Euler's partition theorem and the combinatorics of ℓ-sequences

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Abstract

Euler's partition theorem states that the number of partitions of an integer N into odd parts is equal to the number of partitions of N in which the ratio of successive parts is greater than 1. It was shown by Bousquet-Mélou and Eriksson in [9] that a similar result holds when "odd parts" is replaced by "parts that are sums of successive terms of an ℓ-sequence" and the ratio "1" is replaced by a root of the characteristic polynomial of the ℓ-sequence. This generalization of Euler's theorem is intrinsically different from the many others that have appeared, as it involves a family of partitions constrained by the *ratio* of successive parts.

In this paper, we provide a surprisingly simple bijection for this result, a question suggested by Richard Stanley. In fact, we give a parametrized family of bijections, that include, as special cases, Sylvester's bijection and a bijection for the lecture hall theorem. We introduce *Sylvester diagrams* as a way to visualize these bijections and deduce their properties.

In proving the bijections, we uncover the intrinsic role played by the combinatorics of ℓ-sequences and use this structure to give a combinatorial charaterization of the partitions defined by the ratio constraint. Several open questions suggested by this work are described.

1 Introduction

The main result of this paper is a *simple bijection* that establishes the following generalization of Euler's partition theorem.

Theorem 1 (The ℓ-Euler Theorem [9]) *For integer $\ell \geq 2$, define the sequence $\{a_n^{(\ell)}\}_{n\geq 0}$ by*

$$a_n^{(\ell)} = \ell a_{n-1}^{(\ell)} - a_{n-2}^{(\ell)},$$

with initial conditions $a_0^{(\ell)} = 0$, $a_1^{(\ell)} = 1$. Let c_ℓ be the largest root of the characteristic equation

$$x^2 - \ell x + 1 = 0.$$

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Then the number of partitions of an integer N into parts from the set

$$\{a_0^{(\ell)} + a_1^{(\ell)},\ a_1^{(\ell)} + a_2^{(\ell)},\ a_2^{(\ell)} + a_3^{(\ell)},\ \ldots\}$$

is the same as the number of partitions of N in which the ratio of consecutive (positive) parts is greater than c_ℓ.

Note that $\{a_n^{(2)}\}_{n\geq 0} = (0, 1, 2, 3, 4, 5, \ldots)$ and $c_2 = 1$. Thus, setting $\ell = 2$ in Theorem 1 gives the well-known theorem of Euler: *the number of partitions of an integer N into* **odd** *parts is equal to the number of partitions of N into* **distinct** *parts.*

There have been several generalizations, refinements, and variations of Euler's partition theorem [2, 5, 7, 12, 20, 21, 24, 25, 27, 30], but Theorem 1 is strikingly different. It arose as the limiting case of an unusual enumeration result. Whereas partition identities typically involve relationships between families of partitions characterized by the set of allowable parts, by differences between parts, by rank conditions, etc., Theorem 1 involves a set of partitions constrained by the *ratio* of consecutive parts. With a few exceptions [11, 19, 23], *ratio constraints* did not arise in an interesting way until Bousquet-Mélou and Eriksson discovered the *Lecture Hall Theorem*:

Theorem 2 (The Lecture Hall Theorem [8]) *For $n \geq 0$,*

$$\sum_{\lambda} q^{\lambda_1+\lambda_2+\cdots+\lambda_n} = \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots(1-q^{2n-1})},$$

where the sum is over all sequences $\lambda = (\lambda_1, \lambda_2 \ldots, \lambda_n)$ satisfying

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \frac{\lambda_3}{n-2} \geq \cdots \geq \frac{\lambda_{n-1}}{2} \geq \frac{\lambda_n}{1} \geq 0.$$

These sequences λ are called **lecture hall partitions***.*

Note that taking limits as $n \to \infty$ in Theorem 2 gives Euler's theorem.

There is a generalization of Theorem 2 involving the ℓ-sequences $\{a_n^{(\ell)}\}_{n\geq 0}$. Bousquet-Melou and Eriksson discovered this and proved it (and more) in [9]:

Theorem 3 (The ℓ-Lecture Hall Theorem [9]) *For $\ell \geq 2$ and $n \geq 0$,*

$$\sum_{\lambda} q^{\lambda_1+\lambda_2+\cdots+\lambda_n} = \frac{1}{(1-q^{a_0^{(\ell)}+a_1^{(\ell)}})(1-q^{a_1^{(\ell)}+a_2^{(\ell)}})(1-q^{a_2^{(\ell)}+a_3^{(\ell)}})\cdots(1-q^{a_{n-1}^{(\ell)}+a_n^{(\ell)}})}$$

where the sum is over all sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying

$$\frac{\lambda_1}{a_n^{(\ell)}} \geq \frac{\lambda_2}{a_{n-1}^{(\ell)}} \geq \frac{\lambda_3}{a_{n-2}^{(\ell)}} \geq \ldots \geq \frac{\lambda_{n-1}}{a_2^{(\ell)}} \geq \frac{\lambda_n}{a_1^{(\ell)}} \geq 0.$$

We call these λ **ℓ-lecture hall partitions***.*

Now, as observed in [9], taking limits as $n \to \infty$ in Theorem 3 gives Theorem 1. In notes prepared for the Clay Mathematics Institute [26], Richard Stanley posed the problem of finding a bijection for Theorem 1 when $\ell = 3$ (Problem 127). In this paper we solve that problem for general $\ell \geq 2$. The bijection is surprisingly simple, coinciding with Sylvester's bijection [27] when $\ell = 2$.

The quest for a bijective proof of Theorem 1 should be viewed in the context of ongoing efforts to develop combinatorial tools for enumerating integer sequences constrained by the ratio of consecutive parts [6, 8, 9, 10, 13, 14, 15, 28, 29]. We have developed two new tools.

First, in order to visualize our bijection, we introduce the *Sylvester diagram*. This is a new variation on the fish hook diagrams and modular diagrams that are standard tools in combinatorial proofs of Euler-type theorems (e.g. [4, 7, 12, 22, 25, 27, 30])

Second, we devise an interpretation of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ satisfying $\lambda_i > c_\ell \lambda_{i+1}$ using the combinatorics of the ℓ-sequences $\{a_n^{(\ell)}\}_{n \geq 0}$.

The *Sylvester diagrams* and the *$a^{(\ell)}$-interpretations of partitions* introduced here appear to be both novel and powerful. To illustrate, we will use them to describe and prove simple bijections not only for Theorem 1, but for Theorems 2 and 3 as well.

The paper is organized as follows. In Section 2, we describe the bijections for Theorems 1,2 and 3 and introduce the Sylvester diagrams to illustrate them. In Section 3, we investigate the combinatorics of ℓ-sequences to construct the tools needed to prove the bijections. In Section 4, we prove that the mappings defined in Section 2 are actually bijections. This work suggests many new questions and we describe some of them in Section 5.

2 Bijections and Sylvester diagrams

2.1 Definitions and notation

A *partition* of an integer N is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_t = N$. Each λ_i is a *part* of λ and N is the *weight* of λ. When convenient or necessary, we define $\lambda_i = 0$ for $i > t$. The number of occurrences of j as a part of λ is the *multiplicity* of j in λ.

For positive integer $\ell \geq 2$, define the *ℓ-sequence* $\{a_n^{(\ell)}\}_{n \geq 0}$ by

$$a_n^{(\ell)} = \ell a_{n-1}^{(\ell)} - a_{n-2}^{(\ell)}, \tag{1}$$

with initial conditions $a_0^{(\ell)} = 0$, $a_1^{(\ell)} = 1$. For example,

$$\begin{aligned} \{a_n^{(2)}\}_{n \geq 0} &= 0, 1, 2, 3, 4, 5, 6, \ldots; \\ \{a_n^{(3)}\}_{n \geq 0} &= 0, 1, 3, 8, 21, 55, 144, \ldots. \end{aligned}$$

Define the sequence $\{p_n^{(\ell)}\}_{n \geq 1}$ by

$$p_n^{(\ell)} = a_n^{(\ell)} + a_{n-1}^{(\ell)}. \tag{2}$$

For example,

$$\begin{aligned} \{p_n^{(2)}\}_{n \geq 1} &= 1, 3, 5, 7, 9, 11, \ldots; \\ \{p_n^{(3)}\}_{n \geq 1} &= 1, 4, 11, 29, 76, 199, \ldots. \end{aligned}$$

Let $O^{(\ell)}$ be the set of partitions (of any integer) into parts from the infinite set

$$\{p_1^{(\ell)},\ p_2^{(\ell)},\ p_3^{(\ell)},\ \ldots\}.$$

For example, when $\ell = 2$, $O^{(2)}$ is the set of partitions into odd parts; when $\ell = 3$, there are five partitions of 12 in the set $O^{(3)}$, namely

$$(11,1),\ (4,4,4),\ (4,4,1,1,1,1),\ (4,1,1,1,1,1,1,1,1),\ (1,1,1,1,1,1,1,1,1,1,1,1).$$

Let $O_n^{(\ell)} \subseteq O^{(\ell)}$ be the set of partitions (of any integer) into parts from the finite set

$$\{p_1^{(\ell)},\ p_2^{(\ell)},\ p_2^{(\ell)},\ \ldots,p_n^{(\ell)}\}.$$

So, although there are five partitions of 12 in $O^{(3)}$, there are only four partitions of 12 in $O_2^{(3)}$, since part 11 is no longer allowed.

For fixed ℓ, we can represent a partition $\mu \in O_n^{(\ell)}$ as $\mu = p_n^{m_n} p_{n-1}^{m_{n-1}} \ldots p_1^{m_1}$, where m_i is the multiplicity of $p_i^{(\ell)}$ in μ. So the partitions of 12 in $O^{(3)}$ are $11^1 4^0 1^1 = 11^1 1^1,\ 4^3,\ 4^2 1^4,\ 4^1 1^8,$ and 1^{12}.

For $\ell \geq 2$, let $c_\ell = (\ell + \sqrt{\ell^2 - 4})/2$. That is, c_ℓ is the largest root of the characteristic equation for the ℓ-sequence: $x^2 - \ell x + 1 = 0$. So, e.g.,

$$c_2 = (2 + \sqrt{0})/2 = 1; \qquad c_3 = (3 + \sqrt{5})/2 \approx 2.618.$$

Let $D^{(\ell)}$ be the set of partitions in which the ratio of successive positive parts is greater than c_ℓ.
Examples:

$$\begin{aligned}
(4,3,3) \notin D^{(2)} &\quad \text{since} \quad 3/3 \not> 1 = c_2;\\
(7,6,5) \in D^{(2)} &\quad \text{since} \quad 7/6 > 1 = c_2 \;\text{ and }\; 6/5 > 1 = c_2;\\
(55,21,8,3) \in D^{(3)} &\quad \text{since} \quad 55/21 > c_3,\;\; 21/8 > c_3,\;\; \text{and}\;\; 8/3 > c_3.
\end{aligned}$$

$D^{(2)}$ is the set of partitions into *distinct* parts.

Let $D_n^{(\ell)}$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, called *$\ell$-lecture hall partitions*, satisfying

$$\frac{\lambda_n}{a_n^{(\ell)}} \geq \frac{\lambda_{n-1}}{a_{n-1}^{(\ell)}} \geq \cdots \geq \frac{\lambda_1}{a_1^{(\ell)}} \geq 0.$$

Examples:

$$\begin{aligned}
(7,6,5,0) \notin D_4^{(2)} &\quad \text{since} \quad 7/4 \not\geq 6/3;\\
(10,3,1) \in D_3^{(2)} &\quad \text{since} \quad 10/3 \geq 3/2 \geq 1/1 \geq 0;\\
(55,21,8,3) \notin D_4^{(3)} &\quad \text{since} \quad 55/21 \not\geq 21/8;\\
(55,21,8,3,0) \in D_5^{(3)} &\quad \text{since} \quad 55/55 \geq 21/21 \geq 8/8 \geq 3/3 \geq 0/1 \geq 0.
\end{aligned}$$

To prove Theorem 1, we would like a weight-preserving bijection:

$$\Theta^{(\ell)} \;:\; O^{(\ell)} \;\rightarrow\; D^{(\ell)}.$$

and to prove Theorems 2 and 3, a weight-preserving bijection:

$$\Theta_n^{(\ell)} \;:\; O_n^{(\ell)} \;\rightarrow\; D_n^{(\ell)}.$$

We define the bijections $\Theta^{(\ell)}$ and $\Theta_n^{(\ell)}$ in Section 2.3. First, in Section 2.2, we introduce *Sylvester diagrams* as a tool for visualizing the bijections.

2.2 Sylvester diagrams

With $a_n^{(\ell)}$ and $p_n^{(\ell)}$ defined as in (1) and (2), define $d_n^{(\ell)}$ for $n \geq 1$ by

$$d_n^{(\ell)} = a_n^{(\ell)} - a_{n-1}^{(\ell)}. \tag{3}$$

For example,

$$\begin{aligned} \{d_n^{(2)}\}_{n\geq 1} &= 1, 1, 1, 1, 1, 1, \ldots; \\ \{d_n^{(3)}\}_{n\geq 1} &= 1, 2, 5, 13, 34, 89, \ldots. \end{aligned}$$

To simplify notation, fix $\ell \geq 2$ and let $a_n = a_n^{(\ell)}$, $p_n = p_n^{(\ell)}$, and $d_n = d_n^{(\ell)}$. Note that

$$p_n = a_{n-1} + a_n = (d_1 + d_2 + \cdots + d_{n-1}) + (d_n + d_{n-1} + \cdots + d_2 + d_1).$$

For each $\mu \in O^{(\ell)}$, we associate a *Sylvester diagram* of filled cells (see Figure 1). For each part p_k of μ, there is a horizontal row of $2k-1$ cells filled, in order, with the integers:

$$d_1, d_2, \ldots, d_{k-1}, d_k, d_{k-1}, \ldots, d_2, d_1,$$

(all ones when $\ell = 2$.) Rows are vertically centered with longer rows above shorter rows. For example, when $\ell = 2$, the Sylvester diagram for $\mu = p_4^8 p_3^2 p_2^2 p_1 = 4^8 3^2 3^2 1$ is shown in Figure 1(a). When $\ell = 3$, the Sylvester diagram for $\mu = p_4^8 p_3^2 p_2^2 p_1 = 29^8 11^2 4^2 1$ is shown in Figure 1(b).

Note that a Sylvester diagram is a weighted and "unbent" version of the "fish hook" diagrams used by Sylvester to describe his bijection in [27], (p. 288). If, instead, each row in a Sylvester diagram is folded at a vertical line between the center column and the column to its left (see Figure 2(b)), and if entries in coinciding cells are summed, the result is an ℓ-weighted version of the modular diagram, introduced by MacMahon [22] (pp. 1090-1097), and used by Bessenrodt [7] and others. In the ℓ-weighted modular diagrams, the weights in the cells are from the set $\{d_i + d_{i-1} \mid i \geq 1\}$. The row in the Sylvester diagram corresponding to part p_i of μ is filled with the integers:

$$d_i + d_{i-1}, d_{i-1} + d_{i-2}, \ldots, d_2 + d_1, d_1.$$

When $\ell = 2$, each row has the form $(2, 2, \ldots, 2, 1)$, giving the familiar case of 2-modular diagrams. We note that the Sylvester diagrams defined here give a generalization of 2-modular diagrams that is *fundamentally different* from the (m, c)-generalizations discussed in [7, 25, 30].

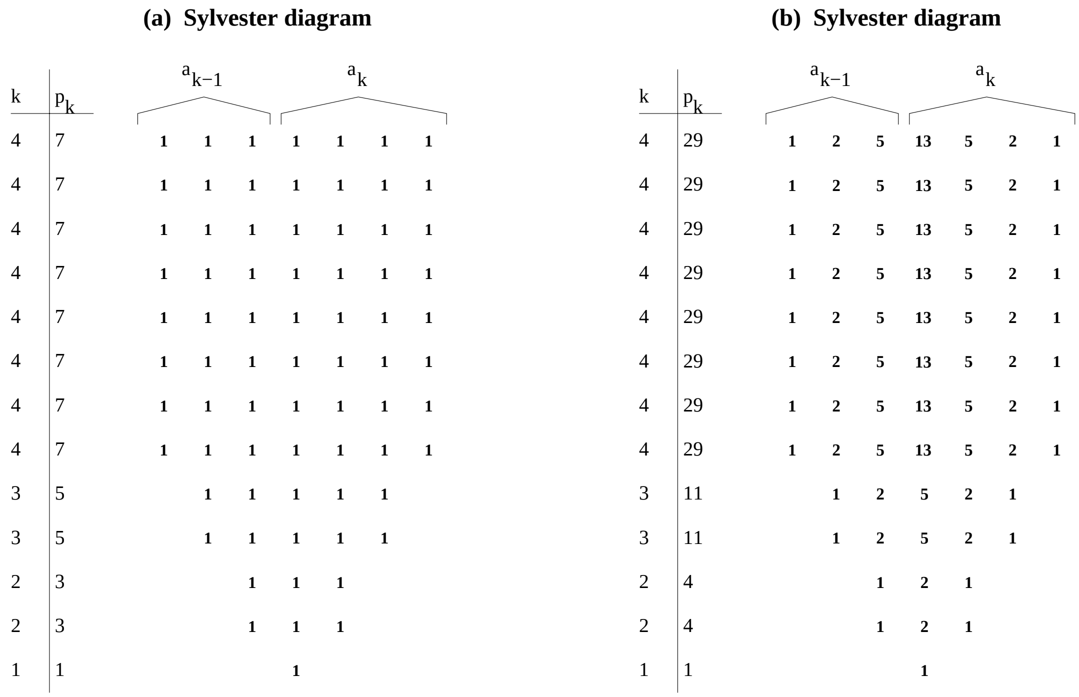

Figure 1: The Sylvester diagram for $\mu = p_4^8 p_3^2 p_2^2 p_1$ when (a) $\ell = 2$ and (b) $\ell = 3$.

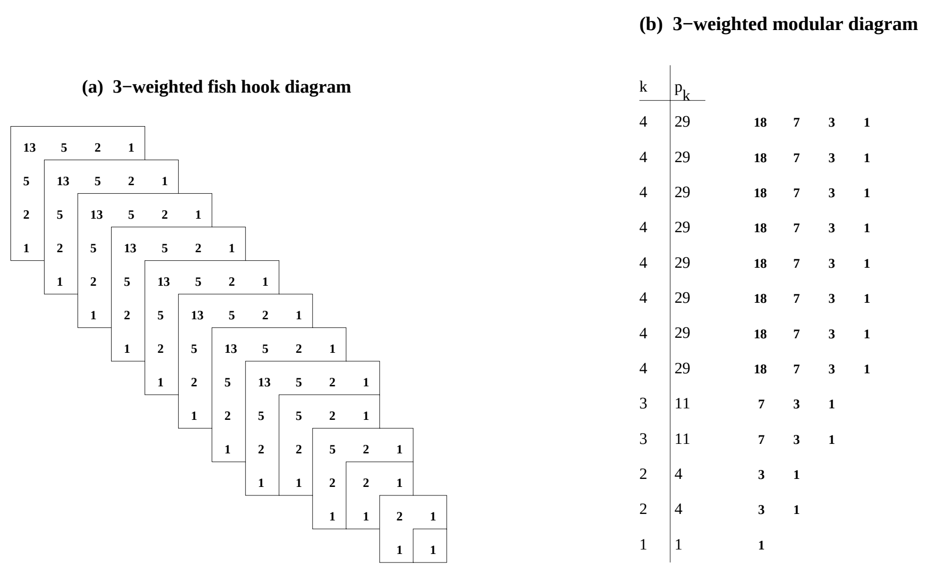

Figure 2: For $\mu = p_4^8 p_3^2 p_2^2 p_1$, the ℓ-weighted fish hook diagram (a) and the ℓ-weighted modular diagram when $\ell = 3$.

2.3 The Bijections

We now describe the bijections for the ℓ-Euler Theorem and the ℓ-Lecture Hall Theorem in such a way to emphasize their simplicity and their similarity. To simplify notation, assume that ℓ is fixed and let $a_n = a_n^{(\ell)}$ for $n \geq 0$ and $p_n = p_n^{(\ell)}$ for $n \geq 1$.

Bijection for the ℓ-Euler Theorem

$$\Theta^{(\ell)}: \quad O^{(\ell)} \quad \rightarrow \quad D^{(\ell)}$$

For $\mu = p_r^{m_r} p_{r-1}^{m_{r-1}} \ldots p_1^{m_1} \in O^{(\ell)}$, define $\Theta^{(\ell)}(\mu)$ to be the sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ obtained from the empty sequence $(0, 0, \ldots)$ by inserting the parts of μ in nonincreasing order according to the following $\Theta^{(\ell)}$ insertion procedure.

To insert p_k into $(\lambda_1, \lambda_2, \ldots)$:
If $k = 1$, then add a_1 to λ_1;
otherwise, if $(\lambda_1 + a_k - a_{k-1}) > c_\ell(\lambda_2 + a_{k-1} - a_{k-2})$, Test(*)
add $a_k - a_{k-1}$ to λ_1, add $a_{k-1} - a_{k-2}$ to λ_2,
and recursively insert p_{k-1} into $(\lambda_3, \lambda_4, \ldots)$ via $\Theta^{(\ell)}$ insertion;
otherwise, add a_k to λ_1, and add a_{k-1} to λ_2.

Observe that by (2), $p_k = a_k + a_{k-1} = (a_k - a_{k-1}) + (a_{k-1} - a_{k-2}) + p_{k-1}$, so, the insertion procedure adds p_k to the weight of λ, making $\Theta^{(\ell)}$ weight-preserving.

Bijection for the ℓ-Lecture Hall Theorem

$$\Theta_n^{(\ell)}: \quad O_n^{(\ell)} \quad \rightarrow \quad D_n^{(\ell)}$$

For $\mu = p_n^{m_n} p_{n-1}^{m_{n-1}} \ldots p_1^{m_1} \in O_n^{(\ell)}$, define $\Theta_n^{(\ell)}(\mu)$ to be the sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ obtained from the empty sequence $(0, 0, \ldots, 0)$ by inserting the parts of μ in nonincreasing order according to the following $\Theta_n^{(\ell)}$ insertion procedure.

To insert p_k into $(\lambda_1, \lambda_2, \ldots, \lambda_n)$:
If $k = 1$, then add a_1 to λ_1;
otherwise, if $\lambda_1 + a_k - a_{k-1} \geq \frac{a_n}{a_{n-1}}(\lambda_2 + a_{k-1} - a_{k-2})$, Test(**)
add $a_k - a_{k-1}$ to λ_1, add $a_{k-1} - a_{k-2}$ to λ_2,
and recursively insert p_{k-1} into $(\lambda_3, \lambda_4, \ldots, \lambda_n)$ via $\Theta_{n-2}^{(\ell)}$ insertion;
otherwise, add a_k to λ_1, and add a_{k-1} to λ_2.

Note that $\Theta_n^{(\ell)}$ differs from $\Theta^{(\ell)}$ only in the following. (i) If $\mu \in O_n^{(\ell)}$, no part of μ is larger than $p_n^{(\ell)}$; (ii) in Test(**), the ratio constant is a_n/a_{n-1} rather than c_ℓ and the inequality is *not* strict; (iii) When Test(**) is passed, the recursive insertion is via $\Theta_{n-2}^{(\ell)}$, a "smaller" version of $\Theta_n^{(\ell)}$. In particular, $\Theta_n^{(\ell)}$ is also weight-preserving.

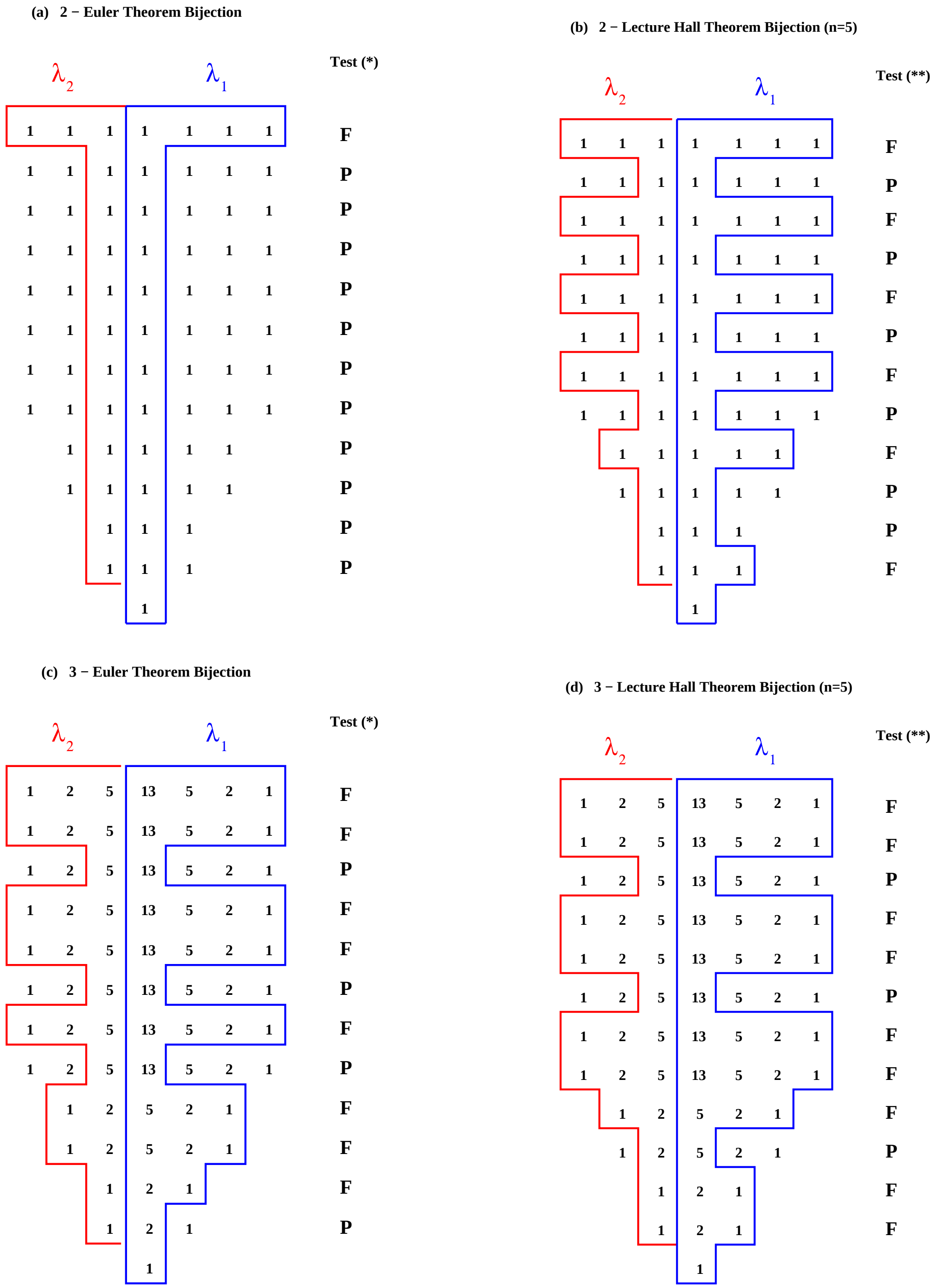

Figure 3: Comparing the bijections when inserting $\mu = p_4^8 p_3^2 p_2^2 p_1$.

We use the Sylvester diagrams of Section 2.2 to illustrate $\Theta^{(\ell)}$ and $\Theta_n^{(\ell)}$ on some examples. It will be shown in Section 4 that $\Theta^{(\ell)}$ and $\Theta_n^{(\ell)}$ are bijections.

Example: Computing $\Theta^{(2)}(\mu)$ for $\mu = 7^8 5^2 3^2 1 \in O^{(2)}$

Initially $\lambda = (0, 0, \ldots)$. The first part inserted is $p_4 = 7 = 4 + 3 = a_4 + a_3$. Recall that $c_2 = 1$. Since $\lambda_1 + a_4 - a_3 = 1 = \lambda_2 + a_3 - a_2$, Test(*) fails, so we add 4 to λ_1 and 3 to λ_2. Insertion of each subsequent part $p_i > 1$ in μ passes Test(*) and causes 1 to be added to λ_1 and to λ_2 and the part p_{i-1} to be inserted recursively. Any remaining parts $p_1 = 1$ in μ add 1 to λ_1. The record of passes and failures of Test(*) for the parts of μ and the contribution of those parts to λ_1 and λ_2 are shown on the Sylvester diagram for μ in Figure 3(a). Thus, for $\lambda = \Theta^{(2)}(\mu)$, $\lambda_1 = 16$ and $\lambda_2 = 14$. By definition of $\Theta^{(2)}$, $(\lambda_3, \lambda_4, \ldots)$ is determined recursively: the entries in Figure 3(a) that contribute to λ_1 and λ_2 are removed; the remaining contents of each row shift toward the center, leaving the Sylvester diagram for $\mu' = 5^7 3^2 \in O^{(2)}$. Then $(\lambda_3, \lambda_4, \ldots) = \Theta^{(2)}(\mu')$ is computed recursively.

Example: Computing $\Theta_5^{(2)}(\mu)$ for $\mu = 7^8 5^2 3^2 1 \in O_5^{(2)}$

Here, intially, $\lambda = (0, 0, 0, 0, 0)$. Again, 7, the first part of μ to be inserted into λ, fails Test(**), leaving $\lambda_1 = 4$, $\lambda_2 = 3$; the second part, again a 7=4+3, passes Test(**), leaving $\lambda_1 = 5$, $\lambda_2 = 4$. But now, the third part of μ, again a 7=4+3, fails Test(**), since

$$(5 + (4 - 3)) = 6 < (5/4)(4 + (3 - 2)) = (25)/4,$$

leaving $\lambda_1 = 9, \lambda_2 = 7$. The record of passes and failures of Test(**) for the parts of μ and the contribution of those parts to λ_1 and λ_2 are shown in the Sylvester diagram for μ in Figure 3(b). Thus, for $\lambda = \Theta_5^{(2)}(\mu)$, $\lambda_1 = 28$ and $\lambda_2 = 21$. By definition of $\Theta_5^{(2)}$, $(\lambda_3, \lambda_4, \lambda_5)$ is determined recursively: the entries in the Sylvester diagram for μ that contribute to λ_1 and λ_2 are removed; the remaining contents of each row shift toward the center, leaving the Sylvester diagram for $\mu' = 5^4 3^1 1^1 \in O_3^{(2)}$. Then $(\lambda_3, \lambda_4, \lambda_5) = \Theta_3^{(2)}(\mu')$ is computed recursively.

Example: Computing $\Theta^{(3)}(\mu)$ for $\mu = 29^8 11^2 4^2 1 \in O^{(3)}$

Initially $\lambda = (0, 0, \ldots)$. The first part inserted is $p_4 = 29 = 21 + 8 = a_4 + a_3$. Recall $c_3 = (3 + \sqrt{5})/2$. Checking Test(*), $\lambda_1 + a_4 - a_3 = 13$ and $\lambda_2 + a_3 - a_2 = 5$, but $13 \le c_3 5$, so Test(*) fails and we add 21 to λ_1 and 8 to λ_2. The second part, again a 29, also fails Test(*), since $21 + 13 \le c_3(8 + 5)$, leaving $\lambda_1 = 42$ and $\lambda_2 = 16$. But now, the third part of μ, again a 29, passes Test(*), since $42 + 13 > c_3(16 + 5)$, leaving $\lambda_1 = 55, \lambda_2 = 21$. The record of passes and failures of Test(*) for the parts of μ and the contribution of those parts to λ_1 and λ_2 are shown on the Sylvester diagram for μ in Figure 3(c). Thus, for $\lambda = \Theta^{(3)}(\mu)$, $\lambda_1 = 166$ and $\lambda_2 = 63$. Next, the entries in the Sylvester diagram for μ that contribute to λ_1 and λ_2 are removed; the remaining contents of each row shift toward the center, leaving the Sylvester diagram for $\mu' = 11^3 1^1 \in O^{(3)}$ and $(\lambda_3, \lambda_4, \ldots) = \Theta^{(3)}(\mu')$ is computed recursively.

Example: Computing $\Theta_5^{(3)}(\mu)$ for $\mu = 29^8 11^2 4^2 1 \in O_5^{(3)}$

This proceeds in the same way as the previous example, except the constant for comparison in the Test(**) is 55/21 rather than c_3. The record of passes and failures of Test(**) for the parts of μ and the contribution of those parts to λ_1 and λ_2 are shown on the Sylvester diagram for μ in Figure 3(d). Note that the pass/fail patterns for $\Theta^{(3)}(\mu)$ and $\Theta_5^{(3)}(\mu)$ agree until insertion of the eighth part. Thus, for $\lambda = \Theta^{(3)}(\mu)$, $\lambda_1 = 166$ and $\lambda_2 = 63$ and $(\lambda_3, \lambda_4, \lambda_5) = \Theta_3^{(3)}(\mu')$, where $\mu' = 11^2 4^1 \in O_3^{(3)}$.

2.4 Observations about the bijections

At first glance, it is not at all clear that $\Theta^{(\ell)}$ and $\Theta_n^{(\ell)}$ are bijections. Although Tests(*) and (**) guarantee that the first two parts of λ satisfy the ratio constraints for $D^{(\ell)}$ and $D_n^{(\ell)}$, respectively, and although an induction assumption can guarantee that the parts $(\lambda_3, \lambda_4, \ldots)$ satisfy the required ratio constraints, there is no explicit guarantee that the ratio λ_2/λ_3 will be sufficiently large. This will be the most delicate part of the proof (in Section 4) that $\Theta^{(\ell)}$ and $\Theta_n^{(\ell)}$ are bijections.

Providing a bijective proof of Theorem 1 is the main contribution of this work. However, defining $\Theta_\infty^{(\ell)} = \Theta^{(\ell)}$, we make the following observations about $\Theta_n^{(\ell)}$.

- When $n = \infty$ and $\ell = 2$, it is not hard to see that $\Theta_n^{(\ell)}$ is Sylvester's bijection for Euler's partition theorem.
- Thus, when $n = \infty$ and $\ell > 2$, $\Theta_n^{(\ell)}$ is a new and different generalization of Sylvester's bijection.
- When n is finite, $\Theta_n^{(\ell)}$ gives the *same* bijection as the one derived by Bousquet-Mélou and Eriksson from their proof in [9]. This may be surprising since it has a much different description. Also, for finite n, and $\ell = 2$, $\Theta_n^{(\ell)}$ is the same as Yee's bijection in [28], but, again, with a much different description.

Consequently, $\Theta_n^{(\ell)}$ gives a *unified view* of Euler's partition theorem, the lecture hall theorem, and their generalizations. Moreover, the *alternating sum statistic*

$$l_{\mathrm{alt}}(\lambda) = \lambda_1 - \lambda_2 + \lambda_3 - \cdots ,$$

introduced by Bessenrodt in [7], becomes transparent. From the recursive description of the bijection, it is clear that for $\mu = p_n^{m_n} \cdots p_1^{m_1} \in O_n^{(\ell)}$, under the mapping $\lambda = \Theta_n^{(\ell)}(\mu)$, each part $p_k = a_k + a_{k-1}$ of μ contributes a net of a_k to odd-indexed parts $\lambda_1, \lambda_3, \ldots$ and a net of a_{k-1} to the even-indexed parts $\lambda_2, \lambda_4, \ldots$, and thus a net of $d_k = a_k - a_{k-1}$ to $l_{\mathrm{alt}}(\lambda)$. Thus

$$l_{\mathrm{alt}}(\lambda) = m_n d_n + \cdots + m_1 d_1,$$

giving

$$\sum_{\lambda \in O_n^{(\ell)}} t^{l_{\mathrm{alt}}(\lambda)} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1 - t^{d_k} q^{p_k}}.$$

Letting $x = tq$ and $y = q/t$ gives

$$\sum_{\lambda \in O_n^{(\ell)}} x^{\lambda_1 + \lambda_3 + \cdots} y^{\lambda_2 + \lambda_4 + \cdots} = \prod_{i=1}^{n} \frac{1}{1 - x^{a_n} y^{a_{n-1}}},$$

a fact first observed in [9].

3 The Combinatorics of ℓ-sequences

In this section, we derive the properties of the ℓ-sequences (1) required to prove that $\Theta^{(\ell)}$ and $\Theta_n^{(\ell)}$ of Section 2.3 are bijections and to interpret the families of partitions involved.

3.1 Basic identities for $a_n^{(\ell)}$

Proposition 1 *For $n \geq k \geq 1$*

$$a_{n-k}^{(\ell)} = a_{n-1}^{(\ell)} a_k^{(\ell)} - a_n^{(\ell)} a_{k-1}^{(\ell)}. \tag{4}$$

Proof. For fixed $n \geq 1$, the proof is by induction on k. This is clear for $k = 1, 2$ from the recurrence for $a_n^{(\ell)}$ and the initial conditions $a_0^{(\ell)} = 0$, $a_1^{(\ell)} = 1$. If, for some k, $2 \leq k < n$, the proposition is true for all integers in $\{1, 2, \ldots, k\}$, then for $k+1$:

$$\begin{aligned} a_{n-1}^{(\ell)} a_{k+1}^{(\ell)} - a_n^{(\ell)} a_k^{(\ell)} &= a_{n-1}^{(\ell)}(\ell a_k^{(\ell)} - a_{k-1}^{(\ell)}) - a_n^{(\ell)}(\ell a_{k-1}^{(\ell)} - a_{k-2}^{(\ell)}) \\ &= \ell(a_{n-1}^{(\ell)} a_k^{(\ell)} - a_n^{(\ell)} a_{k-1}^{(\ell)}) - (a_{n-1}^{(\ell)} a_{k-1}^{(\ell)} - a_n^{(\ell)} a_{k-2}^{(\ell)}) \\ &= \ell a_{n-k}^{(\ell)} - a_{n-(k-1)}^{(\ell)} = a_{n-(k+1)}^{(\ell)}. \end{aligned}$$

□

Setting $n = k+1$ in (4) gives the following, since $a_1 = 1 > 0$.

Corollary 1 *The sequence $\{a_{i+1}^{(\ell)}/a_i^{(\ell)}\}_{i\geq 1}$ is strictly decreasing.*

It follows from Corollary 1 that $\{a_{i+1}^{(\ell)}/a_i^{(\ell)}\}_{i\geq 1}$ converges to a positive real number c_ℓ, which is the largest root of the characteristic equation of the defining recurrence for $a_n^{(\ell)}$:

$$x^2 - \ell x + 1 = 0. \tag{5}$$

Proposition 2 *For $1 \leq i \leq k$,*

$$(\ell - 2)\sum_{j=i}^{k-1} a_j^{(\ell)} + (\ell - 1)a_k^{(\ell)} = a_{k+1}^{(\ell)} - (a_i^{(\ell)} - a_{i-1}^{(\ell)}). \tag{6}$$

Proof. Induction on $k - i$: if $k = i \geq 1$, the right-hand side becomes

$$a_{k+1}^{(\ell)} - (a_k^{(\ell)} - a_{k-1}^{(\ell)}) = \ell a_k^{(\ell)} - a_{k-1}^{(\ell)} - (a_k^{(\ell)} - a_{k-1}^{(\ell)}) = (\ell - 1)a_k^{(\ell)}.$$

Under the assumption that the proposition is true for some i with $1 < i \leq k$,

$$\begin{aligned} (\ell - 2)\sum_{j=i-1}^{k-1} a_j^{(\ell)} + (\ell - 1)a_k^{(\ell)} &= (\ell - 2)a_{i-1}^{(\ell)} + a_{k+1}^{(\ell)} - (a_i^{(\ell)} - a_{i-1}^{(\ell)}) \\ &= a_{k+1}^{(\ell)} - (a_{i-1}^{(\ell)} - a_{i-2}^{(\ell)}). \end{aligned}$$

□

3.2 An interpretation of $a_n^{(\ell)}$

Analogous to the Fibonacci numbers, there are several ways to view $a_n^{(\ell)}$ as counting strings with certain properties. The following interpretation of $a_n^{(\ell)}$ has particular significance for our application.

An ℓ-ary string is a string over the alphabet $\{0, 1, \ldots, \ell-1\}$. Let $T_n^{(\ell)}$ be the set of ℓ-ary strings of length n that do not contain the string $(\ell-1)(\ell-2)^t(\ell-1)$ as a consecutive substring for any $t \geq 0$. For example, $T_1^{(3)} = \{0, 1, 2\}$; $T_2^{(3)} = \{00, 01, 02, 11, 10, 12, 20, 21\}$; $T_3^{(3)}$ contains all 3-ary strings of length 3 except 220, 221, 222, 022, 122, and 212. So, $|T_1^{(3)}| = 3$, $|T_2^{(3)}| = 8$, and $|T_3^{(3)}| = 27-6 = 21$.

Proposition 3 *For $n \geq 0$,*

$$|T_n^{(\ell)}| \quad = \quad a_{n+1}^{(\ell)}. \tag{7}$$

Proof. Since $|T_0^{(\ell)}| = 1 = a_1^{(\ell)}$ and $|T_1^{(\ell)}| = \ell = a_2^{(\ell)}$, the sequences $\{a_n^{(\ell)}\}_{n\geq 0}$ and $\{|T_n^{(\ell)}|\}_{n\geq 1}$ have the same initial conditions. We show that they satisfy the same recurrence.

Let $A_n^{(\ell)}$ be the set of strings in $T_n^{(\ell)}$ that end in $(\ell-1)(\ell-2)^t$ for some $t \geq 0$ and let $B_n^{(\ell)} = T_n^{(\ell)} - A_n^{(\ell)}$. Then for $n \geq 1$,

$A_n^{(\ell)}$ is the union of two sets: the strings in $A_{n-1}^{(\ell)}$ with '$\ell-2$' appended and the strings in $B_{n-1}^{(\ell)}$ with '$\ell-1$' appended, so

$$|A_n^{(\ell)}| = |A_{n-1}^{(\ell)}| + |B_{n-1}^{(\ell)}| = |T_{n-1}^{(\ell)}|.$$

The set $B_n^{(\ell)}$ is the union of two sets: the strings in $A_{n-1}^{(\ell)}$ with an element of $\{0, 1, \ldots, \ell-3\}$ appended and the strings in $B_{n-1}^{(\ell)}$ with an element of $\{0, 1, \ldots, \ell-2\}$ appended, so

$$|B_n^{(\ell)}| = (\ell-2)|A_{n-1}^{(\ell)}| + (\ell-1)|B_{n-1}^{(\ell)}| = (\ell-1)(|A_{n-1}^{(\ell)}| + |B_{n-1}^{(\ell)}|) - |A_{n-1}^{(\ell)}| = (\ell-1)|T_{n-1}^{(\ell)}| - |A_{n-1}^{(\ell)}|.$$

Thus for $n \geq 2$,

$$|T_n^{(\ell)}| = |A_n^{(\ell)}| + |B_n^{(\ell)}| = \ell|T_{n-1}^{(\ell)}| - |A_{n-1}^{(\ell)}| = \ell|T_{n-1}^{(\ell)}| - |T_{n-2}^{(\ell)}|.$$

□

3.3 The ℓ-representation of an integer

Analogous to the case for a binary string of length n written as $b_{n-1}b_{n-2}\ldots b_0$ when it is to be associated with the integer $\sum_{i=0}^{n-1} b_i 2^i$, we will write a string $\alpha \in T_n^{(\ell)}$ as

$$\alpha = b_n b_{n-1} \ldots b_1,$$

since we will associate it with the integer $\sum_{i=1}^{n} b_i a_i^{(\ell)}$. We now show that the interpretation of $a_n^{(\ell)}$ in (7) allows for a unique *ℓ-representation* of any nonnegative integer, up to leading zeroes. (This representation coincides with one of the numeration systems defined by Fraenkel in [18].)

Definition 1 *A string α is ℓ-*admissible *if $\alpha \in T_n^{(\ell)}$ for some $n \geq 0$.*

Note that $b_n \dots b_1$ is ℓ-admissible if and only if $b_1 \dots b_n$ is. We will make use of the following lemma.

Lemma 1 *If $b_n b_{n-1} \dots b_{t+1}(\ell-1)(\ell-2)^{t-1}$ is ℓ-admissible, then so is $b_n b_{n-1} \dots (b_{t+1}+1)$.*

Proof. If $b_n b_{n-1} \dots (b_{t+1}+1)$ is not ℓ-admissible, but $b_n b_{n-1} \dots b_{t+1}(\ell-1)(\ell-2)^{t-1}$ is ℓ-admissible, then for some w with $t+1 \le w \le n$,

$$b_n b_{n-1} \dots b_{t+1} = b_n b_{n-1} \dots b_{w+1}(\ell-1)(\ell-2)^{w-t-1}.$$

But then

$$b_n b_{n-1} \dots b_{t+1}(\ell-1)(\ell-2)^{t-1} = b_n b_{n-1} \dots b_{w+1}(\ell-1)(\ell-2)^{w-t-1}(\ell-1)(\ell-2)^{t-1},$$

which is not ℓ-admissible. □

Proposition 4 *The mapping*

$$f_n : \quad T_n^{(\ell)} \quad \to \quad \{0, 1, \dots, a_{n+1}^{(\ell)} - 1\}$$

defined by

$$f_n(b_n \dots b_1) \ = \ \sum_{i=1}^{n} b_i a_i^{(\ell)}$$

is a bijection.

Proof. By (7), both sets have the same size. We show by induction that f_n is onto, starting with $f_n(0^n) = 0$. For $0 < x < a_{n+1}^{(\ell)}$, assume inductively that $x - 1 = f(b_n \dots b_1)$, where $b_n \dots b_1 \in T_n^{(\ell)}$. If $y = b_n \dots (b_1 + 1) \in T_n^{(\ell)}$, then $f_n(y) = x$. Otherwise, for some t with $1 \le t \le n$,

$$b_n \dots b_1 = b_n \dots b_{t+1}(\ell-1)(\ell-2)^{t-1}.$$

Then, using (6),

$$\begin{aligned} x = 1 + (x-1) \quad &= \quad a_1^{(\ell)} + (\ell-2)\sum_{i=1}^{t-1} a_i^{(\ell)} + (\ell-1)a_t^{(\ell)} + \sum_{j=t+1}^{n} b_j a_j^{(\ell)} \\ &= \quad a_1^{(\ell)} + a_{t+1}^{(\ell)} - (a_1^{(\ell)} - a_0^{(\ell)}) + \sum_{j=t+1}^{n} b_j a_j^{(\ell)} \\ &= \quad a_{t+1}^{(\ell)} + \sum_{j=t+1}^{n} b_j a_j^{(\ell)}. \end{aligned}$$

So $f_n^{-1}(x) = b_n \dots (b_{t+1}+1)0^t$ and by Lemma 1, $f_n^{-1}(x) \in T_n^{(\ell)}$. □

Corollary 2 *If $b_n b_{n-1} \dots b_1$ is ℓ-admissible, then*

$$\sum_{i=1}^{n} b_i a_i^{(\ell)} < a_{n+1}.$$

Definition 2 *For $\ell \geq 2$, an* ℓ-representation *of a nonnegative integer x, denoted $[x]^{(\ell)}$, is an ℓ-admissible string*

$$[x]^{(\ell)} = b_n b_{n-1} \dots b_1$$

satisfying

$$x = \sum_{i=1}^{n} b_i a_i^{(\ell)}.$$

By Proposition 4 such a string $b_n b_{n-1} \dots b_1$ always exists and is unique up to leading zeroes.

In applications to lecture hall theorems, we will be required to represent arbitrarily large integers by a fixed-length string as follows.

Definition 3 *For $\ell \geq 2$ and $n \geq 1$, the* (ℓ, n)-representation *of a nonnegative integer x, denoted $[x]_n^{(\ell)}$, is the unique string*

$$[x]_n^{(\ell)} = b_n b_{n-1} \dots b_1$$

such that

$$\text{(i) } x = \sum_{i=1}^{n} b_i a_i^{(\ell)}$$

and (ii) *$b_{n-1} \dots b_1$ is ℓ-admissible. Note, in contrast to the case for $[x]^{(\ell)}$, $b_n \geq \ell$ is allowed.*

For example, the 3-representation and $(3, 4)$-representation of 100 are, respectively

$$\begin{aligned} [100]^{(3)} &= 12010 = 0012010 \\ [100]_4^{(3)} &= 4200. \end{aligned}$$

3.4 The natural ordering on ℓ-admissible strings

Define the *natural ordering* '$\preceq$' on ℓ-admissible strings by

$$b_n b_{n-1} \dots b_1 \quad \preceq \quad d_m d_{m-1} \dots d_1$$

if and only if, in lexcographic order,

$$0^m b_n b_{n-1} \dots b_1 \quad \leq \quad 0^n d_m d_{m-1} \dots d_1.$$

For strings of the same length, natural order and lexicographic order coincide. Otherwise, strings are "right justified", padded on the left with zeroes to become the same length and then compared lexicographically. We also use '$\succeq$' to denote 'greater than or equal to' in the natural order.

It follows from the proof of Proposition 4 that for any integer x, $[x-1]^{(\ell)} \preceq [x]^{(\ell)}$, giving the following corollary.

Corollary 3 *For nonnegative integers x and y, $x \leq y$ if and only if $[x]^{(\ell)} \preceq [y]^{(\ell)}$.*

It follows that for $\alpha \in T_n^{(\ell)}$,

$$\alpha \quad \preceq \quad (\ell - 1)(\ell - 2)^{n-1}, \tag{8}$$

that is, in the natural ordering on $T_n^{(\ell)}$, $(\ell - 1)(\ell - 2)^{n-1}$ is the largest string.

3.5 Identities for c_ℓ

Recall that c_ℓ is the largest root of (5). In this section we show that several of the identities involving $a_n^{(\ell)}$ have c_ℓ counterparts.

Lemma 2 *For $n \geq 1$ and for $\ell \geq 2$,*

$$\sum_{i=1}^{n-1} (\ell-2)c_\ell^i + (\ell-1)c_\ell^n \quad = \quad c_\ell^{n+1} - c_\ell + 1. \tag{9}$$

Proof. To simplify notation, let $c = c_\ell$. When $n = 1$, it follows from $c^2 - \ell c + 1 = 0$ that

$$(\ell-1)c \quad = \quad c^2 - c + 1.$$

For $n > 1$, when $\ell = 2$, then $c = 1$ and

$$(\ell-1)c^n \quad = \quad 1 \quad = \quad c^{n+1} - c + 1.$$

When $\ell > 2$,

$$\begin{aligned}
\sum_{i=1}^{n-1} (\ell-2)c^i + (\ell-1)c^n &= (\ell-2)\frac{c^n - c}{c-1} + (\ell-1)c^n \\
&= \frac{(\ell-2)(c^n - c) + (c-1)(\ell-1)c^n}{c-1} \\
&= \frac{\ell c^n - 2c^n - \ell c + 2c + \ell c^{n+1} - \ell c^n - c^{n+1} + c^n}{c-1} \\
&= \frac{c^n(\ell c - 1) - c^{n+1} - \ell c + 2c}{c-1} \\
&= \frac{c^{n+2} - c^{n+1} - c^2 + 2c - 1}{c-1} \\
&= c^{n+1} - c + 1,
\end{aligned}$$

where the second last equality follows from $c^2 - \ell c + 1 = 0$. □

Lemma 3 *Let $\ell \geq 2$. For ℓ-admissible $b_n \ldots b_1$ and $d_n \ldots d_1$, if*

$$b_n \ldots b_1 \quad \prec \quad d_n \ldots d_1$$

then

$$\sum_{i=1}^{n} b_i c_\ell^i \quad \leq \quad \sum_{i=1}^{n} d_i c_\ell^i.$$

If $\ell > 2$, the integer inequality is strict.

Proof. If $n = 1$, this is clear. Otherwise, for $n > 1$, let t be the largest index such that $d_t > b_t$. Then

$$\sum_{i=1}^{n} d_i c_\ell^i \; - \; \sum_{i=1}^{n} b_i c_\ell^i \quad \geq \quad c_\ell^t \; - \; \sum_{i=1}^{t-1} b_i c_\ell^i.$$

It remains to show $c_\ell^t - \sum_{i=1}^{t-1} b_i c_\ell^i \geq 0$, with strict inequality when $\ell > 2$. Since $b_{t-1} \ldots b_1 \preceq (\ell-1)(\ell-2)^{t-2}$ and by (9),

$$\sum_{i=1}^{t-1} b_i c_\ell^i \quad \leq \quad \sum_{i=1}^{t-2} (\ell-2) c_\ell^i + (\ell-1) c_\ell^{t-1} \quad = \quad c_\ell^t - c_\ell + 1.$$

So,

$$\sum_{i=1}^{n-1} d_i c_\ell^i \; - \; \sum_{i=1}^{n-1} b_i c_\ell^i \quad \geq c_\ell - 1 \; \geq \; 0,$$

and when $\ell > 2$, the last inequality is strict since $c_\ell > 1$. □

Corollary 4 *If $\ell \geq 2$ and $b_n \ldots b_1$ is ℓ-admissible, then*

$$\sum_{i=1}^{n} b_i c_\ell^i \quad \leq \quad c_\ell^{n+1} - c_\ell + 1. \tag{10}$$

Lemma 4 *For $n \geq 1$, $\ell \geq 2$,*

$$a_n - c_\ell a_{n-1} = c_\ell^{-(n-1)}. \tag{11}$$

Proof. If $n = 1$, then $a_1 = 1 = c_\ell^0$. From the recurrence for a_n and since c_ℓ satisfies (5), for $n > 1$,

$$\begin{aligned} c_\ell(a_n - c_\ell a_{n-1}) \quad &= \quad c_\ell(\ell a_{n-1} - a_{n-2}) - c_\ell^2 a_{n-1} \\ &= \quad (c_\ell \ell - c_\ell^2) a_{n-1} - c_\ell a_{n-2} \\ &= \quad a_{n-1} - c_\ell a_{n-2}. \end{aligned}$$

Since $a_1 - c_\ell a_0 = 1$, iterating gives the result. □

3.6 The ratio constraint

To simplify notation, fix $\ell \geq 2$ and let $a_n = a_n^{(\ell)}$. Theorem 1 refers to integer partitions in which the ratio of consecutive positive parts is *larger than* c_ℓ and Theorems 2 and 3 refer to partitions in which the ratio of the i-th part and its successor is at least a_{n+1-i}/a_{n-i}. We can now interpret these constraints as natural ordering contraints on ℓ-admissible strings. This will be illustrated in Figures 4 and 5. We use the symbol '$\cdot$', as in $u \cdot v$, to denote the concatenation of two strings u, v.

Theorem 4 *For $\ell > 2$, $n \geq 2$, and nonnegative integers x, y,*

$$\begin{aligned} &\text{(a)} \qquad x = \lceil c_\ell \, y \rceil \qquad \textit{iff} \qquad [x]^{(\ell)} = [y]^{(\ell)} \cdot 0; \\ &\text{(b)} \qquad x = \left\lceil \frac{a_n}{a_{n-1}} \, y \right\rceil \qquad \textit{iff} \qquad [x]_n^{(\ell)} = [y]_{n-1}^{(\ell)} \cdot 0 \end{aligned}$$

Proof. (a) Define z by

$$[z]^{(\ell)} \;=\; [y]^{(\ell)} \cdot 0 \;=\; b_n b_{n-1} \ldots b_2 0.$$

Show $0 \le z - c_\ell y < 1$, and therefore $z = \lceil c_\ell\, y \rceil$. Using (11) and (10),

$$\begin{aligned} z - c_\ell y &= \sum_{i=2}^{n} b_i a_i - c_\ell \sum_{i=2}^{n} b_i a_{i-1} \\ &= \sum_{i=2}^{n} b_i (a_i - c_\ell a_{i-1}) \\ &= \sum_{i=2}^{n} b_i \frac{1}{c_\ell^{i-1}} \\ &= \frac{1}{c_\ell^n} \sum_{i=1}^{n-1} b_{n+1-i} c_\ell^i \\ &< \frac{1}{c_\ell^n} c_\ell^n \;=\; 1. \end{aligned}$$

(b) Define z by

$$[z]_n^{(\ell)} \;=\; [y]_{n-1}^{(\ell)} \cdot 0 \;=\; b_n b_{n-1} \ldots b_2 0.$$

It suffices to show $0 \le a_{n-1} z - a_n y < a_{n-1}$. Using (4),

$$\begin{aligned} a_{n-1} z - a_n y &= a_{n-1} \sum_{j=2}^{n} b_j a_j - a_n \sum_{j=2}^{n} b_j a_{j-1} \\ &= \sum_{j=2}^{n} b_j (a_{n-1} a_j - a_n a_{j-1}) \\ &= \sum_{j=2}^{n} b_j a_{n-j}, \end{aligned}$$

This is clearly nonnegative. Re-indexing the sum and using $a_0 = 0$ and Proposition 4 gives

$$\sum_{j=2}^{n} b_j a_{n-j} = \sum_{i=0}^{n-2} b_{n-i} a_i = \sum_{i=1}^{n-2} b_{n-i} a_i < a_{n-1}.$$

□

Since when $\ell = 2$, $[x]^{(2)} = 10^{x-1}$, we have the following corollary for all $\ell \ge 2$.

Corollary 5 *For $\ell \ge 2$, $n \ge 2$, and integers $x, y > 0$,*

$$\text{(a)} \qquad \frac{x}{y} > c_\ell \quad \textit{iff} \;\; [x]^{(\ell)} \succeq [y]^{(\ell)} \cdot 0; \tag{12}$$

$$\text{(b)} \qquad \frac{x}{y} \ge \frac{a_n}{a_{n-1}} \quad \textit{iff} \;\; [x]_n^{(\ell)} \succeq [y]_{n-1}^{(\ell)} \cdot 0. \tag{13}$$

	$a_7^{(3)}$	$a_6^{(3)}$	$a_5^{(3)}$	$a_4^{(3)}$	$a_4^{(3)}$	$a_2^{(3)}$	$a_1^{(3)}$	
	377	144	55	21	8	3	1	λ
$[\lambda_1]^{(3)}$	2	0	0	0	0	0	0	754
$[\lambda_2]^{(3)}$		1	2	0	2	1	0	273
$[\lambda_3]^{(3)}$			1	2	0	1	2	102
$[\lambda_4]^{(3)}$				1	2	0	1	38
$[\lambda_5]^{(3)}$					1	2	0	14
$[\lambda_6]^{(3)}$						1	2	5
$[\lambda_7]^{(3)}$							1	1

$[\lambda_1]^{(3)}$	2 0 0 0 0 0 0
$[\lambda_2]^{(3)} \cdot 0$	1 2 0 2 1 0 0
$[\lambda_3]^{(3)} \cdot 00$	1 2 0 1 2 0 0
$[\lambda_4]^{(3)} \cdot 000$	1 2 0 1 0 0 0
$[\lambda_5]^{(3)} \cdot 0000$	1 2 0 0 0 0 0
$[\lambda_6]^{(3)} \cdot 00000$	1 2 0 0 0 0 0
$[\lambda_7]^{(3)} \cdot 000000$	1 0 0 0 0 0 0

Figure 4: Corollary 5(a): $\lambda \in D^{(3)}$ iff last column is weakly decreasing, lexicographically.

	$a_5^{(3)}$	$a_4^{(3)}$	$a_4^{(3)}$	$a_2^{(3)}$	$a_1^{(3)}$	
	55	21	8	3	1	λ
$[\lambda_1]_5^{(3)}$	5	1	0	2	0	302
$[\lambda_2]_4^{(3)}$		5	1	0	2	115
$[\lambda_3]_3^{(3)}$			5	1	0	43
$[\lambda_4]_2^{(3)}$				4	2	14
$[\lambda_5]_1^{(3)}$					4	4

$[\lambda_1]_5^{(3)}$	5 1 0 2 0
$[\lambda_2]_4^{(3)} \cdot 0$	5 1 0 2 0
$[\lambda_3]_3^{(3)} \cdot 00$	5 1 0 0 0
$[\lambda_4]_2^{(3)} \cdot 000$	4 2 0 0 0
$[\lambda_5]_1^{(3)} \cdot 0000$	4 0 0 0 0

Figure 5: Corollary 5(b): $\lambda \in D_5^{(3)}$ iff last column is weakly decreasing, lexicographically.

Example 2 Recalling the definitions of $D^{(\ell)}$ and $D_n^{(\ell)}$ from Section 2.1,

$$\begin{array}{rll}
(4,3,3) \notin D^{(2)} & \text{since} & [3]^{(2)} = 100 \not\succeq 1000 = [3]^{(2)} \cdot 0 \\
(7,6,5) \in D^{(2)} & \text{since} & [7]^{(2)} = 1000000 = [6]^{(2)} \cdot 0 = [5]^{(2)} \cdot 00 \\
(7,6,5,0) \notin D_4^{(2)} & \text{since} & [7]_4^{(2)} = 1100 \not\succeq 2000 = [6]_3^{(2)} \cdot 0 \\
(10,3,1) \in D_3^{(2)} & \text{since} & [10]_3^{(2)} = 301 \succeq 110 = [3]_2^{(2)} \cdot 0 \succeq [1]_1^{(2)} \cdot 00 = 100 \\
(55,21,8,3) \in D^{(3)} & \text{since} & [55]^{(3)} = 10000 = [21]^{(3)} \cdot 0 = [8]^{(3)} \cdot 00 = [3]^{(3)} \cdot 000 \\
(55,21,8,3) \notin D_4^{(3)} & \text{since} & [55]_4^{(3)} = 2112 \not\succeq 2120 = [21]_3^{(3)} \cdot 0 \\
(55,21,8,3,0) \in D_5^{(3)} & \text{since} & [55]_5^{(3)} = 10000 = [21]^{(3)} \cdot 0 = [8]^{(3)} \cdot 00 = [3]^{(3)} \cdot 000 \succeq [0]^{(3)} \cdot 0000
\end{array}$$

Corollary 5 gives a more combinatorial view of the families satisfying the ratio constraints of Theorems 1-3. In Figure 4, by Corollary 5(a), for the partition λ, represented by the last column of the first table, the ratio of successive parts is at least $c_3 = (3+\sqrt{5})/2$ if and only if in the second table, each row is lexicographically greater than or equal to the row below it.

In Figure 5, by Corollary 5(b), the partition λ, represented by the last column of the first table, is a 3-lecture hall partition in $D_5^{(3)}$ if and only if in the second table, each row is lexicographically greater than or equal to the row below it.

3.7 A closer look at Tests(*) and (**)

In this section we show that Tests(*) and (**) of the insertion procedures for $\Theta^{(\ell)}$ and $\Theta_n^{(\ell)}$ in Section 2.3 are equivalent to inadmissibility tests. This alternate characterization of the tests will be used in Section 4 to prove that $\Theta^{(\ell)}$ and $\Theta_n^{(\ell)}$ are bijections.

To simplify notation, let $[x] = [x]^{(\ell)}$ and $[x]_n = [x]_n^{(\ell)}$. We would like to show the following.

Proposition 5

(a) *Suppose* $[x] \;=\; b_n b_{n-1} \ldots b_k 0^{k-1} \;=\; [y] \cdot 0$. *Then for* $k \geq 2$,

$$(x + a_k - a_{k-1}) > c_\ell(y + a_{k-1} - a_{k-2})$$

if and only if $b_n b_{n-1} \ldots (b_k + 1)$ *is inadmissible.*

(b) *Suppose* $[x]_n \;=\; b_n b_{n-1} \ldots b_k 0^{k-1} \;=\; [y]_{n-1} \cdot 0$. *Then for* $n \geq k \geq 2$,

$$(x + a_k - a_{k-1}) \geq \frac{a_n}{a_{n-1}}(y + a_{k-1} - a_{k-2})$$

if and only if $b_{n-1} \ldots (b_k + 1)$ *is inadmissible.*

This will be a consequence of (12), (13), and the following lemma.

Lemma 5

(a) *Suppose* $[x] \;=\; b_n b_{n-1} \ldots b_k 0^{k-1} \;=\; [y] \cdot 0$. *Then for* $k \geq 2$,

$$[x + a_k - a_{k-1}] \;\succeq\; [y + a_{k-1} - a_{k-2}] \cdot 0$$

if and only if $b_n b_{n-1} \ldots (b_k + 1)$ *is inadmissible. Furthermore, if* $b_n b_{n-1} \ldots (b_k + 1)$ *is inadmissible, then for some* t, $n \geq t \geq k$,

$$[x + a_k - a_{k-1}] = b_n \ldots (b_{t+1} + 1)0^t = [y + a_{k-1} - a_{k-2}] \cdot 0$$

(*letting* $b_{n+1} = 0$) *and otherwise*

$$[x + a_k] = b_n b_{n-1} \ldots (b_k + 1)0^{k-1} = [y + a_{k-1}] \cdot 0.$$

(b) *Suppose* $[x]_n \;=\; b_n b_{n-1} \ldots b_k 0^{k-1} \;=\; [y]_{n-1} \cdot 0$. *Then for* $n \geq k \geq 2$,

$$[x + a_k - a_{k-1}]_n \;\succeq\; [y + a_{k-1} - a_{k-2}]_{n-1} \cdot 0$$

if and only if $b_{n-1} \ldots (b_k + 1)$ *is inadmissible. Furthermore, if* $b_{n-1} \ldots (b_k + 1)$ *is inadmissible, then for some* t, $n > t \geq k$,

$$[x + a_k - a_{k-1}]_n = b_n \ldots (b_{t+1} + 1)0^t = [y + a_{k-1} - a_{k-2}]_{n-1} \cdot 0$$

and otherwise

$$[x + a_k]_n = b_n b_{n-1} \ldots (b_k + 1)0^{k-1} = [y + a_{k-1}]_{n-1} \cdot 0.$$

Proof. (a) First observe that setting $i = 1$ in (6) and rearranging the terms gives

$$a_{k+1} - a_k \;=\; (\ell-2)a_k + \cdots + (\ell-2)a_2 + (\ell-1)a_1. \tag{14}$$

So,

$$[a_k - a_{k-1}] \;=\; (\ell-2)^{k-2}(\ell-1). \tag{15}$$

If $k = 2$ then $[a_{k-1} - a_{k-2}] = 1$, but for $k \geq 3$,

$$[a_{k-1} - a_{k-2}] \;=\; (\ell-2)^{k-3}(\ell-1). \tag{16}$$

So, if $b_n b_{n-1} \ldots (b_k + 1)$ is inadmissible, then for some t, $n \geq t \geq k$,

$$[x] = b_n \ldots b_{t+1}(\ell-1)(\ell-2)^{t-k} : w0^{k-1} = [y] \cdot 0.$$

Thus, by (15), (6), and (16),

$$[x + a_k - a_{k-1}] = b_n \ldots (b_{t+1}+1)0^t = [y + a_{k-1} - a_{k-2}] \cdot 0.$$

Otherwise, $b_n b_{n-1} \ldots (b_k + 1)$ is admissible and

$$[x + a_k]_n = b_n b_{n-1} \ldots (b_k + 1)0^{k-1} = [y + a_{k-1}]_{n-1} \cdot 0.$$

In addition, if $k \geq 3$,

$$\begin{aligned} [x + a_k - a_{k-1}] \;&=\; b_n b_{n-1} \ldots b_k(\ell-2)^{k-2}(\ell-1) \\ &\prec\; b_n b_{n-1} \ldots b_k(\ell-2)^{k-3}(\ell-1)0 \\ &=\; [y + a_{k-1} - a_{k-2}] \cdot 0. \end{aligned}$$

Finally, if $k = 2$ and $b_n b_{n-1} \ldots (b_2 + 1)$ is admissible, then

$$[x + a_2 - a_1] = b_n \ldots b_2(\ell-1) \;\prec\; b_n \ldots (b_2+1)0 \;=\; [y + a_1 - a_0] \cdot 0.$$

The proof of (b) is similar, except $t < n$. □

This implies that we can re-state the bijections as follows.

(Revised) insertion procedure for $\Theta^{(\ell)}$:

To insert p_k into λ, given that $[\lambda_1] = b_n \ldots b_k 0^{k-1} = [\lambda_2] \cdot 0$:
If $k = 1$, then add 1 to λ_1;
otherwise, if for some t, $n \geq t \geq k$, $[\lambda_1] = b_n \ldots b_{t+1}(\ell-1)(\ell-2)^{t-k}0^{k-1}$ Test(*)
  $[\lambda_1] \leftarrow b_n \ldots (b_{t+1}+1)0^t$ and $[\lambda_2] \leftarrow b_n \ldots (b_{t+1}+1)0^{t-1}$;
  and recursively insert p_{k-1} into $(\lambda_3, \lambda_4, \ldots)$ via Θ;
otherwise, $[\lambda_1] \leftarrow b_n \ldots (b_k+1)0^{k-1}$ and $[\lambda_2] \leftarrow b_n \ldots (b_k+1)0^{k-2}$;

There are only three differences in the insertion procedure for $\Theta_n^{(\ell)}$: (i) the (ℓ, n) representation of integers is used, rather than the ℓ representation; (ii) In Test(**), t must be strictly smaller than n; and (iii) the recursive insertion is via Θ_{n-2}.

(Revised) insertion procedure for $\Theta_n^{(\ell)}$:

To insert p_k into λ, given that $[\lambda_1]_n = b_n \ldots b_k 0^{k-1} = [\lambda_2]_n \cdot 0$:
If $k = 1$, then add 1 to λ_1;
otherwise, if for some t, $n > t \geq k$, $[\lambda_1]_n = b_n \ldots b_{t+1}(\ell-1)(\ell-2)^{t-k}0^{k-1}$ Test(**)
$[\lambda_1]_n \leftarrow b_n \ldots (b_{t+1}+1)0^t$ and $[\lambda_2]_n \leftarrow b_n \ldots (b_{t+1}+1)0^{t-1}$;
and recursively insert p_{k-1} into $(\lambda_3, \lambda_4, \ldots)$ via $\Theta_{n-2}^{(\ell)}$;
otherwise, $[\lambda_1]_n \leftarrow b_n \ldots (b_k+1)0^{k-1}$ and $[\lambda_2]_n \leftarrow b_n \ldots (b_k+1)0^{k-2}$;

4 Proving the bijections

It was already observed in Section 2.3 that $\Theta^{(\ell)}$ and $\Theta_n^{(\ell)}$ are weight-preserving. In this section we show they are bijections.

4.1 Bijection $\Theta^{(\ell)}$

To show that $\Theta^{(\ell)} : O^{(\ell)} \to D^{(\ell)}$ is a bijection for Theorem 1, we first show that $\Theta^{(\ell)}(O^{(\ell)}) \subseteq D^{(\ell)}$. Recall from Section 2.1 that $O^{(\ell)}$ is the set of partitions into parts from the infinite set $\{p_1, p_2, \ldots\}$ where $p_k = a_k + a_{k-1}$.

Theorem 5 *Let $\mu = p_r^{m_r} p_{r-1}^{m_{r-1}} \ldots p_k^{m_k} \in O^{(\ell)}$ and let $\lambda = \Theta^{(\ell)}(\mu)$. Then $\lambda \in D^{(\ell)}$. Furthermore, if $k > 1$ then*

(i) *for some ℓ-admissible sequence $b_n \ldots b_k$, $[\lambda_1] = [\lambda_2] \cdot 0 = b_n \ldots b_k 0^{k-1}$;*

(ii) *the last p_k inserted by Θ^ℓ failed test* Test(*) *if and only if $b_k > 0$.*

Proof. To simplify notation, let $\Theta = \Theta^{(\ell)}$. To show the rest, we use induction on $m_r + \cdots + m_k$. If $m_r + \cdots + m_k = 0$, then λ is the empty partition and the theorem holds.

Let $1 \leq k \leq r$ and assume inductively that the theorem holds for λ. We prove that it holds for

$$\lambda' \;=\; \Theta(p_r^{m_r} p_{r-1}^{m_{r-1}} \ldots p_k^{m_k+1}) \;=\; (\lambda'_1, \lambda'_2, \lambda'_3, \ldots).$$

If $k = 1$, this is clear. Otherwise, whether or not the insertion of the last p_k into λ by Θ passes the Test(*), by definition of Θ and by Lemma 5 (a) and Proposition 5 (a), λ'_1 and λ'_2 satisfy (i) and (ii) of the theorem. Thus, by (12), $\lambda'_1 > c_\ell \lambda'_2$.

Also, if p_k passes Test(*), by definition of Θ, p_{k-1} is inserted recursively into $(\lambda_3, \lambda_4, \ldots)$. Note that the insertion of the first part, p_r, by Θ always fails and therefore $(\lambda_3, \lambda_4, \ldots)$ was created by Θ from fewer than the $m_r + \cdots + m_k$ parts of μ. Consequently, by induction, $(\lambda'_3, \lambda'_4, \ldots) \in D^{(\ell)}$. On the other hand, if p_k fails Test(*), then by definition of Θ, $(\lambda'_3, \lambda'_4, \ldots) = (\lambda_3, \lambda_4, \ldots)$. By induction, $(\lambda_3, \lambda_4, \ldots) \in D^{(\ell)}$, and therefore $(\lambda'_3, \lambda'_4, \ldots) \in D^{(\ell)}$.

It remains to show that $\lambda'_2 > c_\ell \lambda'_3$. Throughout the rest of the proof, we take an integer n large enough such that the ℓ-representation of λ_1 is an ℓ-admissible string of length n. Then it follows from (12) that λ_2 and λ_3 have ℓ-representations of length $n-1$ and $n-2$, respectively.

By induction, since $\lambda \in D^{(\ell)}$, $\lambda_2 > c_\ell \lambda_3$, so by (12),

$$[\lambda_2] \;=\; b_{n-1}b_{n-2}\dots b_{k-1}0^{k-2} \;\succeq\; [\lambda_3]\cdot 0. \tag{17}$$

If insertion of p_k into λ by Θ fails Test(*), then using the definition of Θ,

$$[\lambda_2'] \;=\; [\lambda_2 + a_{k-1}] \;\succ\; [\lambda_2] \;\succeq\; [\lambda_3]\cdot 0 \;=\; [\lambda_3']\cdot 0,$$

so by (12), $\lambda_2' > c_\ell \lambda_3'$.

If insertion of p_k into λ by Θ passes Test(*), then, by Lemma 5 (a), and by definition of Θ, λ_2 has the form

$$[\lambda_2] \;=\; b_{n-1}\dots b_{t+1}(\ell-1)(\ell-2)^{t-k+1}0^{k-2}, \tag{18}$$

for some t with $k-1 \le t \le n-1$, and

$$[\lambda_2'] \;=\; [\lambda_2 + a_{k-1} - a_{k-2}] \;=\; b_{n-1}\dots(b_{t+1}+1)0^t, \tag{19}$$

where we define $b_{t+1} = 0$ if $t = n-1$. Also, by definition of Θ, p_{k-1} is inserted recursively into $(\lambda_3, \lambda_4, \dots)$. By induction and (17), since $(\lambda_3, \lambda_4, \dots)$ was formed by Θ from recursive insertion of parts greater than or equal to p_{k-1}, λ_3 has the form

$$[\lambda_3] = d_{n-2}\dots d_{k-1}0^{k-2}, \tag{20}$$

for an ℓ-admissible sequence $d_{n-2}\dots d_{k-1}$. We must show that whether or not p_{k-1} passes the Test(*) for insertion into $(\lambda_3, \lambda_4, \dots)$ by Θ, $[\lambda_2'] \succeq [\lambda_3']\cdot 0$.

By (17) and (20),

$$b_{n-1}b_{n-2}\dots b_{k-1}0^{k-2} \;\succeq\; d_{n-2}\dots d_{k-1}0^{k-1}. \tag{21}$$

If p_{k-1} fails the Test(*) for insertion into $(\lambda_3, \lambda_4, \dots)$ by Θ, then by definition of Θ and by Lemma 5 (a),

$$[\lambda_3'] \;=\; [\lambda_3 + a_{k-1}] \;=\; d_{n-2}\dots(d_{k-1}+1)0^{k-2}. \tag{22}$$

Then by (19), (22), and (21), since $t \ge k-1$,

$$[\lambda_2'] \;=\; b_{n-1}\dots(b_{t+1}+1)0^t \;\succeq\; d_{n-2}\dots(d_{k-1}+1)0^{k-1} \;=\; [\lambda_3']\cdot 0. \tag{23}$$

But if p_{k-1} passes the Test(*) for insertion into $(\lambda_3, \lambda_4, \dots)$ by Θ, then by Lemma 5 (a), λ_3 has the form

$$[\lambda_3] \;=\; d_{n-2}\dots d_{s+1}(\ell-1)(\ell-2)^{s-k-1}0^{k-2}, \tag{24}$$

for some s with $k-1 \le s \le n-2$, and

$$[\lambda_3'] = [\lambda_3 + a_{k-1} - a_{k-2}] = d_{n-2}\dots(d_{s+1}+1)0^s, \tag{25}$$

where we define $d_{s+1} = 0$ if $s = n-2$. Because of (19), (21), and (25), if $t > s$,

$$[\lambda_2'] \;=\; b_{n-1}\dots(b_{t+1}+1)0^t \;\succeq\; d_{n-2}\dots(d_{s+1}+1)0^{s+1} \;=\; [\lambda_3']\cdot 0.$$

On the other hand, if $s \ge t$, consider the prefix $b_{n-1}\dots b_{s+2}b_{s+1}\dots b_{t+1}$ of $[\lambda_2]$ in (18). Since the lexicographically largest ℓ-admissible string of length $s-t+1$, is $(\ell-1)(\ell-2)^{s-t}$, it must be lexicographically greater than or equal to the string $b_{s+1}\dots b_{t+1}$. But observe from (18) that equality cannot hold, since $[\lambda_2]$ is ℓ-admissible. Combining this observation with (24), we have the strict inequality

$$b_{s+1} \dots b_{t+1} \prec (\ell-1)(\ell-2)^{s-t} = d_s \dots d_t.$$

But then because of (17), (18), and (24), we must also have the strict inequality

$$b_{n-1} \dots b_{s+2} \succ d_{n-2} \dots d_{s+1}.$$

Thus

$$[\lambda'_2] = b_{n-1} \dots b_{s+2} \dots (b_{t+1}+1)0^t \succeq d_{n-2} \dots (d_{s+1}+1)0^{s-1}0 = [\lambda'_3] \cdot 0,$$

completing the proof. □

To complete the proof that $\Theta^{(\ell)}$ is a bijection, we prove that it has an inverse.

Theorem 6 *For every nonempty $\lambda \in D^{(\ell)}$ there are unique integers $r \geq k \geq 1$ and nonnegative integers $m_r, m_{r-1}, \dots, m_k$ such that*

$$\lambda = \Theta^{(\ell)}(p_r^{m_r} \dots p_k^{m_k+1}).$$

Proof. Again, let $\Theta = \Theta^{(\ell)}$. Let $\lambda = (\lambda_1, \lambda_2, \dots) \in D^{(\ell)}$. We prove the theorem by induction on $|\lambda_1|$. If $|\lambda_1| = 1$, then $\lambda = (1, 0, 0, \dots) = \Theta(p_1^1)$.

So, assume $|\lambda_1| > 1$. We want to show that λ satisfies the conditions of the theorem. This involves not only identifying k, but

(i) showing that "peeling away" p_k from λ by reversing Θ leaves a partition $\lambda' \in D^{(\ell)}$; and

(ii) showing that there exist unique nonnegative integers $m_r, \dots, m_k$ such that $\lambda' = \Theta(p_r^{m_r} \dots p_k^{m_k})$.

A note about (ii): By induction, if $\lambda' \in D^{(\ell)}$, then λ' is in the image of Θ. However, since Θ processes parts in non-decreasing order of size, we must guarantee that no part smaller than p_k was used by Θ to create λ'.

If $\lambda_1 > \lceil c_\ell \lambda_2 \rceil$, then $(\lambda_1 - 1, \lambda_2, \lambda_3 \dots) \in D^{(\ell)}$, so, by induction, there exist nonnegative integers $m_r, \dots, m_1$ such that $(\lambda_1 - 1, \lambda_2, \dots) = \Theta(p_r^{m_r} \dots p_1^{m_1})$. So, by definition of Θ, $\lambda = \Theta(p_r^{m_r} \dots p_1^{m_1+1})$. Otherwise, $\lambda_1 = \lceil c_\ell \lambda_2 \rceil$, so by Theorem 4, there is a $t \geq 2$ and an ℓ-admissible sequence $b_n \dots b_t$ with $b_t > 0$, such that

$$[\lambda_1] = b_n b_{n-1} \dots b_t 0^{t-1} = [\lambda_2] \cdot 0. \tag{26}$$

Let $\lambda^* = (\lambda_3, \lambda_4, \dots)$. Either λ^* is empty or, by induction, there exist nonnegative integers $m'_{r'}, \dots, m'_s$ such that

$$\lambda^* = \Theta(p_{r'}^{m'_{r'}} \dots p_s^{m'_s+1}) \tag{27}$$

and by Theorem 5 and (12),

$$[\lambda_3] = d_{n-2} \dots d_s 0^{s-1}.$$

Now, by definition of Θ, p_s was the last part inserted (recursively) by Θ to construct λ^* and this was due to insertion (at some point) of the part p_{s+1} into λ' by Θ to construct λ.

If the theorem and (i) and (ii) are to hold for λ and λ', then p_k is the last part inserted to create λ from λ' using Θ. We will see, ultimately, that $k = \min\{t, s+1\}$ (see (31)).

We consider whether p_k passed or failed Test(*) during insertion.

If p_k failed the Test(*), there was no recursive insertion of p_{k-1} to create λ^*, so $k \le s+1$. Also, by Theorem 5, then $b_k > 0$, $b_{k-1} = \cdots = b_1 = 0$, so in (26), $k = t$. Then λ' satisfies

$$\lambda'_1 = \lambda_1 - a_k; \quad \lambda'_2 = \lambda_2 - a_{k-1}; \quad \lambda'_j = \lambda_j, \ j \ge 3.$$

To show $\lambda' \in D^{(\ell)}$, then, by (12), it suffices to show

$$[\lambda'_1] \ \succeq \ [\lambda'_2] \cdot 0 \ \succeq \ [\lambda_3] \cdot 00.$$

Since $t = k$ in (26) and $b_t > 0$, by Lemma 5 (a),

$$[\lambda'_1] \ = \ b_n \ldots (b_t - 1) 0^{t-1} \ = \ [\lambda'_2] \cdot 0. \tag{28}$$

If $\lambda_3 = 0$, the rest is clear, otherwise, since $\lambda \in D^{(\ell)}$,

$$b_n \ldots b_t 0^{t-2} \ = \ [\lambda_2] \ \succeq \ [\lambda_3] \cdot 0 \ \succeq \ d_{n-2} \ldots d_s 0^s,$$

and thus

$$b_n \ldots b_{s+2} \ \succeq \ d_{n-2} \ldots d_s,$$

which implies that

$$[\lambda'_2] \ = \ b_n \ldots (b_t - 1) 0^{t-2} \ \succeq \ d_{n-2} \ldots d_s 0^s \ = \ [\lambda_3] \cdot 0$$

since $t = k \le s+1$.

On the other hand, if p_k passed Test(*), λ^* is the result of recursively inserting p_{k-1} into $(\lambda'_3, \lambda'_4, \ldots)$ by Θ, so $k = s+1$ and (by Theorem 5) $b_k = 0$ so, in (26), $t > k$. Then from (27),

$$(\lambda'_3, \lambda'_4, \ldots) \ = \ \Theta(p_{r'}^{m'_{r'}} \ldots p_{k-1}^{m'_{k-1}}) \in D^{(\ell)}. \tag{29}$$

To show $\lambda' \in D^{(\ell)}$, it remains to show $\lambda'_1 > c_\ell \lambda'_2$ and $\lambda'_2 > c_\ell \lambda'_3$. Since $b_t > 0$ in (26) and since $t > k$, by Lemma 5 (a) and the definition of Θ,

$$\begin{aligned} [\lambda'_1] = [\lambda_1 - (a_k - a_{k-1})] \ &= \ b_n \ldots (b_t - 1)(\ell - 1)(\ell - 2)^{t-k-1} 0^{k-1} \\ &= \ [\lambda_2 - (a_{k-1} - a_{k-2})] \cdot 0 = [\lambda'_2] \cdot 0. \end{aligned} \tag{30}$$

Thus by (12), $\lambda'_1 > c_\ell \lambda'_2$.

As for λ'_3, by definition of Θ, $\lambda'_3 = \lambda_3 - (a_{k-1} - a_{k-2})$ if p_{k-1} passed the Test(*) for the recursive insert and otherwise $\lambda'_3 = \lambda_3 - a_{k-1}$. In either case, $\lambda'_3 \le \lambda_3 - (a_{k-1} - a_{k-2})$, so, since, by Corollary 1, $a_{k-1} > a_{k-2}$ and since $\lambda_2 > c_\ell \lambda_3$ and since $c_\ell \ge 1$,

$$\lambda'_2 \ = \ \lambda_2 - (a_{k-1} - a_{k-2}) \ > \ c_\ell \lambda_3 - (a_{k-1} - a_{k-2}) \ \ge \ c_\ell(\lambda_3 - (a_{k-1} - a_{k-2})) \ge \ c_\ell \lambda'_3.$$

So, $\lambda' \in D^{(\ell)}$ in either case (p_k passed or did not.) Observe that if p_k failed, then $t = k \le s+1$ and if p_k passed, then $t \ge k = s+1$. In either case, then, we have

$$k \ = \min\{t, \ s+1\}. \tag{31}$$

Since $\lambda' \in D^{(\ell)}$, by induction, either λ' is empty or

$$\lambda' = \Theta(p_u^{z_u} \dots p_v^{z_v+1})$$

for some $u \geq v \geq 1$. Finally we must show that $v \geq k$.

If p_v failed Test(*) in creating λ', then by Proposition 5 (a) and Lemma 5 (a), $[\lambda'_1]$ has the form

$$[\lambda'_1] \;=\; e_n \dots (e_v + 1)0^{v-1}$$

for some v. But also $[\lambda'_1]$ has the form (28), where $t = k$ or (30), where $t \geq k$ so, in either case, $v \geq k$.

If p_v passed Test(*) in creating λ', then p_{v-1} was inserted recursively by Θ to create $(\lambda'_3, \lambda'_4 \dots)$, which has the form (29). So, $v - 1 \geq k - 1$, i.e., $v \geq k$. □

This completes the proof that $\Theta^{(\ell)}$ is a bijection for Theorem 1, the ℓ-Euler Theorem.

4.2 Bijection $\Theta_n^{(\ell)}$

Analogous to the $n = \infty$ case in the preceding subsection, the fact that $\Theta_n^{(\ell)} : O_n^{(\ell)} \to D_n^{(\ell)}$ is a bijection follows from the two theorems below.

Theorem 7 *Let $\mu = p_n^{m_n} p_{n-1}^{m_{n-1}} \dots p_k^{m_k} \in O_n$ and let $\lambda = \Theta_n(\mu)$. Then $\lambda \in D_n$. Furthermore, if $k > 1$ then*

(i) *for some ℓ-admissible sequence $b_n \dots b_k$, $[\lambda_1] = [\lambda_2] \cdot 0 = b_n \dots b_k 0^{k-1}$;*

(ii) *the last p_k inserted by $\Theta_n^{(\ell)}$ failed Test(**) if and only if $b_k > 0$.*

Theorem 8 *For every nonempty $\lambda \in D_n^{(\ell)}$ there is a unique integer $k \geq 1$ and nonnegative integers $m_n, m_{n-1}, \dots, m_k$ such that*

$$\lambda = \Theta^{(\ell)}(p_n^{m_n} \dots p_k^{m_k+1}).$$

We omit the proofs, noting that they are essentially the same as the proofs of Theorems 5 and 6, except that we replace r and r' with n, and we apply Lemma 5(b), Proposition 5(b), and equation (13) in place of Lemma 5(a), Proposition 5(a), and equation (12), respectively.

5 Concluding remarks and open questions

Perhaps the first question that comes to mind is the following.

Question 1: Is there an analytic proof of the ℓ-Euler theorem? When $\ell = 2$, the standard approach is to show the equivalence of the generating functions for the sets $O^{(2)}$ and $D^{(2)}$:

$$\prod_{i=1}^{\infty} \frac{1}{1 - q^{2i-1}} = \prod_{i=1}^{\infty} (1 + q^i).$$

To use this approach for $\ell > 2$, we would need the generating function for $D^{(\ell)}$.

Question 2: Corollary 5 in Section 3 implies the following combinatorial characterization of the set $D^{(\ell)}$: *The number of $\lambda \in D^{(\ell)}$ with $\lambda_1 < a_{n+1}^{(\ell)}$ is equal to the number of fillings of a staircase of shape $(n, n-1, \ldots, 1)$ such that* (i) *the filling of each row is an ℓ-admissible sequence and* (ii) *the rows are weakly decreasing, lexicographically.* Can this characterization be used to directly enumerate $D^{(\ell)}$?

Question 3: There are several q-series identities related to Euler's theorem, such as Lebesgue's identity [1, 7], the Roger's-Fine identity [3, 30], and Cauchy's identity [20, 30]. Are there ℓ-analogs of any of these?

Question 4: When $\ell = 2$, several refinements of Euler's theorem follow from Sylvester's bijection, including (i) - (iv) below. What refinements of the ℓ-Euler theorem can be obtained from $\Theta^{(\ell)}$ and $\Theta_n^{(\ell)}$? We have some partial answers.

(i) Bessenrodt showed in [7] that if $\mu \in O^{(2)}$ and $\lambda = \Theta^{(2)}(\mu)$, the length of μ is the same as the alternating sum of λ:

$$l(\mu) = l_{\text{alt}}(\lambda) = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \ldots \tag{32}$$

Bousquet-Mélou and Eriksson showed that this also holds for 2-lecture hall partitions, i.e., when $\mu \in O_n^{(2)}$ and $\lambda = \Theta_n^{(2)}(\mu)$. In fact, this generalizes for $\ell > 2$. As discussed in Section 2.4, if $\mu \in O^{(\ell)}$ and $\mu = p_n^{m_n} \cdots p_1^{m_1}$, where $p_k = a_k + a_{k-1}$ and if $\lambda = \Theta_n^{(\ell)}(\mu)$ or $\lambda = \Theta^{(\ell)}(\mu)$, then it follows easily from our bijection that

$$m_n d_n + \cdots + m_1 d_1 = l_{\text{alt}}(\lambda), \tag{33}$$

where $d_k = a_k - a_{k-1}$. Note that when $\ell = 2$, for all $i \geq 1$, $d_i = 1$ and $m_1 + \ldots + m_n = l(\mu)$. So, (33) generalizes (32) to all $\ell \geq 2$ in both the finite and the infinite case.

(ii) Bessenrodt also showed in [7] that for $\mu \in O^{(2)}$ and $\lambda = \Theta^{(2)}(\mu)$, the length of λ is related to the size of the Durfee square in the 2-modular diagram of μ,

$$\lfloor (l(\lambda) + 1)/2 \rfloor = d_2(\mu). \tag{34}$$

where $d_2(\mu)$ is the largest t such that μ has at least t parts greater than or equal to $2t - 1$. It can be checked that (34) is equivalent to the following:

$$l(\lambda) \leq k \;\text{ iff }\; \sum_{i=\lceil k/2 \rceil + 1}^{\infty} m_i \leq \lfloor k/2 \rfloor, \tag{35}$$

where m_i is the multiplicity of part $p_i = 2i - 1$ in μ. It is open how to generalize (35) when $\ell > 2$. However, in the finite case of $\ell = 2$, i.e., 2-lecture hall partitions, the following analog of (35) appears in [15] (Theorem 5) as a characterization of "truncated lecture hall partitions". If μ is a partition into odd parts less than $2n$, with m_i copies of part $2i - 1$, and if $\lambda = \Theta_n^{(2)}(\mu)$, then

$$l(\lambda) \leq k \;\text{ iff }\; \sum_{i=\lceil k/2 \rceil + 1}^{n - \lfloor k/2 \rfloor} m_i \leq \lfloor k/2 \rfloor. \tag{36}$$

The proof of (36) in [15] used a complex indirect argument involving an analytic component. But our description of $\Theta_n^{(2)}$ in Section 2 now allows a straightforward combinatorial proof (included in the appendix).

(iii) Sylvester showed in [27] that if $\mu \in O^{(2)}$ and $\lambda = \Theta^{(2)}(\mu)$, the number of distinct part sizes occurring in μ is the same as the number of maximal chains in λ. A *chain* is a sequence of consecutive integers. We don't have an analog for $\ell > 2$ or even for 2-lecture all partitions.

(iv) For $\mu \in O^{(2)}$ and $\lambda = \Theta^{(2)}(\mu)$, Fine's theorem [16, 17] relates the size of the largest parts of μ and λ by

$$l(\mu) + (\mu_1 - 1)/2 = \lambda_1,$$

where $l(\mu)$ is the length of μ, i.e., the number of positive parts of μ. Again, here, no analog is known for $\ell > 2$ or even for 2-lecture hall partitions.

Question 5: The Chebyshev polynomials of the second kind, $U_n(x)$, can be defined by the recurrence

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),$$

with initial conditions $U_0(x) = 1$, $U_1(x) = 2x$. So,

$$a_n^{(\ell)} = U_{n-1}(\ell/2).$$

Is the connection between ℓ-sequences and Chebyshev polynomials significant in the context of the ℓ-Euler theorem?

Finally, we note that in [9], Bousquet-Mélou and Eriksson actually proved a (k, ℓ)-generalization of the lecture hall theorem and, taking limits, found a further generalization of Euler's theorem, which can be described as follows.

Theorem 9 (The (k, ℓ)-Euler Theorem [9]) *For integers $k, \ell \geq 2$, define the sequence $\{a_n^{(k,\ell)}\}_{n\geq 0}$ by*

$$a_{2n}^{(k,\ell)} = \ell a_{2n-1}^{(k,\ell)} - a_{2n-2}^{(k,\ell)}; \qquad a_{2n+1}^{(k,\ell)} = k a_{2n}^{(k,\ell)} - a_{2n-1}^{(k,\ell)},$$

with initial conditions $a_0^{(k,\ell)} = 0$ and $a_1^{(k,\ell)} = 1$. Let

$$c_{k,\ell} = \frac{k\ell + \sqrt{k\ell(k\ell - 4)}}{2k}.$$

Then the number of partitions of an integer N into parts from the set

$$\{a_0^{(\ell,k)} + a_1^{(k,\ell)},\ a_1^{(\ell,k)} + a_2^{(k,\ell)},\ a_2^{(\ell,k)} + a_3^{(k,\ell)},\ \ldots\}$$

is the same as the number of partitions λ of N in which the ratio of consecutive (positive) parts λ_i/λ_{i+1} is greater than $c_{k,\ell}$ if i is odd and is greater than $c_{\ell,k}$ if i is even.

Setting $k = \ell$ gives the ℓ-Euler theorem. We have checked that the (k, ℓ) case can be handled similarly, but have not worked through the details of the proof. To insert $p_i = a_i^{(k,l)} + a_{i-1}^{(l,k)}$ into $(\lambda_1, \lambda_2, \ldots)$, Test (*) would become:

If $(\lambda_1 + a_i^{(k,l)} - a_{i-1}^{(k,l)}) > c_{k,\ell}(\lambda_2 + a_{i-1}^{(l,k)} - a_{i-2}^{(l,k)})$,

add $a_i^{(k,l)} - a_{i-1}^{(k,l)}$ to λ_1, add $a_{i-1}^{(l,k)} - a_{i-2}^{(l,k)}$ to λ_2,

and recursively insert p_{i-1} into $(\lambda_3, \lambda_4, \ldots)$.

otherwise, add $a_i^{(k,l)}$ to λ_1, and add $a_{i-1}^{(l,k)}$ to λ_2.

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A Appendix

We use the bijection $\Theta_n^{(\ell)}$ of Section 2.3 to prove the following refinement of the lecture hall theorem.

Theorem 10 [15] *Let* $\mu = p_n^{m_n} p_{n-1}^{m_{n-1}} \dots p_1^{m_1} \in O_n^{(2)}$ *and let* $\lambda = \Theta_n^{(2)}(\mu)$*. Then*

$$l(\lambda) \le k \quad \textit{iff} \quad \sum_{i=\lceil k/2 \rceil+1}^{n-\lfloor k/2 \rfloor} m_i \le \lfloor k/2 \rfloor.$$

We need a lemma that establishes a relationship between the number of passes and fails of Test(**) in $\Theta_n^{(\ell)}$.

Let $\mu = p_n^{m_n} p_{n-1}^{m_{n-1}} \dots p_1^{m_1} \in O_n^{(2)}$. For $2 \le k \le n$, let α_k be the number of times λ failed Test(**) during insertion of the parts p_k and let β_k the number of times the test was passed. Define $\alpha_1 = m_1$ and $\beta_1 = 0$. Then $m_k = \alpha_k + \beta_k$ and

$$\begin{aligned} \lambda_1 &= \sum_{i=1}^{n} (a_i \alpha_i + (a_i - a_{i-1})\beta_i) \\ \lambda_2 &= \sum_{i=2}^{n} (a_{i-1} \alpha_i + (a_{i-1} - a_{i-2})\beta_i). \end{aligned}$$

Also, since every p_n inserted fails Test(**). $\alpha_n = m_n$ and $\beta_n = 0$.

Lemma 6 *If* $\Theta_n^{(2)}$ *is applied to the partition* $\mu = p_n^{m_n} p_{n-1}^{m_{n-1}} \dots p_1^{m_1} \in O_n^{(2)}$*, then for every* k*,* $2 \le k \le n$*, the* α_i, β_i *satisfy*

$$0 \;\le\; \sum_{i-k}^{n} (a_{n-i})\alpha_i - (a_{n-i+1} - a_{n-1})\beta_i \;<\; a_{n-k+1}.$$

Proof. Let $\lambda^{(k)} = \Theta_n^{(2)}(p_n^{m_n} p_{n-1}^{m_{n-1}} \dots p_k^{m_k})$. By Theorem 7, $\lambda^{(k)} \in D_n^{(2)}$ and if $k > 1$, $[\lambda_1^{(k)}]_n = b_n \cdots b_k 0^{k-1} = [\lambda_2^{(k)}]_{n-2} \cdot 0$, for an ℓ-admissible sequence $b_n \cdots b_k$. Thus,

$$\begin{aligned} 0 \;\le\; a_{n-1}\lambda_1^{(k)} - a_n \lambda_2^{(k)} &\le\; a_{n-1} \sum_{i=k}^{n} b_i a_i - a_n \sum_{i=k}^{n} b_i a_{i-1} \\ &=\; \sum_{i=k}^{n} b_i (a_{n-1} a_i - a_n a_{i-1}) \\ &=\; \sum_{i=k}^{n} b_i a_{n-i} \;=\; \sum_{i=0}^{n-k} a_i b_{n-i} \;<\; a_{n-k+1}, \end{aligned}$$

where the second equality follows from (4) and the last inequality from Corollary 2, using $a_0 = 0$. Now writing λ_1 and λ_2 in terms of the α_i and β_i and again applying (4),

$$\begin{aligned} a_{n-1}\lambda_1^{(k)} - a_n\lambda_2^{(k)} &= a_{n-1}(\sum_{i=k}^{n}(a_i\alpha_i + (a_i - a_{i-1})\beta_i)) - a_n(\sum_{i=2}^{n}(a_{i-1}\alpha_i + (a_{i-1} - a_{i-2})\beta_i)) \\ &= \sum_{i=k}^{n}(a_{n-1}a_i - a_n a_{i-1})(\alpha_i + \beta_i) - (a_{n-1}a_{i-1} + a_n a_{i-2})\beta_i \\ &= \sum_{i=k}^{n}(a_{n-i})\alpha_i - (a_{n-i+1} - a_{n-i})\beta_i. \end{aligned}$$

□

Proof of Theorem 10. Proceed by induction on k. If $k = 0$, the theorem says $m_i = 0$ for all i and for $k = 1$ it says $m_i = 0$ unless $i = 1$, so the theorem is true for $k = 0, 1$. When $k = n$, the sum is empty and this becomes the lecture hall theorem.

For $n > k \geq 2$, let $\lambda' = (\lambda_{n-2}, \lambda_{n-3}, \ldots, \lambda_1)$. Then $l(\lambda) \leq k$ if and only if $l(\lambda') \leq k - 2$. Since $\lambda = \Theta_n^{(2)}(\mu)$, by definition of $\Theta_n^{(2)}$ and the β_i, $\lambda' = \Theta_{n-1}^{(2)}(p_{n-2}^{\beta_{n-1}} p_{n-3}^{\beta_{n-2}} \ldots p_1^{\beta_2})$, where $m_i = \alpha_i + \beta_i$. So, by induction, $l(\lambda') \leq k - 2$ if and only if

$$\lfloor k/2 \rfloor - 1 \geq \sum_{i=\lceil k/2 \rceil}^{(n-2)-(\lfloor k/2 \rfloor - 1)} \beta_{i+1} = \sum_{i=\lceil k/2 \rceil + 1}^{n - \lfloor k/2 \rfloor} \beta_i.$$

So we need to show

$$\sum_{i=\lceil k/2 \rceil + 1}^{n-\lfloor k/2 \rfloor} m_i \ \leq \ \lfloor k/2 \rfloor \quad \text{iff} \quad \sum_{i=\lceil k/2 \rceil + 1}^{n-\lfloor k/2 \rfloor} \beta_i \ \leq \ \lfloor k/2 \rfloor - 1.$$

First assume that $\sum_{i=\lceil k/2 \rceil + 1}^{n-\lfloor k/2 \rfloor} \beta_i \ \leq \ \lfloor k/2 \rfloor - 1$. To show that $\sum_{i=\lceil k/2 \rceil + 1}^{n-\lfloor k/2 \rfloor} m_i \ \leq \ \lfloor k/2 \rfloor$, it suffices to show that $\sum_{i=\lceil k/2 \rceil + 1}^{n-\lfloor k/2 \rfloor} \alpha_i \ \leq \ 1$. By Lemma 6,

$$\begin{aligned} n - (\lceil k/2 \rceil + 1) &\geq \sum_{i=\lceil k/2 \rceil + 1}^{n} ((n-i)\alpha_i - \beta_i) \\ &= \sum_{i=\lceil k/2 \rceil + 1}^{n-\lfloor k/2 \rfloor} ((n-i)\alpha_i - \beta_i) + \sum_{i=n-\lfloor k/2 \rfloor + 1}^{n} ((n-i)\alpha_i - \beta_i) \\ &\geq \sum_{i=\lceil k/2 \rceil + 1}^{n-\lfloor k/2 \rfloor} ((n-i)\alpha_i - \beta_i). \end{aligned}$$

Rearranging the terms gives

$$\sum_{i=\lceil k/2 \rceil + 1}^{n-\lfloor k/2 \rfloor} \beta_i \ \geq \ -(n - (\lceil k/2 \rceil + 1)) + \sum_{i=\lceil k/2 \rceil + 1}^{n-\lfloor k/2 \rfloor} (n-i)\alpha_i.$$

Now use the fact that $(n-i) \geq (n-(n-\lfloor k/2 \rfloor))$ for $i \leq n - \lfloor k/2 \rfloor$ and that, since $k \leq n-1$, $\lceil k/2 \rceil + 1 \leq n - \lfloor k/2 \rfloor$ to get

$$\sum_{i=\lceil k/2 \rceil+1}^{n-\lfloor k/2 \rfloor} \beta_i \geq \left(-1 + \sum_{i=t+1}^{n-\lfloor k/2 \rfloor} \alpha_i \right) \lfloor k/2 \rfloor.$$

Putting this together with our assumption about the sum of the β_i gives

$$\lfloor k/2 \rfloor - 1 \geq \lfloor k/2 \rfloor \left(-1 + \sum_{i=\lceil k/2 \rceil+1}^{n-\lfloor k/2 \rfloor} \alpha_i \right)$$

and therefore, $\sum_{i=\lceil k/2 \rceil+1}^{n-\lfloor k/2 \rfloor} \alpha_i \leq 1$.

For the converse, assume that $\sum_{i=\lceil k/2 \rceil+1}^{n-\lfloor k/2 \rfloor} \beta_i \geq \lfloor k/2 \rfloor$. We claim that $\sum_{i=\lceil k/2 \rceil+1}^{n-\lfloor k/2 \rfloor} \alpha_i \geq 1$. To show this, it suffices to show that $\sum_{i=\lceil k/2 \rceil+1}^{n-\lfloor k/2 \rfloor} (n-i)\alpha_i \geq 1$, since then at least one $\alpha_i \geq 1$. First, by Lemma 1,

$$\sum_{n-\lfloor k/2 \rfloor+1}^{n} (n-i)\alpha_i \leq \lfloor k/2 \rfloor - 1 + \sum_{n-\lfloor k/2 \rfloor+1}^{n} \beta_i. \tag{37}$$

Then,

$$\begin{aligned}
\sum_{i=\lceil k/2 \rceil+1}^{n-\lfloor k/2 \rfloor} (n-i)\alpha_i &= \sum_{i=\lceil k/2 \rceil+1}^{n} (n-i)\alpha_i - \sum_{i=n-\lfloor k/2 \rfloor+1}^{n} (n-i)\alpha_i \\
&\geq \sum_{i=\lceil k/2 \rceil+1}^{n} \beta_i - \sum_{i=n-\lfloor k/2 \rfloor+1}^{n} (n-i)\alpha_i \qquad \text{(by Lemma 6)} \\
&\geq \sum_{i=\lceil k/2 \rceil+1}^{n} \beta_i - \left(\lfloor k/2 \rfloor - 1 + \sum_{n-\lfloor k/2 \rfloor+1}^{n} \beta_i \right) \qquad \text{(by (37))} \\
&= \sum_{i=\lceil k/2 \rceil+1}^{n-\lfloor k/2 \rfloor} \beta_i - \lfloor k/2 \rfloor + 1 \\
&\geq \lfloor k/2 \rfloor - \lfloor k/2 \rfloor + 1 = 1,
\end{aligned}$$

where the last inequality follows from our assumption about the sum of the β_i. □