LET ME TELL YOU MY FAVORITE LATTICE-POINT PROBLEM...

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Abstract. This collection was compiled by Bruce Reznick from problems presented at the 2006 AMS/IMS/SIAM Summer Research Conference on *Integer points in polytopes*.

1. Solid-Angle Polynomials

Presented by Matthias Beck (San Francisco State University)

Suppose $\mathcal{P} \subset \mathbb{R}^d$ is a convex rational d-polyhedron. The *solid angle* $\omega_{\mathcal{P}}(\mathbf{x})$ of a point $\mathbf{x}$ (with respect to $\mathcal{P}$) is a real number equal to the proportion of a small ball centered at $\mathbf{x}$ that is contained in $\mathcal{P}$. That is, we let $B_\epsilon(\mathbf{x})$ denote the ball of radius ϵ centered at $\mathbf{x}$ and define

$$\omega_{\mathcal{P}}(\mathbf{x}) := \frac{\operatorname{vol}\left(B_\epsilon(\mathbf{x}) \cap \mathcal{P}\right)}{\operatorname{vol} B_\epsilon(\mathbf{x})}$$

for all positive ϵ sufficiently small. We note that when $\mathbf{x} \notin \mathcal{P}$, $\omega_{\mathcal{P}}(\mathbf{x}) = 0$; when $\mathbf{x} \in \mathcal{P}^\circ$, $\omega_{\mathcal{P}}(\mathbf{x}) = 1$; when $\mathbf{x} \in \partial\mathcal{P}$, $0 < \omega_{\mathcal{P}}(\mathbf{x}) < 1$. We define

$$A_{\mathcal{P}}(t) := \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} \omega_{t\mathcal{P}}(\mathbf{m}) \, ,$$

the sum of the solid angles at all integer points in $t\mathcal{P}$; recalling that $\omega_{\mathcal{P}}(\mathbf{x}) = 0$ if $\mathbf{x} \notin \mathcal{P}$, we can also write

$$A_{\mathcal{P}}(t) = \sum_{\mathbf{m} \in \mathbb{Z}^d} \omega_{t\mathcal{P}}(\mathbf{m}) \, .$$

I. G. Macdonald [3] inaugurated the systematic study of solid-angle sums in integral polytopes in 1971 with the following theorem (see also [1, Chapter 11]).

Theorem (Macdonald)**.** *Suppose $\mathcal{P}$ is a convex integral d-polytope. Then $A_{\mathcal{P}}(t)$ is a polynomial in t of degree d whose leading coefficient is $\operatorname{vol}\mathcal{P}$ and whose constant term is 0. Furthermore, $A_{\mathcal{P}}$ is either even or odd:*

$$A_{\mathcal{P}}(-t) = (-1)^d A_{\mathcal{P}}(t) \, .$$

Open Problems:

- For which integral polytopes $\mathcal{P}$ are all the coefficients of $A_{\mathcal{P}}(t)$ rational?
- Are there solid-angle polynomials with negative coefficients?
- Does the numerator polynomial of the rational generating function of $A_{\mathcal{P}}$ always have nonnegative coefficients, in analogy with Stanley's Nonnegativity Theorem for Ehrhart series?
- Classify all polytopes that have only rational solid angles at their vertices.
- Among all d-simplices Δ, does the regular d-simplex have the property that $\sum_{\mathbf{v} \text{ a vertex of } \Delta} \omega_\Delta(\mathbf{v})$ is a minimum?

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2. What is the maximal number of vertices of a reflexive polytope?

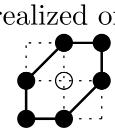
Presented by Benjamin Nill (Freie Universität Berlin)

Introduction. Reflexive polytopes are lattice polytopes containing the origin in their interior such that also the dual polytope is a lattice polytope. So they always appear as dual pairs. The notion of reflexive polytopes was introduced by Batyrev in 1994 to provide a combinatorial framework for constructing mirror symmetric pairs of Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties, [1]. In fixed dimension n only a finite number of reflexive polytopes exist up to unimodular isomorphisms. This follows from results in [9], since any reflexive polytope contains no lattice points in its interior except the origin. Using heavy computer calculations Kreuzer and Skarke succeeded in classifying reflexive polytopes up to dimension four, [6, 7]. They found up to unimodular isomorphisms 16 reflexive polytopes for $n = 2$, 4319 for $n = 3$, and 473800776 for $n = 4$. Contrasting these finiteness results, Haase and Melnikov showed that *any* lattice polytope is isomorphic to a face of some (possibly much higher-dimensional) reflexive polytope, [5].

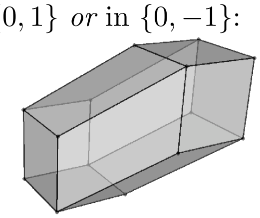
In recent years progress has been made towards a better understanding of the combinatorics and geometry of reflexive polytopes. Still, many elementary questions are open. Here we focus on the number of vertices.

Observations. Using the database of reflexive polytopes [8] we find the following reflexive polytopes with the maximal number of vertices in dimension $n \leq 4$:

- $n = 2$: The maximum is 6 vertices; realized only by the following hexagon $\mathcal{H}$:



- $n = 3$: The maximum is 14 vertices; realized only by the reflexive polytope having vertices with coordinates *either* all in $\{0, 1\}$ *or* in $\{0, -1\}$:



- $n = 4$: The maximum is 36 vertices; realized only by $\mathcal{H} \times \mathcal{H}$.

These observations motivate the following question (note that $14 = \lfloor 6^{3/2} \rfloor$):

Problem. Are there n-dimensional reflexive polytopes having more than $6^{n/2}$ vertices? If not, is $\mathcal{H}^{n/2}$, for n even, the only one with this number of vertices?

It suffices to solve the problem for *even* n, since products of reflexive polytopes are reflexive. We remark that in odd dimension we cannot even state a conjectural sharp bound. One should also note that for $n \geq 6$ there are non-reflexive lattice polytopes with only one interior lattice point and more than $6^{n/2}$ vertices. We thank Günter Ziegler for this observation.

Actually, the problem should be seen as a quest to find, if possible, interesting examples, or even better, explicit constructions of higher-dimensional reflexive polytopes having many vertices that are not simply products of lower-dimensional ones.

Implications. If the bound in the problem would hold, then we would also get sharp upper bounds on the following invariants:

- The number of vertices of an n-dimensional Gorenstein polytope, i.e., of a lattice polytope with some multiple being a reflexive polytope (up to translation). A Gorenstein polytope can also be characterized as a lattice polytope with symmetric δ-vector (sometimes also called h^*-, or h-vector), [2]. An example is the Birkhoff polytope.
- The class number of a Gorenstein toric Fano variety, [10, Sect.5].
- The number of facets of an n-dimensional reflexive polytope, simply due to duality.
- The topological Euler characteristic of a Gorenstein toric Fano variety, [4, p.59].

Results. The following cases have been settled:

- For $n = 2$ a simple explanation for the above bound can be given, [10, Cor.4.2(2)]. However, for $n = 3, 4$ no direct proof is known apart from the computer classification of Kreuzer and Skarke, [6].
- An n-dimensional simplicial reflexive polytope has at most $3n$ vertices, where equality holds only for the dual of $\mathcal{H}^{n/2}$, [3].
- An n-dimensional reflexive polytope where the vertices are the only lattice points on the boundary has $\leq 2^{n+1} - 2$ vertices, and equality implies central-symmetry, [10, Cor.6.3].
- The problem is solved affirmatively for simple reflexive polytopes whose dual contains a centrally symmetric pair of facets, [11]. In particular, $\mathcal{H}^{n/2}$ is, for n even, the only centrally symmetric simple reflexive polytope with the maximal number of $6^{n/2}$ vertices.

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3. My favorite lattice point problem

Presented by Bruce Reznick (University of Illinois)

In 2002, M. D. Choi, T. Y. Lam and I [1] discussed the following combinatorial object, whose original application was to the representation of polynomials in several variables as a sum of squares of polynomials. Let $\mathcal{P}$ in $\mathbb{R}^n$ be a lattice polytope and suppose

$$\mathcal{L}(\mathcal{P}) := \mathcal{P} \cap \mathbb{Z}^n = \{v_1, \dots, v_N\},$$

where $N := N(\mathcal{P}) = \#(\mathcal{L}(\mathcal{P}))$. Then $\mathcal{P}$ is called a *distinct pair-sum* or *dps* lattice polytope if the $N + \binom{N}{2}$ points in $\mathcal{L}(\mathcal{P}) + \mathcal{L}(\mathcal{P})$; namely, $2v_1, \dots, 2v_N;\ v_1 + v_2,\ v_1 + v_3, \dots, v_{N-1} + v_N$, are distinct.

We proved the following results in [1]:

• $\mathcal{P}$ is a dps lattice polytope if and only if $\mathcal{L}(\mathcal{P})$ contains neither three collinear points nor the vertices of a nondegenerate parallelogram;

• $\mathcal{P}$ is a dps lattice polytope if and only if no two vectors of the form $v_i - v_j$, $i < j$, are parallel;

• $N(\mathcal{P}) \le 2^n$.

The proofs of these are fairly simple. The discordant directions of the vectors in $\mathcal{P} - \mathcal{P}$ suggest the alternate description as a *box of cats.* The upper bound 2^n uses a Pigeonhole Principle argument from the 1971 Putnam: no two elements of $\mathcal{L}(\mathcal{P})$ are congruent mod 2. In case $N(\mathcal{P}) = 2^n$, we say that $\mathcal{P}$ is *maximum.*

• For every n, there exists a maximum dps polytope in $\mathbb{R}^n$.

Maximum dps polytopes may be constructed inductively. For any finite set $A \subset \mathbb{R}^n \setminus 0$, there exists $M \in SL_n(\mathbb{Z})$ so that $A \cap M(A) = \emptyset$. Suppose $\mathcal{P}_{n-1} \in \mathbb{R}^{n-1}$ is a maximal dps lattice polytope and suppose $\mathcal{L}(\mathcal{P}_{n-1}) = \{v_1, \dots, v_{2^{n-1}}\}$. Let $A = \{v_i - v_j : i \ne j\}$ and choose M as above. Then

$$\mathcal{P}_n = cvx(\{(v_i, 0)\} \cup \{(M(v_i), 1)\}) \tag{1}$$

is a maximum dps polytope in $\mathbb{R}^n$. (That is, we join two copies of $\mathcal{P}_{n-1}$ on parallel planes, where one is smooshed unimodularly so it doesn't recognize the other.) The essentially unique maximum dps triangle $\mathcal{P}_2$ has vertices $(0, 1), (1, 2)$ and $(2, 0)$. Now let

$$\mathcal{A} = \{(4, 1, 0, 0), (0, 4, 1, 0), (0, 0, 4, 1), (1, 0, 0, 4)\},$$
$$\mathcal{B} = \{(2, 1, 1, 1), (1, 2, 1, 1), (1, 1, 2, 1), (1, 1, 1, 2)\}.$$

If $\mathcal{Q} = cvx(\mathcal{A})$, then $\mathcal{L}(\mathcal{Q}) = \mathcal{A} \cup \mathcal{B}$. The projection of $\mathcal{Q}$ onto its first three coordinates, $\mathcal{P}_3$, gives a maximum dps polyhedron. Also, a direct application of the smoosh to $\mathcal{P}_2$ suggests the unimodular map $M(x, y) = (10x + 3y, 3x + y)$ and the maximum dps polyhedron $\mathcal{P}'_3$ in $\mathbb{R}^3$ with

$$\mathcal{L}(\mathcal{P}'_3) = \{(0, 1, 0), (1, 1, 0), (1, 2, 0), (2, 0, 0), (3, 1, 1), (13, 4, 1), (16, 5, 1), (20, 6, 1)\}.$$

Since translates of dps polytopes are also dps, we may always assume, as above, that $\mathcal{P}$ lies in $\mathbb{R}^n_+$. In this case $s(\mathcal{P})$, the *size* of $\mathcal{P}$, is defined by

$$s(\mathcal{P}) = \max\{v_j^{(1)} + \cdots + v_j^{(n)} :\ v_j = (v_j^{(1)}, \dots, v_j^{(n)}) \in \mathcal{L}(\mathcal{P})\}.$$

If $s = s(\mathcal{P})$, then $\mathcal{P}$ can be viewed as a projection onto the first n coordinates of a polytope in $\mathbb{R}^{n+1}$ which lies in the simplex $s \cdot \Delta_{n+1}$. Let s_n denote the minimum size of any maximum dps polytope in $\mathbb{R}^n$. The examples given show that $s_2 \le s(\mathcal{P}_2) = 3$ and $s_3 \le s(\mathcal{P}_3) = 5$, and it is not difficult to show that these examples are extremal.

Someone asked me in Snowbird, aren't you supposed to have some questions?

• What is an exact, or asymptotic bound for s_n? (Since a maximum dps polytope must have a point with odd positive integral coordinates, we must have $s_n \ge n$; the construction of the unimodular M leads to an ungainly doubly exponential bound that's not worth exploring.)

• What is the possible range for the volume of a maximum dps polytope in $\mathbb{R}^n$?

• How many different "inequivalent" maximum dps polytopes are there in $\mathbb{R}^n$? (These last two questions are open, even for $n = 3$.)

• Are there "nice" constructions of "reasonably symmetric" maximum dps polytopes in $\mathbb{R}^n$ resembling the cyclically constructed $\mathcal{P}_2$ and $\mathcal{P}_3$?

• Is every *maximal* dps polytope *maximum*? That is, if $\mathcal{P}$ is in $\mathbb{R}^n$ and $N(\mathcal{P}) < 2^n$, must $\mathcal{P}$ be contained in a maximum dps polytope? (Again, this question is open even for $n = 3$, although I suspect the answer is "no".)

• Do dps polytopes have interesting Ehrhart polynomials?

If $\mathcal{P}$ is a lattice polytope, then the *lattice width* of $\mathcal{P}$, $w(\mathcal{P})$, is defined to be

$$w(\mathcal{P}) = \min_{0 \neq a \in \mathbb{Z}^n} \Big(\max_{v, v' \in \mathcal{L}(\mathcal{P})} |a \cdot (v - v')| \Big). \tag{2}$$

This computation is also a lattice point problem: $w(\mathcal{P})$ is the smallest positive integer d so that the polytope $\cap\{(v - v')\cdot \ \le d\}$ contains a non-zero lattice point.

• What can be said about the lattice widths of maximum dps polytopes? (It is not hard to show that $w(\mathcal{P}_2) = 2$ and $w(\mathcal{P}_3) = 3$, but every maximum example constructed using the unimodular smoosh has lattice width 1, taking $a = e_n$.)

Sasha Barvinok asked during the Problem Session about dps polytopes as a subset of the unit cube $\{0,1\}^n$. If $\mathcal{A}$ is any such subset, then $\mathcal{L}(cvx(\mathcal{A})) = \mathcal{A}$, so the only condition to be checked is the distinctness of the pair-sums. This turns out to be a special case of a well-studied problem in combinatorics.

• What is the largest dps subset of $\{0,1\}^n$?

Clearly, such a polytope could not be maximum, and as as a trivial upper bound, the $N + \binom{N}{2}$ distinct sums must live in $\{0,1,2\}^n$, so $N < c \cdot 3^{n/2}$. A set is said to have the *B_2-property* if its pair-sums are distinct. Early work on this subject for subsets of $\mathbb{N}$ was done by Erdös and Turán; early work for subsets of $\{0,1\}^n$ was done by Lindström (see [3, 4]). The current record upper bound for the size of a B_2-subset of $\{0,1\}^n$, found by Cohen, Litsyn and Zémor in 2001 [2], is $2^{.5753n}$. (Note that $2^{.5753} \approx 1.49 < \sqrt{3}$.) I thank Zoltán Füredi, Gyula Katona and Doug West for their help in directing me towards this literature.

More interesting for our purposes would be constructions of "large" B_2-sets in $\{0,1\}^n$, but these seem to have received less attention in the literature.

• What does the greedy algorithm yield when applied to finding a B_2-subset of $\{0,1\}^n$, ordered lexicographically?

Finally, we note that each $v \in \{0,1\}^n$ may be naturally interpreted as the characteristic function of a subset of $\{1,\dots,n\}$. The corresponding condition to the B_2-property is that $\mathcal{S} = \{S_i\}$ is a collection of subsets of $\{1,\dots,n\}$ with the property that $S_i \cap S_j$ and $S_i \cup S_j$ determine $\{S_i, S_j\}$.

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4. Counting related pairs in the dominance order on partitions

Presented by Carla Savage (North Carolina State University)

Given two partitions $\lambda = \lambda_1, \lambda_2, \dots$ and $\mu = \mu_1, \mu_2$ of n, with $\lambda_1 \geq \lambda_2 \geq \dots$, $\mu_1 \geq \mu_2 \geq \dots$, and

$$|\lambda| = \lambda_1 + \lambda_2 + \dots = n = \mu_1 + \mu_2 + \dots = |\mu|,$$

λ *dominates* μ if for every i,

$$\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j.$$

So the set D_n of pairs (λ, μ) of partitions of n such that λ dominates μ is just the set of nonnegative integer sequences $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n$ satisfying a certain collection of linear constraints.

- Problem: Count D_n.

This arose as a question of Macdonald [3] (Ch. 1, Sec. 1) about the probability as $n \to \infty$ that λ dominates μ for two randomly chosen partitions λ, μ of n. Boris Pittel showed that this probability does go to 0 [1]. Pittel's approach applies results from probability theory about the joint distribution of the largest parts of a partition and its conjugate, bypassing the problem of counting D_n directly.

One can count D_n for fairly large n with a variation of the recurrence of Axel Kohnert [2].

As a variation, fix n, the number of parts of λ and μ, but remove the constraint $|\lambda| = n = |\mu|$.

- Is it possible to find a generating function for those pairs (λ, μ) of nonnegative integer sequences satisfying $\lambda_1 \geq \dots \geq \lambda_n$, $\mu_1 \geq \dots \geq \mu_n$ and λ dominates μ?

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5. Two problems on lattice points in Minkowski sums of polytopes

Presented by Ivan Soprunov (Cleveland State University)

Let P be an n-dimensional lattice polytope in $\mathbb{R}^n$ (n-polytope for short). It is intuitively clear that if P has many lattice points (or interior lattice points) then the volume of P must be large. The opposite is, however, not true. For example, the 3-simplex with vertices $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(0,0,m)$ has no interior lattice points, but can have arbitrary large volume. Therefore, in general there is no non-trivial lower bound on the number of interior lattice points of P in terms of the volume of P. Nevertheless, if P is a large enough *multiple* of a lattice polytope or, more generally, a *Minkowski sum* of sufficiently many lattice polytopes then such a lower bound exists.

Lower Bound. *Let $P_1, \dots, P_n$ be lattice n-polytopes in $\mathbb{R}^n$ and $P = \sum_{i=1}^n P_i$ their Minkowski sum. Then*

$$\#\{\text{interior}(P) \cap \mathbb{Z}^n\} \geq n!\, \mathrm{V}(P_1, \dots, P_n) - 1, \tag{3}$$

where $\mathrm{V}(P_1, \dots, P_n)$ is the mixed volume of the polytopes.

This inequality follows from the toric version of the Euler–Jacobi theorem due to Khovanskii [3], and Bernstein's theorem [2] on the number of solutions for sparse polynomial systems (see [4] for details). Both theorems are deep non-trivial results in algebraic geometry.

Problem 1. Give a combinatorial proof of (3).

In the case when $P_1, \dots, P_n$ are all equal to the same lattice n-polytope Q (and so $P = nQ$) the situation is well-understood. The inequality (3) becomes

$$\#\big\{\text{interior}(nQ) \cap \mathbb{Z}^n\big\} \ge n!\,\mathrm{V}(Q) - 1, \tag{4}$$

where $\mathrm{V}(Q)$ is the volume of Q.

Here is a short proof of (4) using Stanley's positivity theorem [5]. Let $I_Q(t)$ be the number of interior lattice points in tQ, for $t \in \mathbb{N}$. Then one can write

$$I_Q(t) = \sum_{i=0}^{n} h_i^* \binom{t+i-1}{n},$$

where h_i^* are *non-negative* integers and $h_0^* = 1$. Since the coefficient of the main term of $I_Q(t)$, as $t \to \infty$, is the volume of Q, we get $\sum_{i=0}^{n} h_i^*/n! = \mathrm{V}(Q)$. Therefore, for $t = n$ we obtain

$$\#\big\{\text{interior}(nQ) \cap \mathbb{Z}^n\big\} = I_Q(n) = \sum_{i=1}^{n} h_i^* \binom{n+i-1}{n} \ge \sum_{i=1}^{n} h_i^* = n!\,\mathrm{V}(Q) - 1.$$

It follows from the above proof that the bound (4) is attained at Q if and only if $h_2^* = \cdots = h_n^* = 0$. This characterizes polytopes of *degree* at most 1, in the sense of Batyrev and Nill. In a recent paper [1] they described all possible polytopes Q of degree 0 (simplices of unit volume) and 1 (Lawrence prisms and exceptional simplices).

The above discussion suggests the following definition.

Definition. We say that a collection of n lattice n-polytopes $\{P_1, \dots, P_n\}$ has *mixed degree at most* 1 if the lower bound (3) is attained, and *mixed degree* 0 if $P_1 + \cdots + P_n$ has no interior lattice points.

Example. Let $\Delta(m)$ denote the simplex defined as the convex hull of $n+1$ points $\{0, e_1, \dots, e_{n-1}, me_n\}$, where e_i is the i-th standard basis vector and m is a positive integer.

Consider a collection of n such simplices $\Delta(m_1), \dots, \Delta(m_n)$ and order them so that $m_1 \le \cdots \le m_n$. It is not hard to see that their mixed volume equals m_n. Also one can see that the number of interior lattice points in $\Delta(m_1) + \cdots + \Delta(m_n)$ is exactly $m_n - 1$. (In fact, these lattice points are precisely the points $(1, \dots, 1, k)$ for $1 \le k < m_n$.) Therefore, for any $m_1, \dots, m_n$ the collection $\{\Delta(m_1), \dots, \Delta(m_n)\}$ has mixed degree at most 1, and it has mixed degree 0 if $m_1 = \cdots = m_n = 1$.

Problem 2. Describe collections of n lattice n-polytopes of mixed degree at most 1.

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6. My favorite lattice point problem

Presented by Zhiqiang Xu (Chinese Academy of Sciences, Beijing)

Problem. Give an iterative formulation, which runs in polynomial time for fixed dimension, for counting the lattice points in polytopes.

It is well known that there exists an good iterative formulation for computing the volume of polytopes. We hope to give a similar formulation for counting the lattice points in polytopes. Lasserre and Zeron have given an iterative algorithm for counting lattice points in a convex polytope. But the algorithm is not polynomial time for fixed dimension [2]. Barvinok presented a polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed [1]. But the algorithm is not iterated. In fact, we even do not know whether there exists a such formulation.

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