+1$ constraints in $2m+1$ variables defining Δ_m. The constraint matrix, e.g. when $m=4$, is:

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and its inverse is:

$$C^{-1} = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & 3 & 2 & 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & 3 & 2 & 1 & 1 & 1 & 1 & 0 \\ 3 & 3 & 3 & 2 & 1 & 1 & 1 & 1 & 0 \\ 3 & 3 & 3 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Generalizing this to arbitrary m, the last m columns of C^{-1} give rise to the first product in (11) and the first $m+1$ columns, the second product.

Example 5. Theorem 1 can be used to work backwards from a target generating function. For example, starting with the matrix on the right below and considering its inverse gives another "Euler bijection". Observe that

$$C^{-1} = \begin{bmatrix} 1 & -2 & 2 & -2 & 2 & \dots & (-1)^{n+1}2 \\ 0 & 1 & -2 & 2 & -2 & \dots & (-1)^{n}2 \\ 0 & 0 & 1 & -2 & 2 & \dots & (-1)^{n+1}2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & 2 & 2 & \dots & 2 \\ 0 & 0 & 1 & 2 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{12}$$

Using C as a constraint matrix defines the set S_n of sequences $\lambda_1, \ldots, \lambda_n$ satisfying $\lambda_i \geq 2(\lambda_{i+1} - \lambda_{i+2} + \lambda_{i+3} - \lambda_{i+4} + \cdots)$, where we assume $\lambda_i = 0$ if $i > n$. (It can be checked that the elements of S_n are partitions.) By Theorem 1 and (12), the generating function for S_n is

$$S_n(x_1, x_2, \ldots, x_n) = \frac{1}{(1-x_1)(1-x_1^2x_2)(1-x_1^2x_2^2x_3)\cdots(1-x_1^2x_2^2\cdots x_{n-1}^2x_n)}.$$

So, the number of partitions of N into odd parts in $\{1, 3, \ldots, 2n-1\}$ is equal to number of partitions of N in S_n and the bijection is given by

$$1^{m_1}3^{m_2}5^{m_3}\cdots(2n-1)^{m_n} \rightarrow (m_1 + 2\sum_{j=2}^{n} m_j, \quad m_2 + 2\sum_{j=3}^{n} m_j, \quad m_3 + 2\sum_{j=4}^{n} m_j, \quad \ldots, \quad m_n).$$

4 Conclusion

Theorem 1 provides a uniform treatment of a broad class of partition identities, many of which have been handled elsewhere by a variety of techniques. We propose Theorem 1 as the "method of first attack" since, when it does apply, it gives the generating function and the bijection, for the price of inverting a matrix. It also gives the multivariable generating function, which can be viewed as an encoding of all possible solutions.

It is not hard to find examples beyond the scope of this method. For example, define the *super convex compositions* to be those integer sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying

$$2\lambda_i \geq \lambda_{i-1} + \lambda_{i+1}, \quad 1 \leq i \leq r,$$

where we assume $\lambda_0 = \lambda_{r+1} = 0$. These differ from the super-convex partitions of Example 3 in that here $2\lambda_1 \geq \lambda_2$, but the λ_i need not be nonincreasing. The inverse of the constraint matrix for this system no longer has integer entries. In fact, we offer as an open question the challenge of computing the generating function for this family.

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