

# Five Guidelines for Partition Analysis with Applications to Lecture Hall-type Theorems

Sylvie Corteel  
CNRS PRISM, UVSQ  
45 Avenue des Etats-Unis  
78035 Versailles, France  
syl@prism.uvsq.fr

Sunyoung Lee \*  
Computer Science  
N. C. State University  
Raleigh, NC 27695  
slee7@unity.ncsu.edu

Carla D. Savage †  
Computer Science  
N. C. State University  
Raleigh, NC 27695  
savage@csc.ncsu.edu

August 23, 2005; revised May 19, 2006

## Abstract

Five simple guidelines are proposed to compute the generating function for the nonnegative integer solutions of a system of linear inequalities. In contrast to other approaches, the emphasis is on deriving recurrences. We show how to use the guidelines strategically to solve some nontrivial enumeration problems in the theory of partitions and compositions. This includes a strikingly different approach to lecture hall-type theorems, with new  $q$ -series identities arising in the process. For completeness, we prove that the guidelines suffice to find the generating function for any system of homogeneous linear inequalities with integer coefficients. The guidelines can be viewed as a simplification of MacMahon's partition analysis with ideas from matrix techniques, Elliott reduction, and "adding a slice".

## 1 Introduction

This continues our work in [18, 19] studying nonnegative integer solutions to linear inequalities as they relate to the enumeration of integer partitions and compositions. Define the *weight* of a sequence  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  of integers to be  $|\lambda| = \lambda_1 + \dots + \lambda_n$ . If sequence  $\lambda$  of weight  $N$  has all parts nonnegative, we call it a *composition* of  $N$ ; if, in addition,  $\lambda$  is a nonincreasing sequence, we call it a *partition* of  $N$ .

Given an  $r \times n$  integer matrix  $C = [c_{i,j}]$ , we consider the set  $S_C$  of nonnegative integer sequences  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  satisfying the constraints

$$c_{i,0} + c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \dots + c_{i,n}\lambda_n \geq 0, \quad 1 \leq i \leq r. \quad (1)$$

We seek the full generating function

$$F_C(x_1, x_2, \dots, x_n) = \sum_{\lambda \in S_C} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}, \quad (2)$$

---

\*Research supported in part by NSF grant DMS-0300034

†Research supported in part by NSF grants DMS-0300034 and INT-0230800

which can be viewed as an encapsulation of the solution set  $S_C$ : the coefficient of  $q^N$  in  $F_C(qx_1, qx_2, \dots, qx_n)$  is a listing (as the terms of a polynomial) of all nonnegative integer solutions to (1) of weight  $N$  and the number of such solutions is the coefficient of  $q^N$  in  $F_C(q, q, \dots, q)$ .

Variations of this problem arise in other areas of mathematics, e.g., solving systems of linear equations, finding volume of polytopes, as well as in enumeration. In the papers [18, 19] we demonstrated that in the area of partition and composition enumeration many familiar sets of linear constraints can be easily handled a matrix inversion: for homogeneous systems, if the constraint matrix  $C$  is an  $n \times n$  invertible matrix, and if all entries of  $C^{-1} = B = [b_{i,j}]$  are *nonnegative integers* then by Theorem 1 in [19]:

$$F_C(x_1, x_2, \dots, x_n) = \prod_{j=1}^n \frac{1}{(1 - x_1^{b_{1,j}} x_2^{b_{2,j}} \dots x_n^{b_{n,j}})}.$$

This theorem (in its full generality) suffices to handle the enumeration of such families as Hickerson partitions [22], Santos' interpretation of Euler's family [28], Sellers' generalization of Santos [29, 30], partitions with nonnegative second differences [3], super-concave partitions [31], partitions with  $r$ -th differences nonnegative [3, 14, 33], partitions with mixed difference conditions [3], and examples (0-5) of Pak in [27]. The theorem provides bijections as well as generating functions.

However, it is easy to find simple examples where the "C matrix" technique fails. In Section 2, we propose five simple guidelines for computing the generating function  $F_C$  of a system  $C$  of linear diophantine inequalities. The guidelines can be viewed as a simplification of MacMahon's partition analysis [25], with ideas from matrix methods, Elliott reduction [20], and "adding a slice" (e.g. [23]).

Our focus is on the use of the guidelines to *derive a recurrence* for the generating function  $F_{C_n}$  of an *infinite family*  $\{C_n | n \geq 1\}$  of constraint systems. This is in contrast to the focus of the Omega package [6], a software implementation of partition analysis, well-designed to compute the generating function of a given fixed, finite system of linear constraints. The advantage of a recurrence for  $F_{C_n}$  is a program which computes  $F_{C_n}$  for any given  $n$ . But more significantly, if the recurrence can be solved, it provides a closed form for the generating function for the infinite family.

In Sections 3-6, we show how to use the guidelines of Section 2 strategically to solve some nontrivial enumeration problems in the theory of partitions and compositions. Sections 3 and 4 address well-studied problems, included as "warm-up" exercises to illustrate the approach and the handling of the recurrences that result. Sections 5 and 6 apply the method to the problem of enumerating anti-lecture hall compositions [16] and truncated lecture hall partitions [17], giving a simpler approach than in [16, 17]. For completeness, in Section 7 we prove that the guidelines suffice to find the generating function for the nonnegative integer solutions of any homogeneous system of linear inequalities with integer coefficients.

This work was inspired by the the work of Andrews, Paule, and Riese in the sequence of papers [2, 3, 6, 4, 12, 7, 8, 9, 5, 10, 11], which illustrate many applications of partition analysis. The Omega Package software [6] was an invaluable tool in our early investigations. As illustrated in papers such as [2, 3, 12, 9, 10], recurrences can certainly be derived using partition analysis. However, we found that the task became easier with a simpler set of tools which appear to be no less powerful. In Section 8 we discuss MacMahon's partition analysis and show how the proposed guidelines can be viewed as essential ideas underlying his theory.

## 2 The Five Guidelines

Let  $\mathcal{C}$  be a set of linear constraints in  $n$  variables,  $\lambda_1, \dots, \lambda_n$ , each constraint  $c \in \mathcal{C}$  of the form

$$c: [a_0 + \sum_{i=1}^n a_i \lambda_i \geq 0],$$

for integer values  $a_0, a_1, \dots, a_n$ .

Let  $S_{\mathcal{C}}$  be the set of nonnegative integer sequences  $\lambda = (\lambda_1, \dots, \lambda_n)$  satisfying all constraints in  $\mathcal{C}$ . Since we are only interested here in *nonnegative* integer solutions, we will always assume that  $\mathcal{C}$  contains the constraints  $[\lambda_i \geq 0]$  for  $1 \leq i \leq n$ . Define the *full generating function* of  $\mathcal{C}$  to be:

$$F_{\mathcal{C}}(x_1, \dots, x_n) \triangleq \sum_{\lambda \in S_{\mathcal{C}}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}.$$

If  $c$  is the constraint:  $[a_0 + \sum_{i=1}^n a_i \lambda_i \geq 0]$  define the *negation* of  $c$ ,  $\neg c$ , to be the constraint  $[-a_0 - \sum_{i=1}^n a_i \lambda_i \geq 1]$ . Then any sequence  $(\lambda_1, \dots, \lambda_n)$  satisfies  $c$  or  $\neg c$ , but not both. A constraint  $c$  is *implied* by the set of constraints  $\mathcal{C}$  if  $S_{\mathcal{C} \cup \{c\}} = \emptyset$ . A constraint  $c$  is *redundant* if  $S_{\mathcal{C} \cup \{c\}} = S_{\mathcal{C}}$ .

Let  $\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$  denote the set of constraints which results from replacing  $\lambda_i$  by  $\lambda_i + a\lambda_j$  in every constraint in  $\mathcal{C}$ . Note that if constraint  $c$  is implied by  $\mathcal{C}$  then  $c_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$  is implied by  $\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$ . Thus observe that if  $\mathcal{C}$  contains the constraints  $[\lambda_k \geq 0], 1 \leq k \leq n$  and if  $[\lambda_i - a\lambda_j \geq 0]$  is implied by  $\mathcal{C}$ , then all of the constraints  $[\lambda_k \geq 0], 1 \leq k \leq n$  are also implied by  $\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$ .

**Lemma 1** *Let  $\mathcal{C}$  be a set of linear constraints on variables  $\lambda_1, \dots, \lambda_n$  which contains the constraints  $[\lambda_k \geq 0], 1 \leq k \leq n$ . Let  $a$  be any integer (possibly negative). Suppose  $[\lambda_i - a\lambda_j \geq 0]$  is implied by  $\mathcal{C}$  and let  $\mathcal{C}' = \mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$ . Then*

$$\beta = (\beta_1, \dots, \beta_n) \in S_{\mathcal{C}} \quad \text{iff} \quad \beta' = (\beta_1, \dots, \beta_{i-1}, \beta_i - a\beta_j, \beta_{i+1}, \dots, \beta_n) \in S_{\mathcal{C}'}$$

**Proof.** By the remarks preceding the lemma, the constraints  $\mathcal{C}$  and  $\mathcal{C}'$  guarantee that  $S_{\mathcal{C}}$  and  $S_{\mathcal{C}'}$  contain only nonnegative integer solutions. So, it suffices to show that  $\beta$  satisfies a constraint in  $\mathcal{C}$  iff  $\beta'$  satisfies the corresponding constraint in  $\mathcal{C}'$ .

Let  $c(\lambda) = c_0 + \sum_{t=1}^n c_t \lambda_t$  and assume  $[c(\lambda) \geq 0] \in \mathcal{C}$ . Under the substitution  $\lambda_i \leftarrow \lambda_i + a\lambda_j$ ,  $c(\lambda)$  becomes  $c'(\lambda)$  defined by

$$c'(\lambda) = c_0 + \sum_{t=1}^n c_t \lambda_t + c_i a \lambda_j = c(\lambda) + c_i a \lambda_j$$

and  $[c'(\lambda) \geq 0] \in \mathcal{C}'$ . Thus

$$c(\beta) = c'(\beta) - c_i a \beta_j = c'(\beta'),$$

so  $c(\beta) \geq 0$  iff  $c'(\beta') \geq 0$ . □

Finally, to simplify notation, we will let  $X_n$  refer to the parameter list  $x_1, \dots, x_n$ , so that  $F(X_n)$  denotes  $F(x_1, \dots, x_n)$ . Let  $F(X_n; x_i \leftarrow x_i x_j^a)$  denote the function  $F(X_n)$  with all occurrences of  $x_i$  replaced by  $x_i x_j^a$ .

**Theorem 1** (*The Five Guidelines*)

1. If  $\mathcal{C} = \{[\lambda_1 \geq t]\}$ , for integer  $t \geq 0$ , then

$$F_{\mathcal{C}}(x_1) = \frac{x_1^t}{1 - x_1}.$$

2. If  $\mathcal{C}_1$  is a set of constraints on variables  $\lambda_1, \dots, \lambda_j$  and  $\mathcal{C}_2$  is a set of constraints on variables  $\lambda_{j+1}, \dots, \lambda_n$ , then

$$F_{\mathcal{C}_1 \cup \mathcal{C}_2}(x_1, \dots, x_n) = F_{\mathcal{C}_1}(x_1, \dots, x_j) F_{\mathcal{C}_2}(x_{j+1}, \dots, x_n).$$

3. Let  $\mathcal{C}$  be a set of linear constraints on variables  $\lambda_1, \dots, \lambda_n$  and assume  $\mathcal{C}$  contains the constraints  $[\lambda_i \geq 0], 1 \leq i \leq n$ . Let  $a$  be any integer (possibly negative). If  $[\lambda_i - a\lambda_j \geq 0]$  is implied by  $\mathcal{C}$ ,

$$F_{\mathcal{C}}(X_n) = F_{\mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}}(X_n; x_j \leftarrow x_j x_i^a).$$

4. Let  $c$  be any constraint with the same variables as the set  $\mathcal{C}$ . Then

$$F_{\mathcal{C}}(X_n) = F_{\mathcal{C} \cup \{c\}}(X_n) + F_{\mathcal{C} \cup \{-c\}}(X_n).$$

5. Let  $c \in \mathcal{C}$ . Then

$$F_{\mathcal{C}}(X_n) = F_{\mathcal{C} - \{c\}}(X_n) - F_{\mathcal{C} - \{c\} \cup \{-c\}}(X_n).$$

**Proof.**

1. This is clear since  $F_{\mathcal{C}}(x_1) = x_1^t + x_1^{t+1} + \dots$ .

2. The sequence  $(\lambda_1, \dots, \lambda_n) \in S_{\mathcal{C}_1 \cup \mathcal{C}_2}$  iff  $(\lambda_1, \dots, \lambda_j) \in S_{\mathcal{C}_1}$  and  $(\lambda_{j+1}, \dots, \lambda_n) \in S_{\mathcal{C}_2}$ .

3. Let  $\mathcal{C}' = \mathcal{C}_{\lambda_i \leftarrow \lambda_i + a\lambda_j}$ . By Lemma 1,

$$(\lambda_1, \dots, \lambda_n) \in S_{\mathcal{C}'} \quad \text{iff} \quad (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + a\lambda_j, \lambda_{i+1}, \dots, \lambda_n) \in S_{\mathcal{C}}.$$

So,

$$\begin{aligned} F_{\mathcal{C}'}(X_n; x_j \leftarrow x_j x_i^a) &= \sum_{\lambda \in S_{\mathcal{C}'}} x_1^{\lambda_1} x_2^{\lambda_2} \dots x_{j-1}^{\lambda_{j-1}} (x_j x_i^a)^{\lambda_j} x_{j+1}^{\lambda_{j+1}} \dots x_n^{\lambda_n} \\ &= \sum_{\lambda \in S_{\mathcal{C}'}} x_1^{\lambda_1} x_2^{\lambda_2} \dots x_{i-1}^{\lambda_{i-1}} x_i^{(\lambda_i + a\lambda_j)} x_{i+1}^{\lambda_{i+1}} \dots x_n^{\lambda_n} \\ &= \sum_{\lambda \in S_{\mathcal{C}}} x_1^{\lambda_1} x_2^{\lambda_2} \dots x_i^{\lambda_i} \dots x_n^{\lambda_n} \\ &= F_{\mathcal{C}}(X_n). \end{aligned}$$

4.  $S_{\mathcal{C}}$  can be partitioned into those  $\lambda$  that satisfy  $c$  and those that do not.

5. By guideline 4,  $F_{\mathcal{C} - \{c\}}(X_n) = F_{\mathcal{C} - \{c\} \cup \{c\}}(X_n) + F_{\mathcal{C} - \{c\} \cup \{-c\}}(X_n)$ . Then  $\mathcal{C} - \{c\} \cup \{c\} = \mathcal{C}$ , since  $c \in \mathcal{C}$ .  $\square$

### 3 Minc's Partition Function and Cayley Compositions

Minc's partition function  $\nu(d, N)$  is the number of compositions of  $N$  in which the first part is  $d$  and each part is at most twice the size of the preceding part [26]. For example, in the special case  $d = 1$ , these are called Cayley compositions [15, 1, 12]. In this section we compute the generating function  $\nu(q) = \sum_{d, N \geq 0} \nu(d, N)q^N = q + 2q^2 + 4q^3 + 7q^4 + 13q^5 + 24q^6 + \dots$ . For example, the coefficient of  $q^5$  is 13, since of the 16 compositions of 5, only these three violate the constraints: (1, 4), (1, 3, 1), and (1, 1, 3).

Let  $\mathcal{C}_n$  be the set of constraints  $\mathcal{C}_n = \{\lambda_i \geq \frac{1}{2}\lambda_{i+1} > 0 \mid 1 \leq i < n\}$  and let  $C_n(x_1, \dots, x_n)$  be the generating function of  $\mathcal{C}_n$ . Focusing on the constraint  $c = [\lambda_{n-1} \geq \frac{1}{2}\lambda_n]$ , after noting that  $[\lambda_{n-1} > 0]$  is redundant, we can write  $\mathcal{C}_n$  as

$$\mathcal{C}_n = \left[ \begin{array}{l} \lambda_1 \geq \frac{1}{2}\lambda_2 \\ \lambda_2 \geq \frac{1}{2}\lambda_3 \\ \vdots \\ \lambda_{n-2} \geq \frac{1}{2}\lambda_{n-1} \\ \lambda_{n-1} \geq \frac{1}{2}\lambda_n \\ \lambda_n > 0 \end{array} \right] = \left[ \begin{array}{l} \lambda_1 \geq \frac{1}{2}\lambda_2 \\ \lambda_2 \geq \frac{1}{2}\lambda_3 \\ \vdots \\ \lambda_{n-2} \geq \frac{1}{2}\lambda_{n-1} \\ \lambda_{n-1} \geq \frac{1}{2}\lambda_n \\ \lambda_{n-1} > 0 \\ \lambda_n > 0 \end{array} \right] = \left[ \begin{array}{l} \lambda_1 \geq \frac{1}{2}\lambda_2 \\ \lambda_2 \geq \frac{1}{2}\lambda_3 \\ \vdots \\ \lambda_{n-2} \geq \frac{1}{2}\lambda_{n-1} \\ \lambda_{n-1} > 0 \\ \lambda_n > 0 \end{array} \right] - \left[ \begin{array}{l} \lambda_1 \geq \frac{1}{2}\lambda_2 \\ \lambda_2 \geq \frac{1}{2}\lambda_3 \\ \vdots \\ \lambda_{n-2} \geq \frac{1}{2}\lambda_{n-1} \\ \lambda_n > 2\lambda_{n-1} \\ \lambda_{n-1} > 0 \end{array} \right],$$

where  $c$  has been removed from the next-to-last system, making it  $\mathcal{C}_{n-1} \cup [\lambda_n > 0]$ , and  $c$  has been replaced by  $-c$  in the last system. By guidelines 1 and 2,  $x_n C_{n-1}(x_1, \dots, x_{n-1}) / (1 - x_n)$  is the generating function for  $\mathcal{C}_{n-1} \cup [\lambda_n > 0]$ . Note further that the substitution  $\lambda_n \leftarrow \lambda_n + 2\lambda_{n-1}$  in the last system results  $\mathcal{C}_{n-1} \cup [\lambda_n > 0]$ , so by guideline 3, the last system has generating function  $x_n C_{n-1}(x_1, \dots, x_{n-1} x_n^2) / (1 - x_n)$ . Putting this together with guideline 5 and the initial condition  $C_1(x_1) = x_1 / (1 - x_1)$  gives the recurrence

$$C_n(x_1, \dots, x_n) = \frac{x_n}{1 - x_n} (C_{n-1}(x_1, \dots, x_{n-1}) - C_{n-1}(x_1, \dots, x_{n-2}, x_{n-1} x_n^2)).$$

Let  $C_n(q, s) = C_n(q, q, \dots, q, s)$ . Then the above recurrence gives  $C_1(q, s) = s / (1 - s)$  and for  $n \geq 2$ ,

$$C_n(q, s) = \frac{s}{1 - s} (C_{n-1}(q, q) - C_{n-1}(q, qs^2)).$$

Set  $C(q, s) = \sum_{n=1}^{\infty} C_n(q, s)$  and use the recurrence for  $C_n(q, s)$  to get

$$C(q, s) = \sum_{n=1}^{\infty} C_n(q, s) = \frac{s}{1 - s} + \sum_{n=2}^{\infty} C_n(q, s) = \frac{s}{1 - s} (1 + C(q, q) - C(q, qs^2)).$$

Iterating the recurrence for  $C(q, s)$  gives

$$C(q, s) = (1 + C(q, q)) \sum_{i=1}^{\infty} (-1)^{i-1} \prod_{j=0}^{i-1} \frac{q^{2^j-1} s^{2^j}}{(1 - q^{2^j-1} s^{2^j})}.$$

Let  $C(q) = C(q, q)$ , then

$$\nu(q) = 1 + C(q) = \frac{1}{1 + \sum_{i=1}^{\infty} \frac{(-1)^i q^{2^{i+1}-i-2}}{(1-q)(1-q^3)(1-q^7)\dots(1-q^{2^i-1})}}.$$

## 4 Two-Rowed Plane Partitions

This example illustrates the advantage of guideline 3 of Theorem 1 when  $a < 0$ . The *two-rowed plane partitions* are those integer sequences  $(a_1, b_1, \dots, a_n, b_n)$  satisfying the constraints

$$\mathcal{P}_n = [a_i \geq b_i \geq 0, \quad 1 \leq i \leq n; \quad a_i \geq a_{i+1}, \quad b_i \geq b_{i+1}, \quad 1 \leq i \leq n-1].$$

It is well-known that the generating function for  $\mathcal{P}_n$  is [24]

$$P_n(q) = \frac{1}{(q; q)_n (q^2; q)_n}. \quad (3)$$

In [3], Andrews shows how MacMahon's *partition analysis* can be used to compute  $P_n(q)$  by considering an intermediate family  $\mathcal{G}_n$ . We will use this approach, but with a slight twist, to show how the generating function for  $\mathcal{P}_n$ , can be computed via  $\mathcal{G}_n$  from the guidelines of Theorem 1.

We will use the convention that when a constraint system is represented by a calligraphic letter, its generating function is represented by the corresponding roman letter. Also, to keep notation simple, when the meaning is clear from context, we will use the same letter to refer to multivariable and single variable forms of the generating function.

Define  $\mathcal{G}_n$  to be the set of constraints below:

$$\mathcal{G}_n = \left[ \begin{array}{rcl} a_1 + a_2 + \dots + a_n & \geq & b_1 + b_2 + \dots + b_n \\ a_2 + \dots + a_n & \geq & b_2 + \dots + b_n \\ & \vdots & \vdots \\ a_{n-1} + a_n & \geq & b_{n-1} + b_n \\ a_n & \geq & b_n \\ a_i, b_i \geq 0, & & i = 1, \dots, n \end{array} \right].$$

Denote the full generating functions for  $\mathcal{P}_n$  and  $\mathcal{G}_n$  by

$$P_n(x_1, y_1, \dots, x_n, y_n) \triangleq \sum_{(a_1, b_1, \dots, a_n, b_n) \in S_{\mathcal{P}_n}} x_1^{a_1} y_1^{b_1} \dots x_n^{a_n} y_n^{b_n},$$

$$G_n(x_1, y_1, \dots, x_n, y_n) \triangleq \sum_{(a_1, b_1, \dots, a_n, b_n) \in S_{\mathcal{G}_n}} x_1^{a_1} y_1^{b_1} \dots x_n^{a_n} y_n^{b_n}.$$

Note that  $\mathcal{P}_n$  can be transformed into  $\mathcal{G}_n$  by the sequence of substitutions:

$$a_i \leftarrow a_i + a_{i+1}; \quad b_i \leftarrow b_i + b_{i+1}; \quad i = 1, 2, \dots, n-1.$$

We focus on  $G_n$ . Since for  $1 \leq i \leq n-1$ ,  $a_i - a_{i+1} \geq 0$  and  $b_i - b_{i+1} \geq 0$  in  $\mathcal{P}$ , by guideline 3 of Theorem 1,  $P_n$  is obtained from  $G_n$  by the sequence of substitutions:

$$x_i \leftarrow x_i x_{i-1}; \quad y_i \leftarrow y_i y_{i-1} \quad i = n, n-1, n-2, \dots, 2.$$

Thus

$$P_n(x_1, y_1, \dots, x_n, y_n) = G_n(x_1, y_1, x_1 x_2, y_1 y_2, \dots, x_1 x_2 \dots x_n, y_1 y_2 \dots y_n).$$

In particular, the generating function (3) for two-rowed plane partitions is obtained by setting  $x_i = y_i = q$  in  $P_n$  for  $i = 1, \dots, n$ :

$$P_n(q, q, q, \dots, q) = G_n(q, q, q^2, q^2, \dots, q^n, q^n). \quad (4)$$

Since  $a_n - b_n \geq 0$  in  $\mathcal{G}_n$ , by guideline 3, we can do the substitution  $a_n \leftarrow a_n + b_n$  in  $\mathcal{G}_n$  to get  $\mathcal{F}_n$  and recover  $G_n$  from  $F_n$  as shown below.

$$\mathcal{F}_n = \left[ \begin{array}{ccc} a_1 + a_2 + \dots + a_n & \geq & b_1 + b_2 + \dots + b_{n-1} \\ a_2 + \dots + a_n & \geq & b_2 + \dots + b_{n-1} \\ & \vdots & \vdots \\ a_{n-1} + a_n & \geq & b_{n-1} \\ a_i, b_i \geq 0, & & i = 1, \dots, n \end{array} \right],$$

$$G_n(x_1, y_1, \dots, x_n, y_n) = F_n(x_1, y_1, \dots, x_n, y_n; y_n \leftarrow x_n y_n).$$

Since  $a_{n-1} + a_n \geq 0$  in  $\mathcal{F}_n$ , by guideline 3, we can substitute  $a_{n-1} \leftarrow a_{n-1} - a_n$  in  $\mathcal{F}_n$  to get  $\mathcal{H}_n$  and recover  $F_n$  from  $H_n$  as shown.

$$\mathcal{H}_n = \left[ \begin{array}{ccc} a_1 + a_2 + \dots + a_{n-1} & \geq & b_1 + b_2 + \dots + b_{n-1} \\ a_2 + \dots + a_{n-1} & \geq & b_2 + \dots + b_{n-1} \\ & \vdots & \vdots \\ & a_{n-1} & \geq & b_{n-1} \\ & a_{n-1} & \geq & a_n \\ a_i, b_i \geq 0 & & i = 1, \dots, n \end{array} \right],$$

$$F_n(x_1, y_1, \dots, x_n, y_n) = H_n(x_1, y_1, \dots, x_n, y_n; x_n \leftarrow x_n/x_{n-1}).$$

Summarizing to this point, we have

$$G_n(x_1, y_1, \dots, x_n, y_n) = H_n(x_1, y_1, \dots, x_{n-1}, y_{n-1}, x_n/x_{n-1}, x_n y_n). \quad (5)$$

Now apply guideline 5 to  $\mathcal{H}_n$  using the constraint  $c = [a_{n-1} \geq a_n]$ . Then

$$\mathcal{H}_n = \mathcal{K}_n - \mathcal{L}_n,$$

where

$$\mathcal{K}_n = \mathcal{H}_n - \{[a_{n-1} \geq a_n]\}, \quad \mathcal{L}_n = \mathcal{H}_n - \{[a_{n-1} \geq a_n]\} \cup \{[a_n \geq a_{n-1} + 1]\},$$

that is,

$$\mathcal{K}_n = \left[ \begin{array}{ccc} a_1 + a_2 + \dots + a_{n-1} & \geq & b_1 + b_2 + \dots + b_{n-1} \\ a_2 + \dots + a_{n-1} & \geq & b_2 + \dots + b_{n-1} \\ & \vdots & \vdots \\ & a_{n-1} & \geq & b_{n-1} \\ a_i, b_i \geq 0 & & i = 1, \dots, n \end{array} \right]$$

and

$$\mathcal{L}_n = \left[ \begin{array}{ccc} a_1 + a_2 + \dots + a_{n-1} & \geq & b_1 + b_2 + \dots + b_{n-1} \\ a_2 + \dots + a_{n-1} & \geq & b_2 + \dots + b_{n-1} \\ & \vdots & \vdots \\ & a_{n-1} & \geq & b_{n-1} \\ & a_n & \geq & a_{n-1} + 1 \\ a_i, b_i \geq 0 & & i = 1, \dots, n \end{array} \right],$$

so that

$$H_n(x_1, y_1, \dots, x_n, y_n) = K_n(x_1, y_1, \dots, x_n, y_n) - L_n(x_1, y_1, \dots, x_n, y_n). \quad (6)$$

Now observe that

$$\mathcal{K}_n = \mathcal{G}_{n-1} \cup \{[a_n \geq 0], [b_n \geq 0]\},$$

so by guidelines 1 and 2,

$$K_n(x_1, y_1, \dots, x_n, y_n) = \frac{G_{n-1}(x_1, y_1, \dots, x_{n-1}, y_{n-1})}{(1-x_n)(1-y_n)}. \quad (7)$$

Returning to  $\mathcal{L}_n$ , since  $a_n - a_{n-1} \geq 0$  in  $\mathcal{L}_n$ , we can do the substitution  $a_n \leftarrow a_n + a_{n-1}$ , resulting in

$$(\mathcal{L}_n)_{a_n \leftarrow a_n + a_{n-1}} = \mathcal{G}_{n-1} \cup \{[a_n \geq 1], [b_n \geq 0]\},$$

so by guidelines 1, 2, and 3,

$$L_n(x_1, y_1, \dots, x_n, y_n) = \frac{x_n G_{n-1}(x_1, y_1, \dots, x_{n-1}, y_{n-1}; x_{n-1} \leftarrow x_{n-1} x_n)}{(1-x_n)(1-y_n)}. \quad (8)$$

Combining (6), (7), and (8), we have

$$H_n(x_1, y_1, \dots, x_n, y_n) = \frac{G_{n-1}(x_1, y_1, \dots, x_{n-1}, y_{n-1})}{(1-x_n)(1-y_n)} - \frac{x_n G_{n-1}(x_1, y_1, \dots, x_{n-2}, y_{n-2}, x_{n-1} x_n, y_{n-1})}{(1-x_n)(1-y_n)}.$$

Finally, substituting this expression for  $H_n$  into (5) gives a recurrence for  $G_n$ :

$$G_n(x_1, y_1, \dots, x_n, y_n) = \frac{G_{n-1}(x_1, y_1, \dots, x_{n-1}, y_{n-1}) - \frac{x_n}{x_{n-1}} G_{n-1}(x_1, y_1, \dots, x_{n-2}, y_{n-2}, x_n, y_{n-1})}{(1-x_n/x_{n-1})(1-x_n y_n)}, \quad (9)$$

with initial condition  $G_1(x_1, y_1) = 1/(1-x_1)/(1-x_1 y_1)$ .

Let  $G_n^*(q, s) = G_n(q, q, q^2, q^2, \dots, s, q^n)$ . Then from the recursion (9),

$$G_n^*(q, s) = \frac{G_{n-1}^*(q, q^{n-1}) - (s/q^{n-1}) G_{n-1}^*(q, s)}{(1-s/q^{n-1})(1-sq^n)}.$$

It is straightforward to show by induction that  $G_n^*(q, s)$  satisfies

$$G_n^*(q, s) = \frac{1}{(1-s)(1-sq)(q; q)_{n-1}(q^2; q)_{n-1}}.$$

Substituting  $s = q^n$  gives

$$P_n(q) = G_n(q, q, q^2, q^2, \dots, q^n, q^n) = G_n^*(q, q^n) = \frac{1}{(q; q)_n (q^2; q)_n},$$

the desired generating function for  $2 \times n$  plane partitions.



## 5 Anti-Lecture Hall Compositions

In [16], we considered the set of sequences  $\lambda = (\lambda_1, \dots, \lambda_n)$  satisfying the constraints

$$\mathcal{A}_n = \left[ \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \dots \geq \frac{\lambda_n}{n} \geq 0 \right].$$

We referred to these as *anti-lecture hall compositions* and showed that the generating function is

$$A_n(q) \triangleq \sum_{\lambda \in \mathcal{A}_n} q^{|\lambda|} = \prod_{i=1}^n \frac{1+q^i}{1-q^{i+1}}. \quad (10)$$

Here we show how to apply the guidelines of Theorem 1 to get a recurrence for the full generating function  $A_n(x_1, x_2, \dots, x_n)$  and use it to give an “easy” proof of (10). The idea is easily extended to the *truncated anti-lecture hall compositions* studied in [17]. We start with  $\mathcal{B}_n$ , a slight variation of  $\mathcal{A}_n$ .

**Lemma 2** *The full generating function for the integer sequences defined by the constraints*

$$\mathcal{B}_n = \left[ \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \dots \geq \frac{\lambda_{n-1}}{n-1} \geq \frac{\lambda_n}{1} \geq 0 \right]. \quad (11)$$

*satisfies*

$$B_n(x_1, \dots, x_n) = \frac{A_{n-1}(x_1, \dots, x_{n-1})}{1 - x_1 x_2^2 x_3^3 \cdots x_{n-1}^{n-1} x_n}.$$

**Proof.** The following sequence of substitutions transforms  $\mathcal{B}_n$  into  $\mathcal{A}_{n-1} \cup \{[\lambda_n \geq 0]\}$ , as illustrated in Figure 1:

$$\lambda_i \leftarrow \lambda_i + i\lambda_n, \quad i = n-1, \dots, 1.$$

Note that the constraint  $\lambda_{i-1} \geq (i-1)\lambda_n$  is implied at each stage, so by guidelines 1,2, and 3,  $B_n$  is recovered from  $A_n$  by performing the sequence of substitutions on  $A_{n-1}(x_1, \dots, x_{n-1})/(1-x_n)$ :

$$x_n \leftarrow x_n x_i^i, \quad i = 1, \dots, n-1.$$

□

**Proposition 1** *The full generating function for anti-lecture hall compositions satisfies:*

$$A_n(x_1, \dots, x_n) = \frac{A_{n-1}(x_1, \dots, x_{n-1})}{1-x_n} - A_{n-1}(x_1, \dots, x_{n-2}, x_n x_{n-1}) \left( \frac{1}{1-x_n} - \frac{1}{1-x_1 x_2^2 x_3^3 \cdots x_n^n} \right)$$

*with initial condition*  $A_1(x_1) = 1/(1-x_1)$ .

**Proof.** Using guideline 5 with  $c = [\lambda_{n-1} \geq \frac{n-1}{n}\lambda_n]$ ,

$$A_n(x_1, \dots, x_n) = C_n(x_1, \dots, x_n) - D_n(x_1, \dots, x_n),$$

$$\begin{array}{c}
\left[ \begin{array}{l} \lambda_1 \geq \frac{1}{2}\lambda_2 \\ \lambda_2 \geq \frac{2}{3}\lambda_3 \\ \vdots \\ \lambda_{n-3} \geq \frac{n-3}{n-2}\lambda_{n-2} \\ \lambda_{n-2} \geq \frac{n-2}{n-1}\lambda_{n-1} \\ \lambda_{n-1} \geq (n-1)\lambda_n \\ \lambda_n \geq 0 \end{array} \right] \rightarrow \left[ \begin{array}{l} \lambda_1 \geq \frac{1}{2}\lambda_2 \\ \lambda_2 \geq \frac{2}{3}\lambda_3 \\ \vdots \\ \lambda_{n-3} \geq \frac{n-3}{n-2}\lambda_{n-2} \\ \lambda_{n-2} \geq \frac{n-2}{n-1}\lambda_{n-1} + (n-2)\lambda_n \\ \lambda_{n-1} \geq 0 \\ \lambda_n \geq 0 \end{array} \right] \rightarrow \left[ \begin{array}{l} \lambda_1 \geq \frac{1}{2}\lambda_2 \\ \lambda_2 \geq \frac{2}{3}\lambda_3 \\ \vdots \\ \lambda_{n-3} \geq \frac{n-3}{n-2}\lambda_{n-2} + (n-3)\lambda_n \\ \lambda_{n-2} \geq \frac{n-2}{n-1}\lambda_{n-1} \\ \lambda_{n-1} \geq 0 \\ \lambda_n \geq 0 \end{array} \right] \rightarrow \\
\rightarrow \dots \rightarrow \left[ \begin{array}{l} \lambda_1 \geq \frac{1}{2}\lambda_2 \\ \lambda_2 \geq \frac{2}{3}\lambda_3 + 2\lambda_n \\ \vdots \\ \lambda_{n-3} \geq \frac{n-3}{n-2}\lambda_{n-2} \\ \lambda_{n-2} \geq \frac{n-2}{n-1}\lambda_{n-1} \\ \lambda_{n-1} \geq 0 \\ \lambda_n \geq 0 \end{array} \right] \rightarrow \left[ \begin{array}{l} \lambda_1 \geq \frac{1}{2}\lambda_2 + \lambda_n \\ \lambda_2 \geq \frac{2}{3}\lambda_3 \\ \vdots \\ \lambda_{n-3} \geq \frac{n-3}{n-2}\lambda_{n-2} \\ \lambda_{n-2} \geq \frac{n-2}{n-1}\lambda_{n-1} \\ \lambda_{n-1} \geq 0 \\ \lambda_n \geq 0 \end{array} \right] \rightarrow \left[ \begin{array}{l} \lambda_1 \geq \frac{1}{2}\lambda_2 \\ \lambda_2 \geq \frac{2}{3}\lambda_3 \\ \vdots \\ \lambda_{n-3} \geq \frac{n-3}{n-2}\lambda_{n-2} \\ \lambda_{n-2} \geq \frac{n-2}{n-1}\lambda_{n-1} \\ \lambda_{n-1} \geq 0 \\ \lambda_n \geq 0 \end{array} \right]
\end{array}$$

Figure 1: Transformation of  $\mathcal{B}_n$  into  $\mathcal{A}_{n-1} \cup \{\lambda_n \geq 0\}$  in proof of Lemma 2.

where

$$\mathcal{C}_n = \left[ \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \dots \geq \frac{\lambda_{n-1}}{n-1} \geq 0; \quad \lambda_n \geq 0 \right]; \quad (12)$$

$$\mathcal{D}_n = \left[ \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \dots \geq \frac{\lambda_{n-1}}{n-1} \geq 0; \quad \frac{\lambda_n}{n} > \frac{\lambda_{n-1}}{n-1} \right]. \quad (13)$$

Note that  $\mathcal{C}_n = \mathcal{A}_{n-1} \cup \{\lambda_n \geq 0\}$ , so by guideline 2,  $\mathcal{C}_n$  has generating function

$$C_n(x_1, \dots, x_n) = \frac{A_{n-1}(x_1, \dots, x_{n-1})}{1 - x_n}. \quad (14)$$

Since  $\lambda_n \geq \lambda_{n-1}$  is implied by  $\mathcal{D}_n$  in (13), by guideline 3, substituting  $\lambda_n \leftarrow \lambda_n + \lambda_{n-1}$  in  $\mathcal{D}_n$  gives

$$\mathcal{E}_n = \left[ \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \dots \geq \frac{\lambda_{n-1}}{n-1} \geq 0; \quad \lambda_n > \frac{\lambda_{n-1}}{n-1} \right] \quad (15)$$

and

$$D_n(x_1, \dots, x_n) = E_n(X_n; x_{n-1} \leftarrow x_{n-1}x_n),$$

where  $X_n$  represents the argument list  $x_1, \dots, x_n$ . Using guideline 5 again, with  $c = [\lambda_n > \frac{\lambda_{n-1}}{n-1}]$ , gives

$$E_n(X_n) = C_n(X_n) - B_n(X_n),$$

where  $\mathcal{C}_n$  is (12) and where  $\mathcal{B}_n$  is (11). Putting this all together, we have

$$\begin{aligned} A_n(X_n) &= C_n(X_n) - D_n(X_n) \\ &= C_n(X_n) - E_n(X_n; x_{n-1} \leftarrow x_{n-1}x_n) \\ &= C_n(X_n) - C_n(X_n; x_{n-1} \leftarrow x_{n-1}x_n) + B_n(X_n; x_{n-1} \leftarrow x_{n-1}x_n) \end{aligned}$$

Substituting from (14) and Lemma 2 gives the result.  $\square$

In order to make use of the recurrence of Proposition 1 to prove the generating function (10) for anti-lecture hall compositions, let  $A_n(q, s) \triangleq A_n(q, q, q, \dots, q, s)$ . Then the recurrence of Proposition 1 becomes

$$A_n(q, s) = \frac{A_{n-1}(q, q)}{1-s} - A_{n-1}(q, qs) \frac{s(1-s^{n-1}q^{\binom{n}{2}})}{(1-s)(1-s^nq^{\binom{n}{2}})}, \quad (16)$$

with initial condition  $A_0(q, s) = 1$ . If we were to proceed as with two-rowed plane partitions, we would (i) “guess” the form of  $A_n(q, s)$ , (ii) prove by induction that it satisfies (16), and then (iii) show that setting  $s = q$  gives (10). This would be the easiest proof and it would give a refinement of the anti-lecture hall generating function, enumerating solutions according to both the weight and the size of the last part:

$$\sum_{\lambda \in \mathcal{S}_{\mathcal{A}_n}} q^{|\lambda|} s^{\lambda_n} = A_n(q, qs).$$

Since we have *not* succeeded in guessing  $A_n(q, s)$ , we follow a different approach. Iterating the recurrence of (16) gives:

$$A_n(q, s) = \sum_{i=0}^{n-1} (-1)^i A_{n-1-i}(q, q) s^i q^{\binom{i}{2}} \frac{1-s^{n-i}q^{\binom{n}{2}-\binom{i}{2}}}{(s; q)_{i+1}(1-s^nq^{\binom{n}{2}})}. \quad (17)$$

Now, setting  $s = q$  gives a recurrence independent of  $s$ :

$$A_n(q, q) = \sum_{i=0}^{n-1} (-1)^i A_{n-1-i}(q, q) \frac{q^{\binom{i+1}{2}} - q^{\binom{n+1}{2}}}{(q; q)_{i+1}(1-q^{\binom{n+1}{2}})}. \quad (18)$$

We show by induction that the solution to (18) is

$$A_n(q, q) = \frac{(-q)_n}{(q^2)_n}.$$

Assume inductively that  $A_{n-1-i} = (-q)_{n-1-i}/(q^2)_{n-1-i}$ . Then we need to prove that

$$\frac{B_n(q) - q^{\binom{n+1}{2}} C_n(q)}{1 - q^{\binom{n+1}{2}}} = \frac{(-q)_n}{(q^2)_n};$$

with

$$C_n(q) = \sum_{i=0}^{n-1} (-1)^i \frac{(-q)_{n-1-i}}{(q^2)_{n-1-i}(q)_{i+1}}$$

and

$$B_n(q) = \sum_{i=0}^{n-1} (-1)^i q^{i(i+1)/2} \frac{(-q)_{n-1-i}}{(q^2)_{n-1-i}(q)_{i+1}}$$

We will prove that

$$B_{2n+1}(q) = \frac{(-q)_{2n+1}}{(q^2)_{2n+1}} \quad C_{2n+1}(q) = \frac{(-q)_{2n+1}}{(q^2)_{2n+1}}$$

$$B_{2n}(q) = \frac{(-q)_{2n}}{(q^2)_{2n}} - \frac{q^{\binom{2n+1}{2}}}{(q^2)_{2n}}; \quad C_{2n}(q) = \frac{(-q)_{2n}}{(q^2)_{2n}} - \frac{1}{(q^2)_{2n}}$$

Therefore, we need to prove the following identities for  $C_n$  :

$$\sum_{i=0}^{2n} (-1)^i \frac{(-q)_{2n-i}}{(q^2)_{2n-i}(q)_{i+1}} = \frac{(-q)_{2n+1}}{(q^2)_{2n+1}}. \quad (19)$$

$$\sum_{i=0}^{2n-1} (-1)^i \frac{(-q)_{2n-1-i}}{(q^2)_{2n-1-i}(q)_{i+1}} = \frac{(-q)_{2n}}{(q^2)_{2n}} - \frac{1}{(q^2)_{2n}}. \quad (20)$$

A few  $q$ -series manipulations show that the two previous equations are equivalent to:

$$\sum_{j=0}^n (-1)^j (-1; q)_j \left[ \begin{matrix} n \\ j \end{matrix} \right]_q = (-1)^n \quad (21)$$

Recalling that

$$\left[ \begin{matrix} n \\ j \end{matrix} \right]_q = \frac{(q^{-n})_j (-1)^j q^{nj-j(j-1)/2}}{(q)_j},$$

we see that the identity follows from the case  $a = -1, c \rightarrow \infty$  of  $q$ -Chu Vandermonde summation (1.5.2 in [21]),

$$\sum_{j=0}^n \frac{(a)_j (q^{-n})_j (cq^n/a)^j}{(c)_j (q)_j} = \frac{(c/a)_n}{(c)_n}. \quad (22)$$

Now we need

$$\sum_{i=0}^{2n} (-1)^i q^{\binom{i+1}{2}} \frac{(-q)_{2n-i}}{(q^2)_{2n-i}(q)_{i+1}} = \frac{(-q)_{2n+1}}{(q^2)_{2n+1}}. \quad (23)$$

$$\sum_{i=0}^{2n-1} (-1)^i q^{i(i+1)/2} \frac{(-q)_{2n-1-i}}{(q^2)_{2n-1-i}(q)_{i+1}} = \frac{(-q)_{2n}}{(q^2)_{2n}} + \frac{q^{\binom{2n+1}{2}}}{(q^2)_{2n}}. \quad (24)$$

The same  $q$ -series manipulations show that the two previous equations are equivalent to:

$$\sum_{j=0}^n (-1)^j (-1; q)_j \left[ \begin{matrix} n \\ j \end{matrix} \right]_q q^{\binom{n-j}{2}} = (-1)^n q^{\binom{n}{2}} \quad (25)$$

This follows in a similar way from the “other”  $q$ -Chu Vandermonde summation (1.5.3 in [21]),

$$\sum_{j=0}^n \frac{(a)_j (q^{-n})_j q^j}{(c)_j (q)_j} = \frac{a^n (c/a)_n}{(c)_n}, \quad (26)$$

under the substitutions  $a = -1, c = 0$ . □

## 6 Lecture Hall Partitions

In [13], Bousquet-Mélou and Eriksson studied the set of integer sequences  $\lambda = (\lambda_1, \dots, \lambda_n)$  satisfying the constraints

$$\mathcal{L}_n = \left[ \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \dots \geq \frac{\lambda_n}{1} \geq 0 \right].$$

They referred to these as *lecture hall partitions* and showed that the generating function is

$$L_n(q) \triangleq \sum_{\lambda \in S_{\mathcal{L}_n}} q^{|\lambda|} = \prod_{i=1}^n \frac{1}{1 - q^{2i-1}}. \quad (27)$$

In [2], Andrews showed how to use partition analysis to derive a recurrence for the full generating function of  $\mathcal{L}_n$ . However, substantial new ideas, outside of partition analysis, were required to move from this to the solution (27).

In this section, we show that by strategic application of Theorem 1, we can derive a recurrence for the full generating function of a generalization of  $\mathcal{L}_n$  that will reduce the proof of (27) to a  $q$ -series calculation (albeit nontrivial). Our derivation here via the five guidelines is both simpler and more elementary than the approach in [17] (at the expense of a more challenging  $q$ -series calculation).

In [17], we defined *truncated lecture hall partitions* to be the integer sequences satisfying:

$$\mathcal{L}_{n,k} = \left[ \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \dots \geq \frac{\lambda_k}{n-k+1} \geq 0 \right].$$

We showed that if

$$\bar{\mathcal{L}}_{n,k} = \left[ \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \dots \geq \frac{\lambda_k}{n-k+1} > 0 \right], \quad (28)$$

that is, all parts must be positive, the generating function is

$$\bar{L}_{n,k}(q) = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1}; q)_k}{(q^{2n-k+1}; q)_k}. \quad (29)$$

It can be checked that setting  $k = n$  and dividing by  $q^{\binom{n+1}{2}}$  gives (27).

**Proposition 2** *The generating function for truncated lecture hall partitions (28) satisfies*

$$\begin{aligned} \bar{L}_{n,k}(x_1, \dots, x_k) &= \frac{x_k \bar{L}_{n,k-1}(x_1, \dots, x_{k-1})}{1 - x_k} - \frac{\bar{L}_{n,k-1}(x_1, \dots, x_{k-2}, x_{k-1}x_k)}{1 - x_k} \\ &\quad - \frac{z_{n,k} \bar{L}_{n,k-1}(x_1, \dots, x_{k-2}, x_{k-1}x_k)}{1 - z_{n,k}}. \end{aligned}$$

with  $z_{n,k} = x_1^n x_2^{n-1} \dots x_k^{n-k+1}$ .

**Proof.** Note that  $\lambda_{k-1} > \lambda_k$  is implied by  $\bar{\mathcal{L}}_{n,k}$ , so by guideline 4,  $\bar{\mathcal{L}}_{n,k} = \bar{\mathcal{L}}_{n,k} \cup \{\lambda_{k-1} > \lambda_k\}$ .

Now apply guideline 5 with  $c = [\lambda_{k-1} \geq \frac{n-k+2}{n-k+1}\lambda_k]$  to get  $\bar{\mathcal{L}}_{n,k} = \mathcal{D} - \mathcal{E}$ :

$$\bar{\mathcal{L}}_{n,k} = \begin{bmatrix} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_{k-1} \geq \frac{n-k+2}{n-k+1}\lambda_k \\ \lambda_{k-1} > \lambda_k \\ \lambda_k > 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_{k-1} > \lambda_k \\ \lambda_k > 0 \end{bmatrix} - \begin{bmatrix} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_k > \frac{n-k+1}{n-k+2}\lambda_{k-1} \\ \lambda_{k-1} > \lambda_k \\ \lambda_k > 0 \end{bmatrix} \quad (30)$$

The first system on the right,  $\mathcal{D}$ , implies the constraint  $\lambda_{k-1} > 0$ , so it can be added. Now apply guideline 5 to  $\mathcal{D}$  using  $c = [\lambda_{k-1} > \lambda_k]$  to get:

$$\mathcal{D} = \begin{bmatrix} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_{k-1} > \lambda_k \\ \lambda_{k-1} > 0 \\ \lambda_k > 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_{k-1} > 0 \\ \lambda_k > 0 \end{bmatrix} - \begin{bmatrix} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_k \geq \lambda_{k-1} \\ \lambda_{k-1} > 0 \\ \lambda_k > 0 \end{bmatrix}. \quad (31)$$

The first system on the right of (31) is just  $\bar{L}_{n,k-1} \cup \{[\lambda_k > 0]\}$ . The second system on the right becomes  $\bar{L}_{n,k-1} \cup \{[\lambda_k \geq 0]\}$  after the substitution  $\lambda_k \leftarrow \lambda_k + \lambda_{k-1}$ . So, by Theorem 1 and summarizing so far, we have

$$\bar{L}_{n,k}(x_1, \dots, x_k) = \frac{x_k \bar{L}_{n,k-1}(x_1, \dots, x_{k-1})}{(1-x_k)} - \frac{\bar{L}_{n,k-1}(x_1, \dots, x_{k-1}x_k)}{(1-x_k)} - E(x_1, \dots, x_k), \quad (32)$$

where  $E(x_1, \dots, x_k)$  is the generating function for the last constraint system,  $\mathcal{E}$ , in (30). Apply  $\lambda_{k-1} \leftarrow \lambda_{k-1} + \lambda_k$  to  $\mathcal{E}$  followed by  $\lambda_k \leftarrow \lambda_k + (n-k+1)\lambda_{k-1}$  as illustrated below  $\mathcal{E} \rightarrow \mathcal{E}' \rightarrow \mathcal{F}$ :

$$\begin{bmatrix} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_{k-1} \\ \lambda_k > \frac{n-k+1}{n-k+2}\lambda_{k-1} \\ \lambda_{k-1} > \lambda_k \\ \lambda_k > 0 \end{bmatrix} \rightarrow \begin{bmatrix} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}(\lambda_{k-1} + \lambda_k) \\ \lambda_k > (n-k+1)\lambda_{k-1} \\ \lambda_{k-1} > 0 \\ \lambda_k > 0 \end{bmatrix} \rightarrow \begin{bmatrix} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_k + (n-k+3)\lambda_{k-1} \\ \lambda_k > 0 \\ \lambda_{k-1} > 0 \end{bmatrix}. \quad (33)$$

By guideline 3,

$$\begin{aligned} E(x_1, \dots, x_k) &= E'(x_1, \dots, x_{k-1}, x_{k-1}x_k), \\ E'(x_1, \dots, x_k) &= F(x_1, \dots, x_{k-2}, x_{k-1}x_k^{n-k+1}, x_k), \end{aligned}$$

so

$$E(x_1, \dots, x_k) = F(x_1, \dots, x_{k-2}, x_{k-1}^{n-k+2} x_k^{n-k+1}, x_{k-1}x_k). \quad (34)$$

Finally, starting from  $\mathcal{F}$ , the last set of constraints in (33), perform the following sequence of substitutions

$$\lambda_i \leftarrow \lambda_i + (n - i + 1)\lambda_{k-1}; \quad i = k - 2, \dots, 1,$$

as illustrated below:

$$\begin{aligned} \mathcal{F} \rightarrow & \left[ \begin{array}{l} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} + (n-k+4)\lambda_{k-1} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_k \\ \lambda_k > 0 \\ \lambda_{k-1} > 0 \end{array} \right] \rightarrow \dots \rightarrow \left[ \begin{array}{l} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 + (n-1)\lambda_{k-1} \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_k \\ \lambda_k > 0 \\ \lambda_{k-1} > 0 \end{array} \right] \\ & \rightarrow \left[ \begin{array}{l} \lambda_1 \geq \frac{n}{n-1}\lambda_2 + n\lambda_{k-1} \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_k \\ \lambda_k > 0 \\ \lambda_{k-1} > 0 \end{array} \right] \rightarrow \left[ \begin{array}{l} \lambda_1 \geq \frac{n}{n-1}\lambda_2 \\ \lambda_2 \geq \frac{n-1}{n-2}\lambda_3 \\ \vdots \\ \lambda_{k-3} \geq \frac{n-k+4}{n-k+3}\lambda_{k-2} \\ \lambda_{k-2} \geq \frac{n-k+3}{n-k+2}\lambda_k \\ \lambda_k > 0 \\ \lambda_{k-1} > 0 \end{array} \right] = \mathcal{G}. \quad (35) \end{aligned}$$

The resulting system of constraints,  $\mathcal{G}$ , in (35) can be viewed as  $\mathcal{L}_{n,k-1}$ , where  $\lambda_{k-1}$  has been replaced by  $\lambda_k$ , together with the constraint  $[\lambda_{k-1} > 0]$ . Thus,

$$G(x_1, \dots, x_k) = \frac{x_{k-1}L_{n,k-1}(x_1, \dots, x_{k-2}, x_k)}{(1 - x_{k-1})}. \quad (36)$$

By guideline 3, the generating function for  $\mathcal{F}$  is obtained from  $G$  by the sequence of substitutions

$$x_{k-1} \leftarrow x_{k-1}x_i^{n-i+1}; \quad i = 1 \dots k - 2,$$

giving

$$F(x_1, \dots, x_k) = G(x_1, \dots, x_{k-2}, x_1^n x_2^{n-1} \dots x_{k-2}^{n-k+3} x_{k-1}, x_k). \quad (37)$$

Returning to  $E$  in (34) and using (36) and (37),

$$\begin{aligned} E(x_1, \dots, x_k) &= F(x_1, \dots, x_{k-2}, x_{k-1}^{n-k+2} x_k^{n-k+1}, x_{k-1}x_k) \\ &= G(x_1, \dots, x_{k-2}, x_1^n x_2^{n-1} \dots x_{k-2}^{n-k+3} x_{k-1}^{n-k+2} x_k^{n-k+1}, x_{k-1}x_k) \\ &= \frac{x_1^n x_2^{n-1} \dots x_k^{n-k+1} L_{n,k-1}(x_1, \dots, x_{k-2}, x_{k-1}x_k)}{1 - x_1^n x_2^{n-1} \dots x_k^{n-k+1}}. \quad (38) \end{aligned}$$

Combining (38) with (32) gives the result.  $\square$

Let  $\bar{L}_{n,k}(q, s) = \bar{L}_{n,k}(q, q, \dots, q, s)$ . Setting  $x_k = s$  and  $x_i = q$  for  $i < k$  in Proposition 2 gives

$$\bar{L}_{n,k}(q, s) = \frac{s}{1-s} \bar{L}_{n,k-1}(q, q) - \bar{L}_{n,k-1}(q, sq) \left( \frac{1}{1-s} + \frac{z_{n,k}}{1-z_{n,k}} \right), \quad (39)$$

where  $z_{n,k} = s^{n-k+1} q^{\binom{n+1}{2} - \binom{n-k+2}{2}}$ . One would hope to prove (29) now by finding a closed form for  $\bar{L}_{n,k}(q, s)$ , proving that it satisfies the recurrence (39) and then setting  $s = q$  to get (29). Since we were unable to guess  $\bar{L}_{n,k}(q, s)$ , we proceed as for anti-lecture hall compositions to iterate the recurrence (39) and get

$$\bar{L}_{n,k}(q, s) = \sum_{j \geq 1} (-1)^{j-1} \frac{sq^{j-1}}{(s; q)_j} \cdot \frac{1 - s^{n-k+j} q^{(n-k+j)(j-2) + \binom{n+1}{2} - \binom{n-k+j}{2}}}{1 - s^{n-k+1} q^{\binom{n+1}{2} - \binom{n-k+2}{2}}} \cdot \bar{L}_{n,k-j}(q, q).$$

Now setting  $s = q$  we need only a single argument:

$$\bar{L}_{n,k}(q) = \sum_{j \geq 1} (-1)^{j-1} \frac{q^j}{(q; q)_j} \cdot \frac{1 - q^{k(n-k+j) + \binom{k-j+1}{2}}}{1 - q^{\binom{n+1}{2} - \binom{n-k+1}{2}}} \cdot \bar{L}_{n,k-j}(q).$$

It remains to prove that this recurrence is satisfied by (29). We defer the details until a later report; our main point was to show that strategic application of the guidelines reduce the truncated lecture hall theorem to a  $q$ -series computation.

## 7 The Five Guidelines Suffice

Let  $\mathcal{C}$  be the set of inequalities

$$c_{i,0} + c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \dots + c_{i,n}\lambda_n \geq 0, \quad 1 \leq i \leq r. \quad (40)$$

and let  $S_{\mathcal{C}}$  be the set of nonnegative integer sequences satisfying all constraints in  $\mathcal{C}$ . In this section we show that the five guidelines of Theorem 1 are powerful enough to find the generating function of  $S_{\mathcal{C}}$  for any integers  $c_{i,j}$ . We will assume that all constraints are *homogeneous*, i.e., that  $c_{i,0} = 0$ . Otherwise, introduce a new variable  $\lambda_0$  and let  $\mathcal{C}'$  be the same as  $\mathcal{C}$ , except that for every  $i$ , the  $i$ th constraint is now:

$$c_{i,0}\lambda_0 + c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \dots + c_{i,n}\lambda_n \geq 0.$$

Then  $F_{\mathcal{C}}(x_1, \dots, x_n)$  is the coefficient of  $x_0$  in  $F_{\mathcal{C}'}(x_0, x_1, \dots, x_n)$ . We also generalize the claim a bit to allow any of the constraints of  $\mathcal{C}$  to be equalities.

**Theorem 2** *The five guidelines of Theorem 1 are sufficient to find the full generating function for any homogeneous system of linear inequalities and equalities.*

**Proof.** Let  $\mathcal{C}$  be a homogeneous system of linear inequalities and equalities with variables  $\lambda_1, \dots, \lambda_n$ . Since we require nonnegative integer solutions, we can assume that for each variable  $\lambda_i$ ,  $\mathcal{C}$  contains a constraint  $b_i$  of the form  $[\lambda_i \geq 0]$  or  $[\lambda_i = 0]$ . Call these constraints  $b_i$  *basic*. Write  $\mathcal{C}$  in the form

$$\mathcal{C} = [c_1, c_2, \dots, c_r; b_1, b_2, \dots, b_n],$$



where  $c_1, c_2, \dots, c_r$  is an ordered list of the non-basic constraints in  $\mathcal{C}$ . If  $r = 0$ , all constraints are basic and the generating function follows from guidelines 1 and 2 (and  $F_{[\lambda_i=0]}(x_i) = 1$ ).

Otherwise, define:

- $M$ : the largest positive coefficient of  $c_1$  (0, if none);
- $e_{max}$ : the number of occurrences of  $M$  among the coefficients of  $c_1$ ;
- $m$ : the smallest negative coefficient of  $c_1$  (0, if none);
- $e_{min}$ : the number of occurrences of  $m$  among the coefficients of  $c_1$ .

When  $r > 0$  we show that we can use the guidelines to reduce the computation of the generating function of  $\mathcal{C}$  to the computation of the generating function of one or more systems  $\mathcal{C}'$  in which at least one of the statistics  $\{r, M, e_{max}, |m|, e_{min}\}$  has been reduced.

If  $m = 0$ , all coefficients of  $c_1$  are nonnegative, so  $c_1$  is redundant and can be deleted. Otherwise, if  $M = 0$ , all coefficients of  $c_1$  are nonpositive and so we get an equivalent system replacing  $\lambda_j$  by 0 in  $c_1, \dots, c_r$  and setting  $b_j = [\lambda_j = 0]$ . In so doing we have decreased  $|m|$  or  $e_{min}$ .

Otherwise,  $m < 0$  and  $M > 0$ ; we do a version of *Elliott reduction* [20]. Let  $i$  and  $j$  be such that  $m$  is the coefficient of  $\lambda_i$  in  $c_1$  and  $M$  is the coefficient of  $\lambda_j$ . We would like to use guideline 3 and reduce to a system with smaller  $M$  or  $e_{max}$  or  $|m|$  or  $e_{min}$ . First use guideline 4 with  $c = [\lambda_i \geq \lambda_j]$ :

$$F_{\mathcal{C}}(X_n) = F_{\mathcal{C} \cup [\lambda_i \geq \lambda_j]}(X_n) + F_{\mathcal{C} \cup [\lambda_j > \lambda_i]}(X_n).$$

For the first term,  $F_{\mathcal{C} \cup [\lambda_i \geq \lambda_j]}(X_n)$ , do the substitution  $\lambda_i \leftarrow \lambda_i + \lambda_j$  into constraints  $c_1, c_2, \dots, c_r$  in  $\mathcal{C}$ . This decreases the coefficient of  $\lambda_j$ , thereby decreasing  $M$  or  $e_{max}$ . By guideline 3, the substitution  $x_j \leftarrow x_j x_i$  in the generating function of the resulting constraint system gives  $F_{\mathcal{C} \cup [\lambda_i \geq \lambda_j]}(X_n)$ .

For the second term,  $F_{\mathcal{C} \cup [\lambda_j > \lambda_i]}(X_n)$ , if  $b_j = [\lambda_j = 0]$ , there are no solutions and the generating function is 0. Otherwise,  $b_j = [\lambda_j \geq 0]$ . Substitute  $\lambda_j \leftarrow \lambda_j + \lambda_i$  into constraints  $c_1, c_2, \dots, c_r$  in  $\mathcal{C}$  to get  $\mathcal{C}'$ . This increases the coefficient of  $\lambda_i$ , thereby decreasing  $|m|$  or  $e_{min}$ . Substituting  $\lambda_j \leftarrow \lambda_j + \lambda_i$  into  $[\lambda_j > \lambda_i]$  gives  $[\lambda_j > 0]$ . By guideline 3,

$$F_{\mathcal{C} \cup [\lambda_j > \lambda_i]}(X_n) = F_{\mathcal{C}' \cup [\lambda_j > 0]}(X_n; x_i \leftarrow x_i x_j).$$

However, we disallow strict inequalities. So, use guideline 5 with  $c = [\lambda_j > 0]$  and observe that  $b_j = [\lambda_j \geq 0] \in \mathcal{C}'$ . Let  $\mathcal{C}''$  denote  $\mathcal{C}'$  with  $b_j \leftarrow [\lambda_j = 0]$ . Then

$$F_{\mathcal{C}' \cup [\lambda_j > 0]}(X_n) = F_{\mathcal{C}'}(X_n) - F_{\mathcal{C}' \cup [\lambda_j \leq 0]}(X_n) = F_{\mathcal{C}'}(X_n) - F_{\mathcal{C}''}(X_n).$$

□

(Note that we have optimized the proof for simplicity at the expense of algorithmic efficiency.)

It follows from Theorem 2 and its proof that the full generating function of (40) can be built up from the functions  $1/(1-x_i)$  by a finite number of additions, subtractions, and substitutions. We get then as a corollary the following well-known result: The full generating function for the nonnegative integer solutions to any system of linear inequalities in  $n$  variables with integer coefficients has the form

$$\frac{p(x_1, \dots, x_n)}{(1-\alpha_1)(1-\alpha_2) \cdots (1-\alpha_t)},$$

where  $t \geq 0$ ,  $p$  is a polynomial in  $x_1, \dots, x_n$  and each  $\alpha_i$  is a monomial in  $x_1, \dots, x_n$ .

## 8 Relationship to MacMahon's Partition Analysis

We give a brief introduction to partition analysis in order to highlight the fact that the guidelines of Theorem 1 underlie the work of MacMahon. Indeed, they were distilled from partition analysis by a study of MacMahon's work in [25] and its application by Andrews, Paule and Riese in the series of papers [2, 3, 6, 4, 12, 7, 8, 9, 5, 10, 11].

Consider the set of constraints  $\mathcal{C} = \{c_1, \dots, c_r\}$  where

$$c_i = [a_{i,1}\lambda_1 + \dots + a_{i,n}\lambda_n \geq 0].$$

We seek the full generating function for the set  $S_{\mathcal{C}}$  of nonnegative integer sequences  $\lambda = (\lambda_1, \dots, \lambda_n)$  satisfying  $c_i, 1 \leq i \leq r$ :

$$F_{\mathcal{C}}(x_1, \dots, x_n) = \sum_{\lambda \in S_{\mathcal{C}}} x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}.$$

The method of *partition analysis*, developed by MacMahon in [25] is to view the problem as follows. Let

$$P(x_1, x_2, \dots, x_n, c_1, c_2, \dots, c_n) = \prod_{j=1}^n \frac{1}{1 - x_j c_1^{a_{1,j}} c_2^{a_{2,j}} \dots c_r^{a_{r,j}}}. \quad (41)$$

Expanding  $P$  gives

$$\begin{aligned} P(x_1, x_2, \dots, x_n, c_1, c_2, \dots, c_n) &= \sum_{\lambda_1, \dots, \lambda_n \geq 0} \prod_{j=1}^n (x_j c_1^{a_{1,j}} c_2^{a_{2,j}} \dots c_r^{a_{r,j}})^{\lambda_j} \\ &= \sum_{\lambda_1, \dots, \lambda_n \geq 0} \left( x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} \prod_{i=1}^r c_i^{a_{i,1}\lambda_1 + \dots + a_{i,n}\lambda_n} \right). \end{aligned} \quad (42)$$

Observe that  $\lambda \in S_{\mathcal{C}}$  iff in the term corresponding to  $\lambda$  in the sum (42) every  $c_i, 1 \leq i \leq r$ , has nonnegative exponent. Thus  $F_{\mathcal{C}}(x_1, \dots, x_n)$  is recovered from  $P(x_1, x_2, \dots, x_n, c_1, c_2, \dots, c_n)$  by deleting all terms in which some  $c_i$  has a negative exponent and then setting  $c_1 = c_2 = \dots = c_r = 1$ . MacMahon uses the *Omega operator* to express this process:

$$F_{\mathcal{C}}(x_1, \dots, x_n) = \underset{\geq}{\Omega} P(x_1, x_2, \dots, x_n, c_1, c_2, \dots, c_n).$$

The core of partition analysis is a system of *Omega*-rules designed to be applied strategically to transform  $P(x_1, x_2, \dots, x_n, c_1, c_2, \dots, c_n)$  step-by-step into  $F_{\mathcal{C}}(x_1, \dots, x_n)$ . This view converts the combinatorial problem into an algebraic one, opening the possibility, for example, of a partial fraction decomposition of (41) to assist in the transformation from  $P$  to  $F$ . A list of basic Omega-rules appears in [25](pp. 103-106) and [2].

This approach has proven both powerful and systematic in the computer solution of systems of inequalities. However, for deriving recurrences for infinite families, we found that a return to some of the basic underlying ideas simplified the process. We note the roots of guidelines 3-5 of Theorem 1 in the work of MacMahon [25].

Our use of guideline 3 (which performs limited column operations on the constraint matrix) is used to much the same effect as the following Omega-rule:

$$\underset{\geq}{\Omega} \frac{1}{(1 - x_i c)(1 - \frac{x_j}{c^a})} = \frac{1}{(1 - x_i)(1 - x_j x_i^a)}.$$

The utility of guideline 4 was recognized by MacMahon. He writes in [25], p. 103, “A very useful principle is that of adding an inequality which is afortiori true.” It is also used in a decomposition shown at the beginning of [25], Section 379, p. 131. Guideline 5 is one of MacMahon’s Omega-rules, found in [25], Section 351, p. 104 (slightly transformed):

$$\underset{\geq}{\Omega} P(c) = P(1) - \underset{\geq}{\Omega} P(1/c).$$

## 9 Concluding Remarks

The “five guidelines” of Theorem 1 provide a unified setting for computing the full generating function for many challenging families of constraints. However, even though they are guaranteed to be sufficient to find the generating function for any homogeneous linear system, we are not necessarily guaranteed to be able to use them to devise a *recurrence* for a parametrized family of constraint sets.

In continuing work we consider the case when all constraints have the form  $\lambda_i \geq \lambda_j$  or  $\lambda_i > \lambda_j$ , forming a directed graph. We show how to get a recurrence by strategically manipulating the diagrams. Many examples are presented, including two- and three-rowed plane partitions, plane partitions with diagonals, plane partition diamonds, and hexagonal plane partitions.

Finally, we note that in [32], Xin offers a speed-up to the Omega package for implementing MacMahon’s partition analysis. Xin’s method uses the theory of iterated Laurent series and partial fraction decompositions.

**Acknowledgement.** We are grateful to the referee for a careful reading of the manuscript and detailed suggestions to improve the presentation.

## References

- [1] George E. Andrews. The Rogers-Ramanujan reciprocal and Minc’s partition function. *Pacific J. Math.*, 95(2):251–256, 1981.
- [2] George E. Andrews. MacMahon’s partition analysis. I. The lecture hall partition theorem. In *Mathematical essays in honor of Gian-Carlo Rota (Cambridge, MA, 1996)*, volume 161 of *Progr. Math.*, pages 1–22. Birkhäuser Boston, Boston, MA, 1998.
- [3] George E. Andrews. MacMahon’s partition analysis. II. Fundamental theorems. *Ann. Comb.*, 4(3-4):327–338, 2000. Conference on Combinatorics and Physics (Los Alamos, NM, 1998).
- [4] George E. Andrews and Peter Paule. MacMahon’s partition analysis. IV. Hypergeometric multisums. *Sém. Lothar. Combin.*, 42:Art. B42i, 24 pp. (electronic), 1999. The Andrews Festschrift (Maratea, 1998).
- [5] George E. Andrews, Peter Paule, and Axel Riese. MacMahon’s partition analysis. IX.  $k$ -gon partitions. *Bull. Austral. Math. Soc.*, 64(2):321–329, 2001.
- [6] George E. Andrews, Peter Paule, and Axel Riese. MacMahon’s partition analysis: the Omega package. *European J. Combin.*, 22(7):887–904, 2001.
- [7] George E. Andrews, Peter Paule, and Axel Riese. MacMahon’s partition analysis. VI. A new reduction algorithm. *Ann. Comb.*, 5(3-4):251–270, 2001. Dedicated to the memory of Gian-Carlo Rota (Tianjin, 1999).

- [8] George E. Andrews, Peter Paule, and Axel Riese. MacMahon's partition analysis. VII. Constrained compositions. In *q-series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000)*, volume 291 of *Contemp. Math.*, pages 11–27. Amer. Math. Soc., Providence, RI, 2001.
- [9] George E. Andrews, Peter Paule, and Axel Riese. MacMahon's partition analysis. VIII. Plane partition diamonds. *Adv. in Appl. Math.*, 27(2-3):231–242, 2001. Special issue in honor of Dominique Foata's 65th birthday (Philadelphia, PA, 2000).
- [10] George E. Andrews, Peter Paule, and Axel Riese. MacMahon's partition analysis X: Plane partitions with diagonals. 2004. SFB Report n. 2004-2, J. Kepler University, Linz.
- [11] George E. Andrews, Peter Paule, and Axel Riese. MacMahon's partition analysis XI: Hexagonal plane partitions. 2004. SFB Report n. 2004-4, J. Kepler University, Linz.
- [12] George E. Andrews, Peter Paule, Axel Riese, and Volker Strehl. MacMahon's partition analysis. V. Bijections, recursions, and magic squares. In *Algebraic combinatorics and applications (Gößweinstein, 1999)*, pages 1–39. Springer, Berlin, 2001.
- [13] Mireille Bousquet-Mélou and Kimmo Eriksson. Lecture hall partitions. *Ramanujan J.*, 1(1):101–111, 1997.
- [14] Rod Canfield, Sylvie Corteel, and Pawel Hitczenko. Random partitions with non negative  $r^{\text{th}}$  differences. *Adv. Applied Maths*, 27:298–317, 2001.
- [15] A. Cayley. On a problem in the partition of numbers. *Philosophical Magazine*, 13:245–248, 1857. reprinted in: The Collected Mathematical Papers of A. Cayley, Vol. III, Cambridge University Press, Cambridge, 1890, 247-249).
- [16] Sylvie Corteel and Carla D. Savage. Anti-lecture hall compositions. *Discrete Math.*, 263(1-3):275–280, 2003.
- [17] Sylvie Corteel and Carla D. Savage. Lecture hall theorems,  $q$ -series and truncated objects. *J. Combin. Theory Ser. A*, 108(2):217–245, 2004.
- [18] Sylvie Corteel and Carla D. Savage. Partitions and compositions defined by inequalities. *Ramanujan J.*, 8(3):357–381, 2004.
- [19] Sylvie Corteel, Carla D. Savage, and Herbert S. Wilf. A note on partitions and compositions defined by inequalities. *Integers*, 5(1):A24, 11 pp. (electronic), 2005.
- [20] E. B. Elliott. On linear homogeneous diophantine equations. *Quarterly Journal of Pure and Applied Mathematics*, 34:348–377, 1903.
- [21] George Gasper and Mizan Rahman. *Basic hypergeometric series*, volume 35 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1990. With a foreword by Richard Askey.
- [22] D. R. Hickerson. A partition identity of the Euler type. *Amer. Math. Monthly*, 81:627–629, 1974.
- [23] Arnold Knopfmacher and Helmut Prodinger. On Carlitz compositions. *European J. Combin.*, 19(5):579–589, 1998.
- [24] Percy A. MacMahon. Memoir on the theory of the partitions of numbers VI - Partitions in two-dimensional space. *Phil. Trans. Roy. Soc. London Ser. A*, 211:345–373, 1912. Reprinted in *Percy Alexander MacMahon: Collected Papers*. ed. George E. Andrews. Vol. 1, pp. 1404-1434. MIT Press, Cambridge, Mass., 1978.
- [25] Percy A. MacMahon. *Combinatory analysis, vol. 2*. Cambridge University Press, Cambridge, 1916. Reprinted: Chelsea Publishing Co., New York, 1960.
- [26] H. Minc. A problem in partitions: Enumeration of elements of a given degree in the free commutative entropic cyclic groupoid. *Proc. Edinburgh Math. Soc. (2)*, 11:223–224, 1958/1959.
- [27] Igor Pak. Partition identities and geometric bijections. *Proc. Amer. Math. Soc.*, 132(12):3457–3462 (electronic), 2004.

- [28] Jose Plínio O. Santos. On a new combinatorial interpretation for a theorem of Euler. *Adv. Stud. Contemp. Math. (Pusan)*, 3(2):31–38, 2001.
- [29] James A. Sellers. Extending a recent result of Santos on partitions into odd parts. *Integers*, 3:A4, 5 pp. (electronic), 2003.
- [30] James A. Sellers. Corrigendum to: “Extending a recent result of Santos on partitions into odd parts” [*Integers* **3** (2003), A4, 5 pp. (electronic)]. *Integers*, 4:A8, 1 pp. (electronic), 2004.
- [31] Jan Snellman and Michael Paulsen. Enumeration of concave integer partitions. *J. Integer Seq.*, 7(1):Article 04.1.3, 10 pp. (electronic), 2004.
- [32] Guoce Xin. A fast algorithm for MacMahon’s partition analysis. *Electron. J. Combin.*, 11(1):Research Paper 58, 20 pp. (electronic), 2004.
- [33] Doron Zeilberger. Sylvie Corteel’s one line proof of a partition theorem generated by Andrews-Paule-Riese’s computer. *Shalosh B. Ekhad’s and Doron Zeilberger’s Very Own Journal*, 1998.