Contemporary Mathematics

Enumeration of Integer Solutions to Linear Inequalities Defined by Digraphs

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1. Introduction

This is the fourth in a series of papers [**CS04, CSW05, CLS07**] studying nonnegative integer solutions to linear inequalities as they relate to the enumeration of integer partitions and compositions. In this paper we consider solutions $(\lambda_1, \ldots, \lambda_n)$ to a system $\mathcal{C}$ of inequalities in which every constraint is of the form $\lambda_i \geq \lambda_j$ (or $\lambda_i > \lambda_j$). In this case, $\mathcal{C}$ can be modeled by a directed graph (digraph) in which the vertices are labeled $1, \ldots, n$ and there is an edge (or strict edge) from i to j if $\mathcal{C}$ contains the constraint $\lambda_i \geq \lambda_j$ (or $\lambda_i > \lambda_j$). Many familiar systems can be modeled in this way, including, ordinary partitions and compositions, plane partitions, monotone triangles, and plane partition diamonds and generalizations.

Our focus is on the use of a particular set of tools to *derive a recurrence* for the generating function F_{G_n} for an *infinite family* $\{G_n | n \geq 1\}$ of constraint systems represented by digraphs. The recurrence can be viewed as a program which computes F_{G_n} for any given n. More significantly, if it can be solved, it provides a closed form for the generating function for the infinite family.

The challenge becomes one of applying the tools *strategically* to get a recurrence for the *graph* G_n. Ultimately we want one where the associated recurrence for the generating function F_{G_n} can be solved.

For the "digraph method" offered in this paper, we start with the five guidelines for partition analysis from [**CLS07**], reviewed in Section 2, and derive some special tools tailored to computing the generating function of a directed graph in Section 3. In the remaining sections, we show how to use the digraph method to solve some nontrivial enumeration problems in the theory of partitions and compositions that can be modeled as directed graphs. It can't hurt to jump ahead to the example of Section 4 to see the ultimate goal of the tools in Sections 2 and 3.

This work was inspired by the work of Andrews, Paule, and Riese in the sequence of papers [**And98, And00, APR01b, AP99, APRS01, APR01c, APR01d, APR01e, APR01a, APR04, AP07**] which contain several examples of digraphs whose generating functions are computed using partition analysis and

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the Omega Package. We found that the task became easier with the simpler tools of Sections 2 and 3. These tools can be viewed as a simplification of MacMahon's partition analysis ([**Mac60**], Section VIII) and a generalization of "adding a slice" (e.g. [**KP98**]). The Omega Package software [**APR01b**], implementing partition analysis, was an invaluable tool in our early investigations. Recent speedups of Xin to partition analysis are described in [**Xin04**]

2. Five Guidelines

Let $\mathcal{C}$ be a set of linear constraints in n variables, $\lambda_1, \ldots, \lambda_n$, each constraint $c \in \mathcal{C}$ of the form

$$c: \quad [a_0 + \sum_{i=1}^{n} a_i \lambda_i \geq 0],$$

for integer values $a_0, a_1, \ldots, a_n$.

Let $S_{\mathcal{C}}$ be the set of nonnegative integer sequences $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying all constraints in $\mathcal{C}$. Since we are only interested here in *nonnegative* integer solutions, *we will always assume that* $\mathcal{C}$ *contains the constraints* $[\lambda_i \geq 0]$ *for* $1 \leq i \leq n$. Define the *full generating function of* $\mathcal{C}$ to be:

$$F_{\mathcal{C}}(x_1, \ldots, x_n) \triangleq \sum_{\lambda \in S_{\mathcal{C}}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}.$$

If c is the constraint: $[a_0 + \sum_{i=1}^{n} a_i \lambda_i \geq 0]$ define the *negation* of c, $\neg c$, to be the constraint $[-a_0 - \sum_{i=1}^{n} a_i \lambda_i \geq 1]$. Then any sequence $(\lambda_1, \ldots, \lambda_n)$ satisfies c or $\neg c$, but not both. A constraint c is *implied by* the set of constraints $\mathcal{C}$ if $S_{\mathcal{C} \cup \{\neg c\}} = \emptyset$. A constraint c is *redundant* if $S_{\mathcal{C} \cup \{c\}} = S_{\mathcal{C}}$.

Let $\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$ denote the set of constraints which results from replacing λ_i by $\lambda_i + a\lambda_j$ in every constraint in $\mathcal{C}$. Note that if constraint c is implied by $\mathcal{C}$ then $c_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$ is implied by $\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$. Thus observe that if $\mathcal{C}$ contains the constraints $[\lambda_i \geq 0], 1 \leq i \leq n$ and if $[\lambda_i - a\lambda_j \geq 0]$ is implied by $\mathcal{C}$, then all of the constraints $[\lambda_i \geq 0], 1 \leq i \leq n$ are also implied by $\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$.

Finally, to simplify notation, we will let X_n refer to the parameter list $x_1, \ldots, x_n$, so that $F(X_n)$ denotes $F(x_1, \ldots, x_n)$. Let $F(X_n; x_i \leftarrow x_i x_j^a)$ denote the function $F(X_n)$ with all occurrences of x_i replaced by $x_i x_j^a$.

The following was proved in [**CLS07**]. It was shown there that these five guidelines suffice to find the generating function for any system of homogeneous linear inequalities.

Theorem 2.1. *(The Five Guidelines)* [**CLS07**]

1. If $\mathcal{C}$ *contains only the constraint* $[\lambda_1 \geq t]$, *for integer* $t \geq 0$, *then*

$$F_{\mathcal{C}}(x_1) = \frac{x_1^t}{1 - x_1}.$$

2. If $\mathcal{C}_1$ *is a set of constraints on variables* $\lambda_1, \ldots, \lambda_j$ *and* $\mathcal{C}_2$ *is a set of constraints on variables* $\lambda_{j+1}, \ldots, \lambda_n$, *then*

$$F_{\mathcal{C}_1 \cup \mathcal{C}_2}(x_1, \ldots, x_n) = F_{\mathcal{C}_1}(x_1, \ldots x_j) F_{\mathcal{C}_2}(x_{j+1}, \ldots, x_n).$$

3. Let $\mathcal{C}$ *be a set of linear constraints on variables* $\lambda_1, \ldots, \lambda_n$ *and assume* $\mathcal{C}$ *contains the constraints* $[\lambda_i \geq 0], 1 \leq i \leq n$. *Let a be any integer (possibly negative). If* $[\lambda_i - a\lambda_j \geq 0]$ *is implied by* $\mathcal{C}$,

$$F_{\mathcal{C}}(X_n) = F_{\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}}(X_n; x_j \leftarrow x_j x_i^a).$$

4. Let c be any constraint with the same variables as the set $\mathcal{C}$. The solutions to $\mathcal{C}$ can be partitioned into those satisfying c and those violating c, so

$$F_{\mathcal{C}}(X_n) = F_{\mathcal{C}\cup\{c\}}(X_n) + F_{\mathcal{C}\cup\{\neg c\}}(X_n).$$

5. Let $c \in \mathcal{C}$. The solutions to $\mathcal{C}$ are those solutions to $\mathcal{C} - c$ that do not violate c, so

$$F_{\mathcal{C}}(X_n) = F_{\mathcal{C}-\{c\}}(X_n) - F_{\mathcal{C}-\{c\}\cup\{\neg c\}}(X_n).$$

3. Digraph Rules

Let $G = (V, E)$ be a digraph with $V = \{1, \ldots, n\}$ and with certain edges in E designated as *strict*. Let S_G be the set of nonnegative integer sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying $\lambda_i \geq \lambda_j$ for every edge (i, j) in G and $\lambda_i > \lambda_j$ for every strict edge (i, j) in G. We seek the generating function

$$F_G(x_1, \ldots, x_n) \triangleq \sum_{\lambda \in S_G} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}.$$

THEOREM 3.1. *For $i \in V$, let $G + (n+1, i)$ denote the graph obtained from G by adding vertex $n+1$ and an edge from $n+1$ to i. Then*

$$F_{G+(n+1,i)}(x_1, \ldots, x_{n+1}) = \frac{F_G(x_1, \ldots, x_{i-1}, x_i x_{n+1}, x_{i+1}, \ldots, x_n)}{1 - x_{n+1}}.$$

If the inequality corresponding to edge $(n+1, i)$ is to be strict, the generating function on the right-hand side is multiplied by x_{n+1}.

PROOF. For every integer $j \geq 0$, $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in S_G$ if and only if $(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_i + j) \in S_{G+(n+1,i)}$. So,

$$\begin{aligned} F_{G+(n+1,i)}(x_1, \ldots, x_{n+1}) &= \sum_{\lambda \in S_{G+(n+1,i)}} x_1^{\lambda_1} \cdots x_{n+1}^{\lambda_{n+1}} \\ &= \sum_{j=0}^{\infty} \sum_{\lambda \in S_G} x_1^{\lambda_1} \cdots x_n^{\lambda_n} x_{n+1}^{\lambda_i + j} \\ &= \sum_{j=0}^{\infty} x_{n+1}^j \sum_{\lambda \in S_G} x_1^{\lambda_1} \cdots (x_i x_{n+1})^{\lambda_i} \cdots x_n^{\lambda_n} \\ &= \frac{1}{1 - x_{n+1}} F_G(x_1, \ldots, x_{i-1}, x_i x_{n+1}, x_{i+1}, \ldots, x_n). \end{aligned}$$

If $(n+1, i)$ is a strict edge, then the sum over j starts at $j = 1$ rather than $j = 0$ and we are left with an x_{n+1} in the numerator at the end. □

For $i, j \in V$, we use the notation $j \rightsquigarrow_G i$ to mean that there is a directed path from j to i in G. (Note $i \rightsquigarrow_G i$ for all $i \in V$). If v is a vertex of G, then $G - v$ denotes the graph obtained from G by deleting v and all its incident edges.

THEOREM 3.2. *Let G be an acyclic digraph with vertex set $\{1, \ldots, n+1\}$. Suppose $j \rightsquigarrow_G n+1$ for every vertex j of G. Then*

$$F_G(x_1, \ldots, x_{n+1}) = \frac{F_{G-(n+1)}(x_1, \ldots, x_n)}{(1 - x_1 x_2 \cdots x_n x_{n+1})}.$$

Proof. Since G is acyclic, it is possible to produce a listing $i_1, i_2, \ldots, i_{n+1}$ of the vertices of G satisfying: if $(i_j, i_k) \in E(G)$ then $j < k$. Necessarily $i_{n+1} = n+1$. Let $\mathcal{C}$ be the set of constraints corresponding to the edges of G. Perform the sequence of substitutions

$$\lambda_{i_j} \leftarrow \lambda_{i_j} + \lambda_{n+1}, \quad j = 1, \ldots, n$$

on the constraints in $\mathcal{C}$. Let $\mathcal{C}_j$ denote the resulting set of constraints after the jth substitution, with $\mathcal{C}_0 = \mathcal{C}$. We claim that the following must be true:

(i) $\lambda_{i_t} \geq \lambda_{n+1}$ is implied by $\mathcal{C}_j$ for $t > j$, and

(ii) constraint $\lambda_{i_s} \geq \lambda_{i_t}$ from $\mathcal{C}$ appears now in $\mathcal{C}_j$ as

$$\begin{aligned} \lambda_{i_s} &\geq 0 && \text{if } \ s \leq j \ \text{ and } \ t = n+1; \\ \lambda_{i_s} + \lambda_{n+1} &\geq \lambda_{i_t} && \text{if } \ s \leq j < t < n+1; \\ \lambda_{i_s} &\geq \lambda_{i_t} && \text{otherwise.} \end{aligned}$$

We prove the claim by induction on j. Initially, with $j = 0$, (i) holds since $j \rightsquigarrow_G n+1$ for every vertex j of G and (ii) holds since $j < s$. Let $j \geq 1$ and assume that (i) and (ii) hold after the jth iteration. During iteration $j+1$, the substitution $\lambda_{i_{j+1}} \leftarrow \lambda_{i_{j+1}} + \lambda_{n+1}$ is done to all occurrences of $\lambda_{i_{j+1}}$ in $\mathcal{C}_j$. Note at this time that $\lambda_{i_{j+1}} \geq \lambda_{n+1}$ (by (i)), so guideline 3 of Theorem 2.1 can be used to recover the generating function of $\mathcal{C}_j$ from the generating function of $\mathcal{C}_{j+1}$ by the substitution $x_{n+1} \leftarrow x_{n+1} x_{i_{j+1}}$. By (ii), any constraint with $\lambda_{i_{j+1}}$ on the left-hand-side has the form $\lambda_{i_{j+1}} \geq \lambda_{i_t}$ and becomes $\lambda_{i_{j+1}} \geq 0$ if $t = n+1$ and otherwise becomes $\lambda_{i_{j+1}} + \lambda_{n+1} \geq \lambda_{i_t}$. Again by (ii), any constraint with $\lambda_{i_{j+1}}$ on the right-hand-side has the form $\lambda_{i_s} + \lambda_{n+1} \geq \lambda_{i_{j+1}}$ and this become $\lambda_{i_s} \geq \lambda_{i_{j+1}}$. No other constraint is altered during this iteration, so (i) and (ii) are preserved.

Now observe that after iteration n, condition (ii) implies that condition $\lambda_{i_s} \geq \lambda_{i_t}$ from $\mathcal{C}$ appears now in $\mathcal{C}_n$ as $\lambda_{i_s} \geq 0$ if $t = n+1$ and otherwise as $\lambda_{i_s} \geq \lambda_{i_t}$, unchanged. Thus $\mathcal{C}_n$ is the system of constraints corresponding to the graph $G - (n+1)$ together with the isolated vertex $n+1$ representing the constraint $\lambda_{n+1} \geq 0$. By guidelines 1 and 2 of Theorem 2.1,

$$F_{\mathcal{C}_n}(x_1, x_2, \ldots, x_n, x_{n+1}) = \frac{F_{G-(n+1)}(x_1, x_2, \ldots, x_n)}{1 - x_{n+1}}.$$

Now using guideline 3 of Theorem 2.1 we successively recover the generating function for C_{j-1} from the one for C_j by

$$F_{\mathcal{C}_{j-1}}(x_1, x_2, \ldots, x_n, x_{n+1}) = F_{\mathcal{C}_j}(x_1, x_2, \ldots, x_n, x_{n+1} x_{i_j}),$$

so that finally we have $F_G = F_{\mathcal{C}} = F_{\mathcal{C}_0}$ as

$$F_G(x_1, x_2, \ldots, x_n, x_{n+1}) = F_{\mathcal{C}_n}(x_1, x_2, \ldots, x_n, x_1 x_2 \cdots x_n x_{n+1}),$$

and the result follows. □

4. Plane Partitions with Double Diagonal

We illustrate the "digraph method" on the graph H_n of Figure 1 to find a closed form for the generating function

$$f_{H_n}(x_1, x_2, \ldots, x_{2n}) = \sum_{\lambda \in S_{H_n}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{2n}^{\lambda_{2n}}.$$

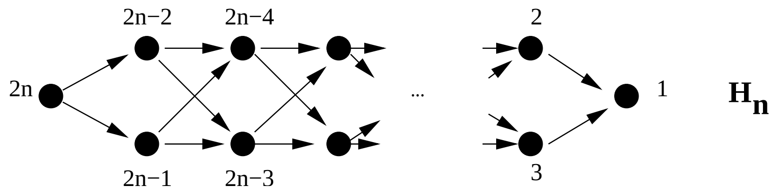

FIGURE 1. Plane partitions with double diagonal

We call these graphs *plane partitions with double diagonal* because they can be obtained from the plane partitions with diagonals of [**APR04**] by adding an additional diagonal.

This example was suggested to us by Sylvie Corteel, who conjectured the form of $f_{H_n}(q, q, \ldots, q)$. S_{H_n} is the set of sequences $\lambda_1, \ldots, \lambda_{2n}$ satisfying $\lambda_i \geq \lambda_j$ if (i, j) is an edge in H_n. This example illustrates the power of Theorem 2.1(4) in conjunction with Theorem 3.1.

The elements of S_{H_n} can be partitioned into those with $\lambda_{2n-1} \geq \lambda_{2n-2}$ and those with $\lambda_{2n-2} > \lambda_{2n-1}$, so by Theorem 2.1(4),

$$f_{H_n} = f_J + f_K,$$

where J and K are the graphs in Figure 2.

In J, the edges representing constraints $(2n, 2n-2)$, $(2n-1, 2n-3)$, and $(2n-1, 2n-4)$ are redundant. Removing them gives J' in Figure 2. Similarly, removing redundant constraints from K gives J' in Figure 2, and now

$$f_{H_n} = f_{J'} + f_{K'}.$$

The graph J' can be obtained from H_{n-1} by adding the edges $(2n-1, 2n-2)$ and $(2n, 2n-1)$. Thus by Theorem 3.1, to get the generating function $f_{J'}$, start with the generating function for H_{n-1}:

$$H_{n-1}(x_1, x_2, \ldots, x_{2n-3}, x_{2n-2}).$$

To add edge $(2n-1, 2n-2)$, replace x_{2n-2} by $x_{2n-2}x_{2n-1}$ throughout and divide by $(1 - x_{2n-1})$:

$$\frac{H_{n-1}(x_1, x_2, \ldots, x_{2n-3}, x_{2n-2}x_{2n-1})}{1 - x_{2n-1}}.$$

To add edge $(2n, 2n-1)$, replace x_{2n-1} by $x_{2n-1}x_{2n}$ throughout and divide by $(1 - x_{2n})$ and the result is $f_{J'}$:

$$f_{J'} = \frac{H_{n-1}(x_1, x_2, \ldots, x_{2n-3}, x_{2n-2}x_{2n-1}x_{2n})}{(1 - x_{2n-1}x_{2n})(1 - x_{2n})}.$$

The graph K' can be obtained from H_{n-1} by first relabeling vertex $2n-2$ in H_{n-1} as $2n-1$ and then adding the strict edge $(2n-2, 2n-1)$ and the edge $(2n, 2n-2)$. Thus, to get the generating function $f_{K'}$, start with the generating function for H_{n-1} with x_{2n-2} relabeled as x_{2n-1}:

$$H_{n-1}(x_1, x_2, \ldots, x_{2n-3}, x_{2n-1}).$$

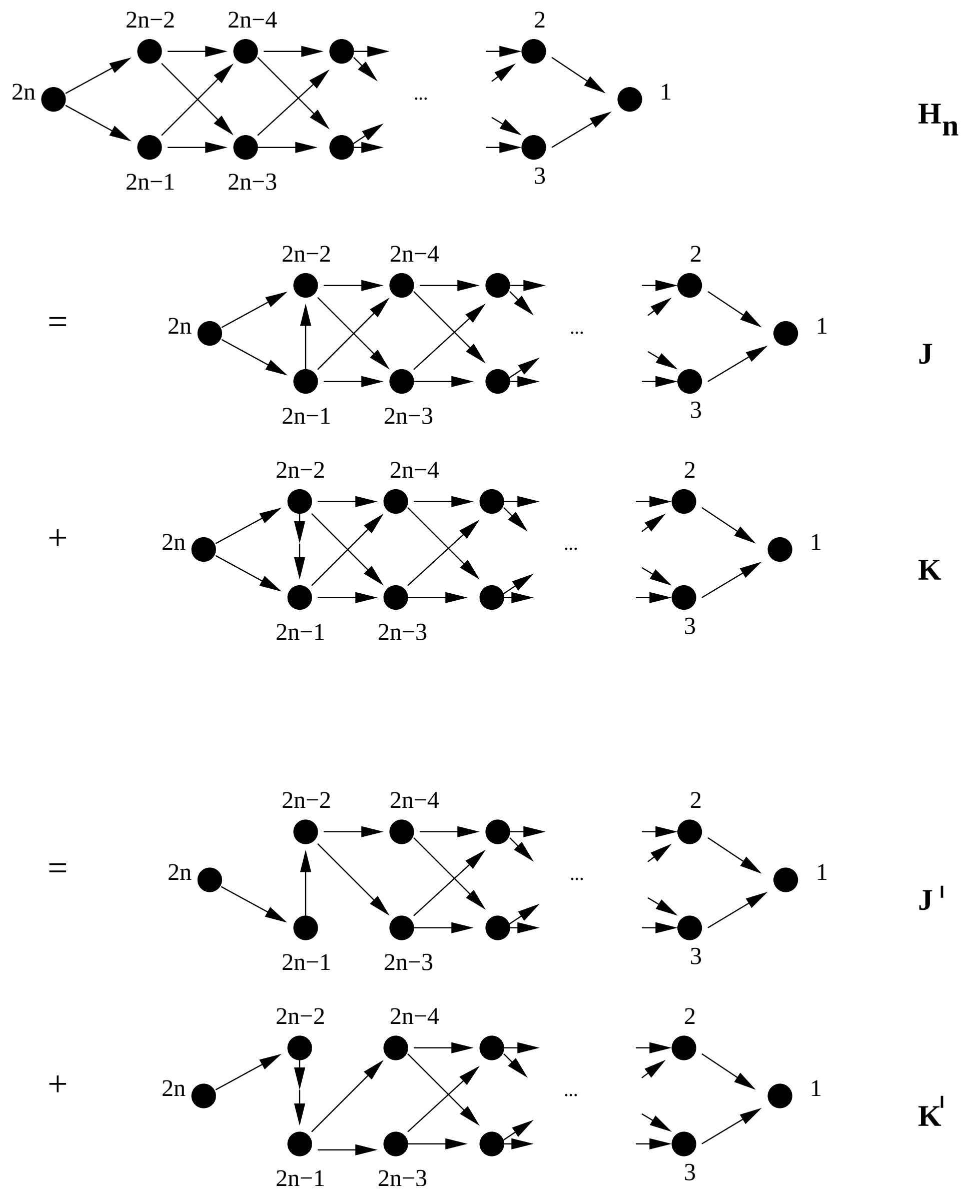

FIGURE 2. The recurrence for plane partitions with double diagonal

Because of the strict edge $(2n-2, 2n-1)$, replace x_{2n-1} by $x_{2n-1}x_{2n-2}$ throughout, multiply by x_{2n-2} and divide by $(1 - x_{2n-2})$:

$$\frac{x_{2n-2}H_{n-1}(x_1, x_2, \ldots, x_{2n-3}, x_{2n-1}x_{2n-2})}{1 - x_{2n-2}}.$$

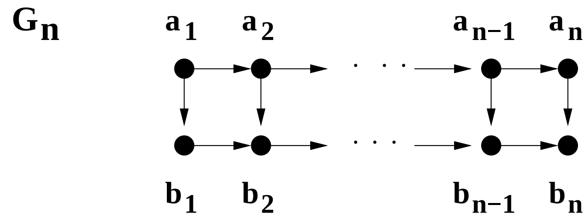

FIGURE 3. Two-rowed plane-partitions.

Because of the edge $(2n, 2n-2)$,replace x_{2n-2} by $x_{2n-2}x_{2n}$ throughout and divide by $(1-x_{2n})$ and the result is $f_{K'}$:

$$f_{K'} = \frac{x_{2n-2}x_{2n}H_{n-1}(x_1, x_2, \ldots, x_{2n-3}, x_{2n-2}x_{2n-1}x_{2n})}{(1-x_{2n-2}x_{2n})(1-x_{2n})}.$$

The resulting recurrence for H_n is
$H_n(x_1, x_2, \ldots, x_{2n}) =$

$$\begin{aligned} & \frac{H_{n-1}(x_1, x_2, \ldots, x_{2n-3}, x_{2n-2}x_{2n-1}x_{2n})}{(1-x_{2n})(1-x_{2n-1}x_{2n})} \\ + & \frac{x_{2n-2}x_{2n}H_{n-1}(x_1, x_2, \ldots, x_{2n-3}, x_{2n-2}x_{2n-1}x_{2n})}{(1-x_{2n})(1-x_{2n-2}x_{2n})} \\ = & \frac{(1-x_{2n}^2x_{2n-1}x_{2n-2})H_{n-1}(x_1, x_2, \ldots, x_{2n-3}, x_{2n-2}x_{2n-1}x_{2n})}{(1-x_{2n})(1-x_{2n-1}x_{2n})(1-x_{2n-2}x_{2n})} \end{aligned}$$

It is straightforward to show that the solution to the recurrence is

$$H_n(x_1, x_2, \ldots, x_{2n}) = \frac{(1-x_{2n}^2x_{2n-1}x_{2n-2})\prod_{i=1}^{n-2}(1-\prod_{j=0}^{2i}x_{2n-j}^2\prod_{j=2i+1}^{2i+2}x_{2n-j})}{\prod_{i=1}^{2n}(1-x_ix_{i+1}\ldots x_{2n})\prod_{i=1}^{n-1}(1-x_{2i}x_{2i+2}x_{2i+3}\ldots x_{2n})}.$$

Setting even indexed variables to x and odd to y:

$$H_n(x, y, x, y, \ldots) = \frac{(x^{2n-3}y^{2n-1}; (xy)^{-2})_{\lfloor n/2 \rfloor}}{(xy; xy)_n(y; xy)_n(y^2; x^2y^2)_{\lfloor n/2 \rfloor}},$$

and setting $x = y = q$:

$$H_n(q) = \frac{(1+q^2)(1+q^4)\cdots(1+q^{2(n-1)})}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^{2n})},$$

confirming Corteel's conjecture. An alternate "digraph" proof appears in the thesis of D'Souza [**D'S05**]. In [**AP07**], partition analysis is used to compute the generating function for a chain of copies of H_n and it is shown that when the diagrams are "broken", by deleting a source vertex, interesting congruence properties emerge.

5. Two-rowed Plane Partitions

We can use the digraph method to prove MacMahon's generating function [**Mac12**] for the two-rowed plane partitions defined by Figure 3:

$$P_{2\times n}(q) \triangleq \sum_{\lambda \in S_{G_n}} q^{|\lambda|} = \frac{1}{(q;q)_n(q^2;q)_n},$$

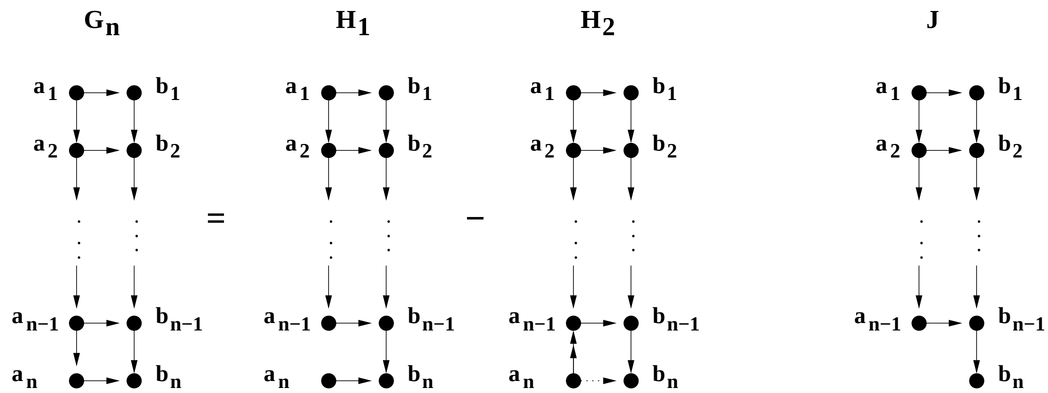

FIGURE 4. The recurrence for two-rowed plane-partitions.

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. This is easier than both Andrews' approach with partition analysis in [**And00**] and our approach with the five guidelines in [**CLS07**]. This example illustrates the power of Theorem 2.1(5) and Theorem 3.2 used in conjunction with Theorem 3.1.

Let G_n be the first graph in Figure 4. Then S_{G_n} is the set of two-rowed plane partitions. Define

$$f_{G_n}(x_1, y_2, x_2, y_2, \ldots, x_n, y_n) \triangleq \sum_{(a_1,b_1,\ldots,a_n,b_n)\in S_{G_n}} x_1^{a_1} y_1^{b_1} x_2^{a_2} y_2^{b_2} \cdots x_n^{a_n} y_n^{b_n}.$$

We derive a recurrence for f_{G_n}. Using guideline 5 of Theorem 2.1 with edge $e = (a_{n-1}, a_n)$ of G_n, we have $f_{G_n} = f_{H_1} - f_{H_2}$, where H_1 and H_2 are shown in Figure 4. Note that in H_2 the edge (a_n, a_{n-1}) is strict and the edge (a_n, b_n) is redundant (so it can be deleted). Graph H_1 is obtained from graph J in Figure 4. by adding edge (a_n, b_n); and J is obtained from G_{n-1} by adding edge (b_{n-1}, b_n). Thus, we get f_{H_1} from f_J using Theorem 3.1 and f_J from $f_{G_{n-1}}$ using Theorem 3.2. As for H_2, after deleting (a_n, b_n), it is obtained from J by adding strict edge (a_n, a_{n-1}), so we apply Theorem 3.1 to get f_{H_2} from f_J. Putting this all together gives

(5.1)

$$f_{G_n}(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) =$$
$$\frac{f_{G_{n-1}}(x_1, y_1, x_2, y_2, \ldots, x_{n-1}, y_{n-1}) - x_n f_{G_{n-1}}(x_1, y_1, x_2, y_2, \ldots, x_{n-2}, y_{n-2}, x_{n-1}x_n, y_{n-1})}{(1-x_n)(1-x_1y_1x_2y_2\cdots x_ny_n)},$$

with initial condition $f_{G_1}(x_1, y_1) = 1/(1-x_1)/(1-x_1y_1)$.

Define

$$f_n(q,s) = f_{G_n}(q, q, q, q, \ldots, q, q, s, q) = \sum_{\lambda=(a_1,b_1,\ldots,a_n,b_n)\in S_{G_n}} q^{|\lambda|}(s/q)^{a_n}.$$

Then the recurrence for f_{G_n} gives

$$f_n(q,s) = \frac{f_{n-1}(q,q) - s f_{n-1}(q,sq)}{(1-s)(1-sq^{2n-1})}, \tag{5.2}$$

with initial condition $f_1(q,s) = 1/(1-s)/(1-qs)$. It is straightforward to verify by induction that the solution to this recurrence is

$$f_n(q,s) = \frac{P_{2\times(n-1)}(q)}{(1-sq^{n-1})(1-sq^n)}.$$

Then observe that setting $s = q$ gives $f_n(q,q) = P_{2\times n}(q)$.

6. Three-rowed Plane Partitions

We can use a similar strategy to derive MacMahon's generating function [**Mac12**] for three-rowed plane partitions, defined by the constraint graph G_n in Figure 5:

$$P_{3\times n}(q) = F_{G_n}(q) = \frac{1}{(q;q)_n(q^2;q)_n(q^3;q)_n}. \tag{6.1}$$

We compute

$$F_{G_n}(x_1,\ldots,x_{3n}) = \sum x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_{3n}^{\lambda_{3n}},$$

where the sum is over all sequences $(\lambda_1,\lambda_2,\ldots,\lambda_{3n})$ of nonnegative integers satisfying the constraints of G_n.

Using the intermediate graph H_n defined in Figure 5, observe that

$$G_n = (H_{n-1} + (3n, 3n-1)) - (H_{n-1} + (3n, 3n-3)^*),$$

where $(3n, 3n-3)^*$ denotes a strict edge and

$$H_{n-1} = ((G_{n-1}+(3n-5,3n-2))+(3n-1,3n-2))-((G_{n-1}+(3n-5,3n-2))+(3n-1,3n-4)^*),$$

where $(3n-1, 3n-4)^*$ is strict. Then, letting X_m denote the argument list $x_1,\ldots,x_m$, by Theorems 2.1, 3.1 and 3.2 we have the following mutual recursion

Proposition 6.1.

$$F_{G_n}(X_{3n}) = \frac{F_{H_{n-1}}(X_{3n-1}; x_{3n-1} \leftarrow x_{3n-1}x_{3n}) - x_{3n}F_{H_{n-1}}(X_{3n-1}; x_{3n-3} \leftarrow x_{3n-3}x_{3n})}{(1-x_{3n})},$$

$$F_{H_{n-1}}(X_{3n-2}) = \frac{F_{G_{n-1}}(X_{3n-3}) - x_{3n-1}F_{G_{n-1}}(X_{3n-3}; x_{3n-4} \leftarrow x_{3n-4}x_{3n-1})}{(1-x_1x_2\cdots x_{3n-1})(1-x_{3n-1})}$$

with initial conditions

$$F_{G_1} = \frac{1}{(1-x_3)(1-x_2x_3)(1-x_1x_2x_3)}; \quad F_{H_0} = \frac{1}{(1-x_2)(1-x_1x_2)}.$$

Define
$g_n(q,s,t) = F_{G_n}(X_{3n}; x_{3n} \leftarrow s, x_{3n-1} \leftarrow t; x_i \leftarrow q, otherwise)$ and
$h_{n-1}(q,s,t) = F_{H_{n-1}}(q,q,q,\ldots,q,q,s,q,t)$.
Then from Proposition (6.1),

$$\begin{aligned} g_n(q,s,t) &= \frac{h_{n-1}(q,q,st) - sh_{n-1}(q,qs,t)}{(1-s)} \\ h_n(q,s,t) &= \frac{g_n(q,s,q) - tg_n(q,s,qt)}{(1-stq^{3n})(1-t)}, \end{aligned}$$

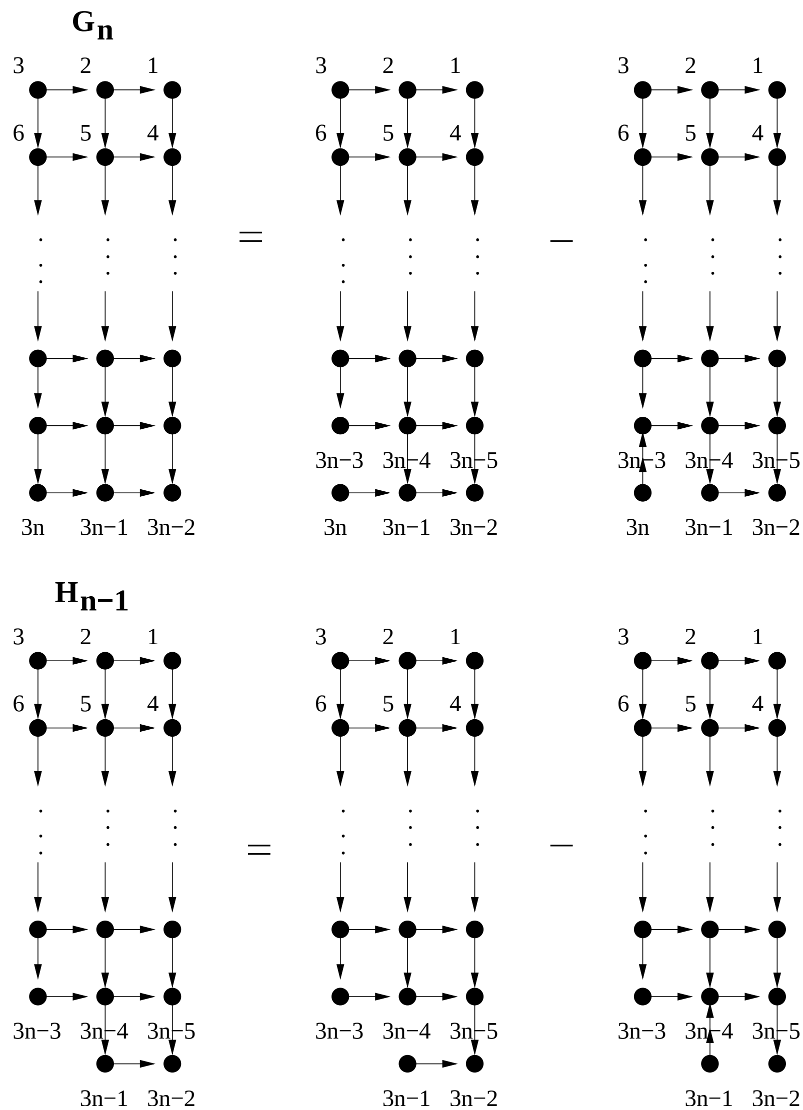

FIGURE 5. The mutual recurrence for three-rowed plane partitions.

It is straightforward to check with Maple and prove by induction that the solutions to these recurrences are:

$$g_n(q,s,t) = \frac{P_{3\times(n-1)}(q)W_n(q,s,t)}{(1-sq^n)(1-sq^{n+1})(1-sq^{n-1})(1-stq^{2n})(1-stq^{2n-1})(1-stq^{2n-2})}$$

$$h_n(q,s,t) = \frac{P_{3\times(n-1)}(q)(1-s^2tq^{3n})}{(1-stq^{2n-1})(1-stq^{2n})(1-stq^{2n+1})(1-sq^{n-1})(1-sq^n)(1-sq^{n+1})},$$

where $P_{3\times n}(q)$ is (6.1) and $W_n = W_n(q,s,t)$ is

$$W_n = 1 - s^3tq^{6n-1} - stq^{3n-1} + s^2q^{3n} - s^2t(q^{3n-1} - q^{4n-2} - q^{4n-1} - q^{4n}) - s(q^{2n-1} + q^{2n} + q^{2n+1} - q^{3n}).$$

A nice form results when we let $t = q$:

$$h_n(q,s,q) = P_{3\times(n-1)}(q)\frac{1-s^2q^{3n+1}}{(1-sq^{2n})(1-sq^{2n+1})(1-sq^{2n+2})(1-sq^{n-1})(1-sq^n)(1-sq^{n+1})}$$

$$g_n(q,s,q) = P_{3\times(n-1)}(q)\frac{1}{(1-sq^n)(1-sq^{n+1})(1-sq^{n-1})}$$

and $g_n(q,s,q)$ becomes $P_{3\times n}(q)$ when $s = q$.

7. Concluding Remarks

7.1. Automating the Process. The digraph method can be used in a similar, straightforward way, to find the generating function for many other families, including plane partition diamonds [**APR01e, CS03**], plane partitions with diagonal [**APR04**], hexagonal plane partitions [**AP07**], vertex-joined enriched hexagons [**AP07**], and up-down compositions [**Pro00**]. In ongoing work, we are applying it to get a recurrence for the generating function for $2 \times 2 \times n$ solid partitions.

From the examples of Sections $4-6$, it becomes apparent that once a recursive description of the digraph is specified, deriving a recurrence for the generating function becomes mechanical. In fact, D'Souza has written a program that takes as input a recursive description of a directed graph G and outputs not only a recurrence for F_G, but a Maple program to compute it [**D'S05**]. His program determines the full generating function recurrence (of the form (5.1), but can also automatically determine a finite-variable recurrence (of the form (5.2)). Many examples are presented in his thesis [**D'S05**].

On the other hand, the digraph method is not entirely mechanical. The place where strategy is required is in finding a recursive description of a digraph that will lead to a simple, solvable, generating function recurrence.

7.2. Relationship with P-partitions. In some sense, enumerating the solutions to linear inequalities defined by a digraph G with n vertices should be easy. We know from Stanley's theory of P-partitions [**Sta86**] that the generating function has the form

$$F_G(q) = \frac{\sum_{\pi\in L(P)} q^{\mathrm{maj}(\pi)}}{(q;q)_n}. \tag{7.1}$$

Here, P is the poset obtained from G by reversing the order relation; P is given a natural labeling $\{1,\ldots,n\}$, consistent with the partial order; $L(P)$ is the set of linear extensions of P; and $\mathrm{maj}(\pi)$ is the sum of the descent positions in π.

However, counting the number of linear extensions is #P-complete [**BW91**], so we do not expect to have an efficient method to compute $F_G(q)$, or even expect that it would have a compact representation. We can expect a connection between directed graphs whose generating functions have a nice form and posets in which the number of linear extensions have a nice form. The best example of this can be found in families with *hook length formulas*, such as reverse plane partitions [**FRT54, Sta71**], forests [**Knu75, BW89**], and d-complete posets [**Pro99**].

Although the families with hook length formulas are limited [**Pro03**], a graph with a recursive structure should at least have a recursively defined generating function. The recurrence itself could be sufficient to prove, for example, divisibility properties of the generating function, as illustrated in [**AG78**] for the q-tangent numbers (which arise as the generating function for up-down compositions).

For directed graphs G with a recursive structure, to what extent can $F_G(q)$ be identified? When can we get a recurrence for $F_G(q)$? What properties of $F_G(q)$ can be deduced from the recurrence? The digraph method was developed as a tool to investigate these questions.

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