On multi-color partitions and the generalized Rogers-Ramanujan identities

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Abstract

Basil Gordon, in the sixties, and George Andrews, in the seventies, generalized the Rogers-Ramanujan identities to higher moduli. These identities arise in many areas of mathematics and mathematical physics. One of these areas is representation theory of infinite dimensional Lie algebras, where various known interpretations of these identities have led to interesting applications. Motivated by their connections with Lie algebra representation theory, we give a new interpretation of a sum related to generalized Rogers-Ramanujan identities in terms of multi-color partitions.

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1 Introduction

The celebrated Rogers-Ramanujan identities and their generalizations (see [G], [An1]) have influenced current research in many areas of mathematics and physics (see [An2, BeM2]). These identities can be expressed as:

$$\prod_{n\not\equiv 0,\pm 2(\bmod 5)} (1-q^n)^{-1} = \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)} \tag{1}$$

and

$$\prod_{n\not\equiv 0,\pm 1(\bmod 5)} (1-q^n)^{-1} = \sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\dots(1-q^n)}. \tag{2}$$

Identities (1) and (2) have a natural combinatorial interpretation in terms of partitions, which was generalized by Gordon ([G], [An1], Theorem 7.5). A partition of a positive integer n is a finite, non-increasing sequence of positive integers, called parts, whose sum is n.

Theorem 1 [Gordon] *For $M = 2k+1$ and $0 < r \leq k$, the number of partitions of n of the form $(\pi_1, \pi_2, \dots, \pi_l)$, where $\pi_j - \pi_{j+k-1} \geq 2$ and at most $r-1$ of the parts are 1 is equal to the number of partitions of n into parts not congruent to 0, r, or $-r$ modulo M.*

Setting $r = 2$ and $M = 5$ in Theorem 1 gives (1) and $r = 1$, $M = 5$ gives (2).

About twenty years ago it was observed that these identities play an important role in the representation theory of affine Lie algebras via its principal characters [LM]. In 1978 Lepowsky and Wilson [LW1] gave the first explicit realization of the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$. This led to a new algebraic structure called the Z-algebra [LW2] which gave a formal foundation to study systematically the connection between affine Lie algebras and combinatorial identities. In particular, Lepowsky and Wilson [LW2, LW3, LW4] used the Z-algebra structure to construct the integrable highest weight representations of the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$ and gave a Z-algebraic proof of the Rogers-Ramanujan identities and Z-algebraic interpretation of the generalized Rogers-Ramanujan identities.

One can also use (generalized) Rogers-Ramanujan identities to construct explicitly integrable representations of other affine Lie algebras. See for example [BM], [M1], [M2], [Ma], [X] for these developments. In this connection the Z-operators still play an important role. These operators act on a certain space $\Omega(V)$ called the vacuum space associated with the representation space V [LW3] and are parameterized by the set of roots of the associated simple Lie algebra $\mathfrak{g}$. However, on V many of these Z-operators are scalar multiples of each other. For example, let us consider the affine Lie algebra $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(5)$ and its integrable highest weight representation $V(\lambda)$, with highest weight $\lambda = \Lambda_0 + \Lambda_2$, where Λ_i are the fundamental weights of the Lie algebra $\hat{\mathfrak{g}}$. The principal character of $V(\lambda)$ is

$$\chi(V(\lambda)) \;=\; F \prod_{n\geq 1, n\not\equiv 0,\pm 3\,(\bmod 7)} (1-q^n)^{-1},$$

where $F = \prod_{n\geq 1, n\not\equiv 0 \bmod 5}(1-q^n)^{-1}$. In this case there are two independent families of Z-operators:

$$Z(\beta,z) = \sum_{i\in\mathbb{Z}} Z(\beta,i)z^{-i}, \qquad Z(\beta,i)\in \mathrm{End}V(\lambda)$$

for $\beta=\beta_1$ and $\beta_1+\beta_2$, where $\{\beta_1,\beta_2,\beta_3,\beta_4\}$ are the simple roots of $\mathfrak{sl}(5)$ corresponding to the principal Cartan subalgebra $\mathfrak{a}$ (see [M3]). In [M3] $V(\Lambda_0+\Lambda_2)$ has been constructed using only one set of Z-operators $Z(\beta_1,z)$ and Gordon's generalization of the Rogers-Ramanujan identities with $r=3$ and $M=7$:

$$\prod_{n\not\equiv 0,\pm 3\,(\bmod 7)} (1-q^n)^{-1} \;=\; \sum_{n\geq 0} a(n)q^n, \tag{3}$$

where $a(n)$ denotes the number of partitions of n such that the outer two of any three consecutive parts differ by at least 2 and at most two parts are 1. However, from the representation theory point of view it would be more natural to construct the representation using both families of operators $Z(\beta_1,z)$ and $Z(\beta_1+\beta_2,z)$. It is expected that this would correspond to another expansion of the left-hand side of (3), namely

$$\prod_{n\not\equiv 0,\pm 3(\bmod 7)} (1-q^n)^{-1} \;=\; \sum_{n_1\geq n_2\geq 0} \frac{q^{n_1^2+n_2^2}}{(q)_{n_1-n_2}(q)_{n_2}}, \tag{4}$$

where we let $(a)_n = \prod_{k=0}^{n-1}(1-aq^k)$. The expansion (4) is a special case of Andrews' and Bressoud's analytic generalization of the Rogers-Ramanujan identities:

$$\prod_{n\not\equiv 0,\pm r(\bmod 2k+s)} (1-q^n)^{-1} \;=\; \sum_{n_1\geq\ldots\geq n_{k-1}\geq 0} \frac{q^{n_1^2+n_2^2+\ldots+n_{k-1}^2+n_r+n_{r+1}+\ldots+n_{k-1}}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q^{2-s};q^{2-s})_{n_{k-1}}}, \tag{5}$$

where $s=0,1$. Andrews ([An1], Theorem 7.8) derived this generalization for the case of odd modulus ($s=1$) and Bressoud [Br1] for the case of even modulus ($s=0$). In [FS] Feigin and Stoyanovsky used the representation of [LP] in the homogeneous gradation to give certain combinatorial interpretations of the multisum side of the generalized Rogers-Ramanujan identities. Later Georgiev [Ge] and also Meurman and Primc ([MP1], [MP2], [MP3] and [P]) related the sum sides of various generalized Rogers-Ramanujan type expressions to multi-color partitions by attaching colors to different roots in Z-algebra type constructions of the homogeneous irreducible highest weight representations of certain affine Lie algebras. It is clear from this work that the language of multi-color partitions are suitable for Z-algebraic constructions and interpretations of Rogers-Ramanujan type identities.

It is well-known that the product side of (5) can also be written as

$$\prod_{n\not\equiv 0,\pm r(\bmod M)} (1-q^n)^{-1} = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{j[(M)j+M-2r]/2} \tag{6}$$

a result which follows from the Jacobi Triple Product Identity. In this paper, we provide a combinatorial description of the sum side of (6) in terms of multi-color partitions with the hope that this will give new insights into the Z-operator constructions in the principal gradation. Our work builds on an interpretation of the sum side of (6) in terms of partitions with bounds on successive ranks due to Andrews [An3] and Bressoud [Br2]. Andrews and Bressoud showed that the sum side of (6) is the generating function for $|A_n(M,r)|$, where $A_n(M,r)$ is the set of all partitions of n whose successive ranks lie in the interval $[-r+2, M-r-2]$ (to be discussed in more detail in Section 2). Our main theorem establishes a bijection between $A_n(M,r)$ and a family of multi-color partitions as described below.

Definition 1 *For $t \geq 1$, a t-*color partition *of n is a pair (α, c_α) where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ is a partition of n and c_α is a function which assigns to each $i \in \{1, 2, \ldots, l\}$ one of the colors $\{1, 2, \ldots, t\}$ so that if $\alpha_i = \alpha_{i+1}$ then $c_\alpha(i) \leq c_\alpha(i+1)$. We say that $c_\alpha(i)$ is the* color *of the i-th part of α.*

For example, $(8_2, 8_3, 5_1, 4_1, 4_2, 4_3, 3_2, 2_1)$ is a 3-color partition of 38, where the subscript of a part denotes its color.

Our main theorem is stated below and proved in Section 2 as Theorem 3.

Main Theorem *For integers r, M, and k satisfying $0 < r \leq M/2$ and $k = \lfloor M/2 \rfloor$, let $C_n(M,r)$ be the set of $k-1$-color partitions (α, c_α) of n satisfying the following three conditions. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$.*

(i) (Initial Conditions) *For $1 \leq i \leq l$,*

$$\alpha_i > \begin{cases} |2c_\alpha(i) - r + 1| & \text{if } \alpha_i \equiv r (\mathrm{mod} 2) \\ |2c_\alpha(i) - r| & \text{otherwise} \end{cases} .$$

(ii) (Color Difference Conditions) *For $1 \leq i < l$,*

$$\alpha_i - \alpha_{i+1} \geq \begin{cases} 2 + |2(c_\alpha(i) - c_\alpha(i+1))| & \text{if } \alpha_i \equiv \alpha_{i+1} (\mathrm{mod} 2) \\ 2 + |2(c_\alpha(i) - c_\alpha(i+1)) - 1| & \text{if } \alpha_i \not\equiv \alpha_{i+1} \equiv r (\mathrm{mod} 2) \\ 2 + |2(c_\alpha(i) - c_\alpha(i+1)) + 1| & \text{if } \alpha_{i+1} \not\equiv \alpha_i \equiv r (\mathrm{mod} 2) \end{cases} .$$

(iii) (Parity Condition on Last Color when M is Even) *For $1 \leq i \leq l$, if M is even and $c_\alpha(i) = k-1$, the last color, then*

$$\alpha_i \not\equiv r \pmod 2.$$

Then $|A_n(M,r)| = |C_n(M,r)|$, *and in particular,*

$$\sum_{n=0}^{\infty} |C_n(M,r)| q^n \quad = \sum_{n=0}^{\infty} |A_n(M,r)| q^n \quad = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{j[(M)j+M-2r]/2}$$

Our results suggest several interesting problems. Our motivating problem is to make use of the multi-color partition interpretation to construct natural realizations of integrable representations of affine Lie algebras, via a correspondence between the $k-1$ colors and the parameters $n_1, \ldots, n_{k-1}$ of (5). However, another intriguing problem is to establish a direct bijection between the multi-color partitions and the partitions counted by the sum side of the Rogers-Ramanujan identity (5), possibly through $A_n(M,r)$. We note some related work for the sum side. In [An4], Andrews gave a combinatorial interpretation of the sum side of (5) in terms of Durfee dissection partitions. Another result, due to Burge [Bu1, Bu2] and formulated in terms of lattice paths by Bressoud [Br3], interprets the sum side of (5) as the number of lattice paths of weight n starting at $(0, k-r)$ which have no peak of height k or greater. (Steps allowed in the lattice path are: $(x,y) \to (x+1, y+1)$; $(x,0) \to (x+1,0)$; and, if $y > 0$, $(x,y) \to (x+1, y-1)$. The weight of a lattice path is the sum of the x-coordinates of its peaks.)

Although we have never seen a multi-color interpretation of (5), together with its conditions, made explicit, multi-color interpretations of other identities of the Rogers-Ramanujan type appear, for example, in [AAB], [AAG], [AA], [AB], [Ge], [MP1], [MP2], [MP3] and [P]. In Section 3 we note that ideas implicit in the papers [AA] and [AB] give rise to an alternative multi-color interpretation of (5).

In Section 4, we give an example of a context in which the multi-color interpretation seems quite natural. In [FQ], Foda and Quano derive a finitization of a form of the generalized Rogers-Ramanujan identities. We show that the corresponding refinement for our multi-color partitions is simply an additional constraint on the size of the largest part.

2 Multi-color partitions

In this section, we give a new combinatorial interpretation of the product side of (5). Our main tool will be a combinatorial generalization of the Rogers-Ramanujan identities, due to George Andrews, which involves the *successive ranks* of a partition.

A partition $\pi = (\pi_1, \pi_2, \ldots, \pi_l)$ can be visualized by its Ferrers diagram, an array of dots, where π_i is the number of left justified dots in the ith row. The largest square subarray of dots in this diagram is the Durfee square and the Durfee square size, denoted

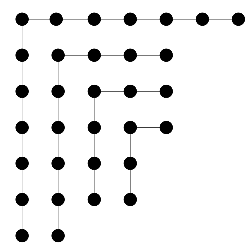

Figure 1: The Ferrers diagram of $\pi = (7, 5, 5, 5, 4, 4, 2)$ with successive ranks $0, -2, -1, -1$, and the four angles indicated, illustrating that $\alpha(\pi) = (13, 9, 6, 4)$.

by $d(\pi)$, is the length of a side. Flipping the diagram along its main diagonal, one obtains the dual diagram, associated with its dual partition $\pi' = (\pi'_1, \pi'_2, \ldots, \pi'_{\pi_1})$, where π'_i is the number of indices j with $\pi_j \geq i$. The sequence of successive ranks of π is the sequence $(\pi_1 - \pi'_1, \pi_2 - \pi'_2, \ldots, \pi_d - \pi'_d)$, where $d = d(\pi)$.

Let π be a partition of n with successive ranks $r_1, r_2, \ldots, r_d$. For $1 \leq i \leq d = d(\pi)$, let

$$\alpha_i = \pi_i + \pi'_i - 2(i-1) - 1 = \pi_i + \pi'_i - 2i + 1,$$

i.e., α_i is the number of dots on the i-th "angle" of π. Denote by $\alpha(\pi)$ the partition with parts: α_1, α_2, $\cdots$, α_d. Note that $\alpha(\pi)$ is a partition of n such that

$$\alpha_i \equiv r_i + 1 \pmod 2. \tag{7}$$

(See Figure 1.)

Lemma 1 *For* $1 \leq i \leq d(\pi)$, $\alpha_i > |r_i|$.

Proof. Since $i \leq d(\pi)$, we have $\pi_i \geq i$ and $\pi'_i \geq i$, so

$$\alpha_i \;=\; \pi_i + \pi'_i - 2i + 1 \;=\; r_i + 2\pi'_i - 2i + 1 \;\geq\; r_i + 2i - 2i + 1 \;=\; r_i + 1$$

and

$$\alpha_i \;=\; \pi_i + \pi'_i - 2i + 1 \;=\; -r_i + 2\pi_i - 2i + 1 \;\geq\; -r_i + 2i - 2i + 1 \;=\; -r_i + 1$$

□

Lemma 2 *For* $1 \leq i < d(\pi)$,

$$\alpha_i - \alpha_{i+1} \;\geq\; 2 + |r_i - r_{i+1}|.$$

Proof. If $r_i \geq r_{i+1}$, then since $\pi'_i \geq \pi'_{i+1}$,

$$\pi_i - \pi_{i+1} \;\geq\; \pi_i - \pi_{i+1} - (\pi'_i - \pi'_{i+1}) \;=\; r_i - r_{i+1}.$$

Thus

$$\alpha_i - \alpha_{i+1} \;= \pi_i - \pi_{i+1} + \pi'_i - \pi'_{i+1} + 2 \;\geq\; r_i - r_{i+1} + 2.$$

Similarly, If $r_i \leq r_{i+1}$, then since $\pi_i \geq \pi_{i+1}$,

$$\pi'_i - \pi'_{i+1} \;\geq\; \pi'_i - \pi'_{i+1} - (\pi_i - \pi_{i+1}) \;=\; -r_i + r_{i+1}.$$

Thus

$$\alpha_i - \alpha_{i+1} \;= \pi'_i - \pi'_{i+1} + \pi_i - \pi_{i+1} + 2 \;\geq\; -r_i + r_{i+1} + 2.$$

□

Definition 2 *Call a partition* type 1 *if successive parts differ by at least 2.*

Corollary 1 *For any partition π, $\alpha(\pi)$ is a type 1 partition.*

Proof. By Lemma 2, $\alpha_i - \alpha_{i+1} \geq 2 + |r_i - r_{i+1}| \geq 2$. □

Remark 1 *The number of partitions of n with all successive ranks in $[0, 1]$ is equal to the number of type 1 partitions of n. The map $\pi \to \alpha(\pi)$ is a bijection.*

Remark 2 *The number of partitions of n with all successive ranks in $[1, 2]$ is equal to the number of type 1 partitions of n in which every part is larger than 1. The map $\pi \to \alpha(\pi)$ is again a bijection.*

We can now state Andrews' generalization of the Rogers-Ramanujan identities. The theorem below was established by Andrews for odd moduli M [An3] and was generalized to even moduli by Bressoud [Br2].

Theorem 2 [Andrews-Bressoud] *For integers M, r, satisfying $0 < r \leq M/2$, let $A_n(M, r)$ be the set of partitions of n with all successive ranks in the interval $[-r + 2, M - r - 2]$. Then $|A_n(M, r)|$ is equal to the number of partitions of n with no part congruent to 0, r, or $-r$ modulo M.*

The smallest M in the above theorem is 4, as when $M = 3$ and $r = 1$, the interval $[-r+2, M-r-2]$ and the set $A_n(M, r)$ are both empty. To see that Theorem 2 generalizes the Rogers-Ramanujan identities, note that when $r = 2$ and $M = 5$, the theorem says that the number of partitions of n with all ranks in $[0, 1]$ is equal to the number of partitions of n using no part congruent to 0, 2 or 3 modulo 5. By Remark 1 and Theorem 1, this is the first Rogers-Ramanujan identity (1). When $r = 1, M = 5$, the Andrews-Bressoud theorem says that the number of partitions of n with all ranks in $[1, 2]$ is equal to the number of partitions of n using no part congruent to 0, 1 or 4 modulo 5. By Remark 2 and Theorem 1, this is the second Rogers-Ramanujan identity (2).

In fact, what Andrews and Bressoud prove is:

$$\sum_{n=0}^{\infty} |A_n(M, r)| q^n = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{j[(M)j+M-2r]/2} = \prod_{n \not\equiv 0, \pm r (\bmod M)} (1 - q^n)^{-1} \quad (8)$$

where the first equality (the hard part) follows by a sieve argument and the second equality by application of the Jacobi Triple Product identity.

We now show that the partitions defined by Andrews' rank conditions are equivalent to certain classes of multi-color partitions.

Theorem 3 *For integers r, M, and k satisfying $0 < r \le M/2$ and $k = \lfloor M/2 \rfloor$, let $C_n(M, r)$ be the set of $k-1$-color partitions (α, c_α) of n satisfying the following three conditions. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$.*

(i) (Initial Conditions) *For $1 \le i \le l$,*

$$\alpha_i > \begin{cases} |2c_\alpha(i) - r + 1| & \text{if } \alpha_i \equiv r (\text{mod} 2) \\ |2c_\alpha(i) - r| & \text{otherwise} \end{cases} .$$

(ii) (Color Difference Conditions) *For $1 \le i < l$,*

$$\alpha_i - \alpha_{i+1} \ge \begin{cases} 2 + |2(c_\alpha(i) - c_\alpha(i+1))| & \text{if } \alpha_i \equiv \alpha_{i+1} (\text{mod} 2) \\ 2 + |2(c_\alpha(i) - c_\alpha(i+1)) - 1| & \text{if } \alpha_i \not\equiv \alpha_{i+1} \equiv r (\text{mod} 2) \\ 2 + |2(c_\alpha(i) - c_\alpha(i+1)) + 1| & \text{if } \alpha_{i+1} \not\equiv \alpha_i \equiv r (\text{mod} 2) \end{cases} .$$

(iii) (Parity Condition on Last Color when M is Even) *For $1 \le i \le l$, if M is even and $c_\alpha(i) = k - 1$, the last color, then*

$$\alpha_i \not\equiv r \pmod 2.$$

Then the number of partitions in the two sets $A_n(M, r)$ and $C_n(M, r)$ are the same.

Proof. This follows once we establish a bijection between $A_n(M, r)$ and $C_n(M, r)$. Let π be a partition of n with ranks $r_1, r_2, \ldots r_{d(\pi)}$, and assume that all ranks lie in the interval $[-r + 2, M - r - 2]$. For $1 \le i \le d$, let $\alpha_i = \pi_i + \pi'_i - 2i + 1$, as before, and let $\alpha(\pi)$ be the partition of n defined by $\alpha_1, \alpha_2, \ldots, \alpha_d$.

Color the parts of α as follows: for $1 \le i \le d$,

$$c_\alpha(i) = \begin{cases} (r_i + r - 1)/2 & \text{if } \alpha_i \equiv r (\text{mod} 2) \\ (r_i + r)/2 & \text{otherwise.} \end{cases} \quad (9)$$

We show that the $k-1$-color partition (α, c_α) satisfies conditions (i)-(iii) of the theorem.

Condition(i). By Lemma 1, $\alpha_i > |r_i|$ and from (9),

$$r_i = \begin{cases} 2c_\alpha(i) - r + 1 & \text{if } \alpha_i \equiv r (\text{mod} 2) \\ 2c_\alpha(i) - r & \text{otherwise.} \end{cases} \tag{10}$$

Condition(ii). By Lemma 2, $\alpha_i - \alpha_{i+1} \geq 2 + |r_i - r_{i+1}|$ and the conditions follow from (10).

Condition(iii). If $M = 2k$ and $c_\alpha(i) = k - 1$, then by definition of c_α,

$$r_i \in \{-r + 2k - 2,\ -r + 2k - 1\} = \{M - r - 2, M - r - 1\}.$$

But $r_i \neq M - r - 1$ since $r_i \in [-r + 2, M - r - 2]$. It follows then that $r_i = M - r - 2$ and therefore that

$$\begin{aligned} \alpha_i &= \pi_i + \pi_i' - 2i + 1 = r_i + 2\pi_i' - 2i + 1 \\ &= M - r - 1 + 2(\pi_i' - i) \equiv r + 1 \pmod 2. \end{aligned}$$

To show this is a bijection, given r, M, and k satisfying the conditions of the theorem, let (α, c_α) be a $k-1$ color partition of n, with $\alpha = (\alpha_1, \ldots, \alpha_l)$, satisfying (i), (ii), and (iii) of the theorem. We define an inverse map which sends α to a partition π of n with d angles, where the ith angle of π has width x_i defined by

$$x_i = \begin{cases} (-r + 2c_\alpha(i) + \alpha_i + 2)/2 & \text{if } \alpha_i \equiv r (\text{mod} 2) \\ (-r + 2c_\alpha(i) + \alpha_i + 1)/2 & \text{otherwise} \end{cases} \tag{11}$$

and height $y_i = \alpha_i - x_i + 1$. We must verify that π is a partition, i.e. that $x_1 > x_2 > \ldots > x_d \geq 1$ and $y_1 > y_2 > \ldots > y_d \geq 1$ (clearly it has weight n), and that the ith rank $r_i = x_i - y_i$ lies in the interval $[-r + 2, M - r - 2]$ and furthermore that it satisfies (9).

We first verify that for $1 \leq i \leq l$, $x_i \geq 1$ and $y_i \geq 1$. If $\alpha_i \equiv r (\text{mod} 2)$, then using (11) and condition (i),

$$x_i = (-r + 2c_\alpha(i) + \alpha_i + 2)/2 > (-r + 2c_\alpha(i) + | - 2c_\alpha(i) + r - 1| + 2)/2 \geq 1/2$$

and

$$y_i = (\alpha_i + r - 2c_\alpha(i))/2 > (|2c_\alpha(i) - r + 1| + r - 2c_\alpha(i))/2 \geq 1/2.$$

Similarly, if $\alpha_i \not\equiv r (\text{mod} 2)$,

$$x_i = (-r + 2c_\alpha(i) + \alpha_i + 1)/2 > (-r + 2c_\alpha(i) + | - 2c_\alpha(i) + r| + 1)/2 \geq 1/2$$

and

$$y_i = (\alpha_i + r - 2c_\alpha(i) + 1)/2 > (|2c_\alpha(i) - r| + r - 2c_\alpha(i) + 1)/2 \geq 1/2.$$

Checking the rank of the ith angle

$$x_i - y_i = 2x_i - \alpha_i - 1 = \begin{cases} -r + 2c_\alpha(i) + 1 & \text{if } \alpha_i \equiv r (\text{mod} 2) \\ -r + 2c_\alpha(i) & \text{otherwise} \end{cases},$$

which satisfies (9).

Now we verify that for $1 \leq i < l$, $x_i > x_{i+1}$ and $y_i > y_{i+1}$. We check each of three cases using (11) and condition (ii) of the theorem. If $\alpha_i \equiv \alpha_{i+1} (\text{mod} 2)$, then

$$x_i - x_{i+1} = (\alpha_i - \alpha_{i+1} + 2(c_\alpha(i) - c_\alpha(i+1)))/2 \geq 1,$$

$$y_i - y_{i+1} = (\alpha_i - \alpha_{i+1} - 2(c_\alpha(i) - c_\alpha(i+1)))/2 \geq 1.$$

If $\alpha_i \not\equiv \alpha_{i+1} \equiv r (\text{mod} 2)$, then

$$x_i - x_{i+1} = (\alpha_i - \alpha_{i+1} + 2(c_\alpha(i) - c_\alpha(i+1)) - 1)/2 \geq 1,$$

$$y_i - y_{i+1} = (\alpha_i - \alpha_{i+1} - 2(c_\alpha(i) - c_\alpha(i+1)) + 1)/2 \geq 1.$$

Finally, if $\alpha_{i+1} \not\equiv \alpha_i \equiv r (\text{mod} 2)$, then

$$x_i - x_{i+1} = (\alpha_i - \alpha_{i+1} + 2(c_\alpha(i) - c_\alpha(i+1)) + 1)/2 \geq 1,$$

$$y_i - y_{i+1} = (\alpha_i - \alpha_{i+1} - 2(c_\alpha(i) - c_\alpha(i+1)) - 1)/2 \geq 1.$$

□

The following result is an immediate consequence of Theorem 3 and (8).

Corollary 2

$$\sum_{n=0}^{\infty} |C_n(M, r)| q^n = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{j[(M)j+M-2r]/2} = \prod_{n \not\equiv 0, \pm r (\bmod M)} (1 - q^n)^{-1}.$$

□

See Figure 2 for an example of the bijection when $M = 7, r = 1$, and $n = 10$. Figure 3 shows the bijection when $M = 8, r = 3$, and $n = 10$.

We note the idea of *unbending angles* has also been used recently by Alladi and Berkovich [AlB] to obtain weighted partition identities in the special cases $M = 6, 7$.

3 Remarks on Alternative Colorings

An alternative coloring is to color the part α_i, derived from the ith angle with rank r_i, by:

$$c'_\alpha(i) = \begin{cases} r_i - k + r & \text{if } r_i > k - r \\ -r_i + k - r & \text{otherwise} \end{cases} \tag{12}$$

and proceed to formulate the conditions required to make this mapping a bijection between $(k-1)$-color partitions of n which satisfy the conditions and the set $A_n(M,r)$ of partitions of n with all ranks in $[-r+2, M-r-1]$. We did not proceed in this direction since an advantage of our coloring is that the color depends only on r and r_i and *not on* k. This may make it easier to establish a direct connection between the multi-sum and our multi-color interpretation. The hope is that for our particular goals, this interpretation will be more fruitful in the Lie algebra setting.

However, we would like to note that the coloring (12) is the coloring which would follow by applying coloring ideas in Agarwal and Andrews (for other families) [AA] to the partition family $A_n(M,r)$. This same multi-coloring, c'_α, arises by applying the coloring scheme in Agarwal and Bressoud [AB] (where it was used on a different family) to the lattice path interpretation of $A_n(M,r)$ described in Bressoud [Br3], namely, for each peak of height y at location x in the lattice path, create a part of size x with color y. This can be shown to be equivalent to the coloring (12) using the bijection between the lattice path and rank interpretations described by Bressoud in [Br3].

4 Finitized Rogers-Ramanujan Identities

In view of (5) and the second equality in (8), the generalized Rogers-Ramanujan identities can be written as

$$\frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{j[(M)j+M-2r]/2} = \sum_{n_1 \geq \ldots \geq n_{k-1} \geq 0} \frac{q^{n_1^2+n_2^2+\ldots+n_{k-1}^2+n_r+n_{r+1}+\ldots+n_{k-1}}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q^{2-s};q^{2-s})_{n_{k-1}}}, \tag{13}$$

where $s=0$ if M is even and $s=1$ if M is odd. In fact, this is the form of interest in recent work relating Rogers-Ramanujan identities to applications in statistical mechanics and conformal field theory. In these applications, the left-hand side is the *bosonic form* and the right-hand side is the *fermionic form*. In the general case, the bosonic form is associated with a character of the minimal model of a Virasoro algebra. In [LW3] and later in [LP] a generalized fermionic Pauli exclusion principle was discovered and discussed in connection with higher level representations of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. Using similar ideas the sum side of the generalized Rogers-Ramanujan identities has recently been studied in several papers, and it has been shown that under certain restrictions, every bosonic form has a fermionic form and, even more, there is a corresponding finitization [BeM1, BMS].

Motivated by these connections to physics, finite approximations of the identities (13) have been derived in [FQ, Ki] with the most general appearing in [BMS]. We will show that the multi-color partition interpretation provides a natural interpretation of the left-hand side for a particular finitization.

We first make an observation. For $0 < r \leq M/2$ and $u, v \geq 0$, let $F_n(M, r, u, v)$ be the set of partitions of n with all ranks in the interval $[-r+2, M-r-2]$ and whose Ferrers diagrams are contained in a $v \times u$ rectangle.

Lemma 3 *For $0 < r \leq M/2$ and $u, v \geq 0$, the set $F_n(M, r, u, v)$ is in one-to-one correspondence with the set of $k-1$-color partitions (α, c_α) of n in $C_n(M, r)$ with the following additional constraint*

$$(iv) \quad (u+v-1) - \alpha_1 \geq |2c_\alpha(1) - r + (v-u) + \frac{1}{2}| - \frac{1}{2}.$$

Proof. Under our map in the proof of Theorem 3, the partitions in $F_n(M, r, u, v)$ are mapped into $k-1$ color partitions (α_i) satisfying (i), (ii), and (iii) in Theorem 3. Since this map was shown to be a bijection, it suffices to show that its image is characterized by the extra condition (iv). So, for a $k-1$ color partition α satisfying (i)-(iii), we derive necessary and sufficient conditions on α to guarantee that under the inverse mapping its first angle has width $x_1 \leq u$ and height $y_1 \leq v$. The inverse map (11) has

$$x_1 = \lfloor(-r + 2c_\alpha(1) + \alpha_1 + 2)/2\rfloor$$

so $x_1 \leq u$ is equivalent to

$$\frac{-r + 2c_\alpha(1) + \alpha_1 + 1}{2} \leq u$$

which in turn is equivalent to

$$2c_\alpha(1) - r + (v-u) \leq (u+v-1) - \alpha_1. \tag{14}$$

On the other hand, the height $y_1 = \alpha_1 - x_1 + 1$ must satisfy $y_1 \leq v$ which is equivalent to $x_1 \geq -v + \alpha_1 + 1$, so we require

$$\frac{-r + 2c_\alpha(1) + \alpha_1 + 2}{2} \geq -v + \alpha_1 + 1$$

which is equivalent to

$$-2c_\alpha(1) + r - (v-u) - 1 \leq (u+v-1) - \alpha_1. \tag{15}$$

Combining (14) and (15) gives the lemma. □

In [FQ], Foda and Quano derive the following finitized Rogers-Ramanujan identity, proved independently by Kirilov in [Ki]:

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j[(2k+1)j+2k+1-2r]/2} \left[\begin{array}{c} N \\ \lfloor \frac{N-k+r-(2k+1)j}{2} \rfloor \end{array} \right]_q \tag{16}$$

$$= \sum q^{n_1^2+\cdots+n_{k-1}^2+n_r+\cdots+n_{k-1}} \prod_{j=1}^{k-1} \left[\begin{array}{c} N-2(n_1+\cdots+n_{j-1})-n_j-n_{j+1}-a_{rj}^{(k)} \\ n_j-n_{j+1} \end{array} \right]_q,$$

where the sum runs over $n_1 \geq \cdots \geq n_{k-1} \geq 0$ such that $2(n_1+\cdots+n_{k-1}) \leq N-k+r$ and $a_{ij}^{(k)}$ is the ij-entry of the $k \times (k-1)$ matrix

$$A^{(k)} = \begin{pmatrix} 1 & 2 & \cdots & k-1 \\ 0 & 1 & \cdots & k-2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

When $N \to \infty$, the identity reduces to (13) in the case of odd M and $s=1$.

Foda and Quano prove that the left-hand side of (16) is the generating function for the set of partitions $F_n(M,r,u,v)$ in the special case that $M=2k+1$, $u=\lfloor (N+k-r+1)/2 \rfloor$, and $v=\lfloor (N-k+r)/2 \rfloor$. (A more general form of this result appeared earlier in [ABBBFV].) Combining this with Lemma (3) we get the following.

Corollary 3 *For $M=2k+1$ and $r \leq k$, the left-hand side of (16) is the generating function for the set of $k-1$-color partitions (α, c_α) of n in $C_n(M,r)$ with the following additional constraint*

$$(iv) \quad (N-1)-\alpha_1 \geq \begin{cases} |2c_\alpha(1)-k+\frac{1}{2}|-\frac{1}{2}, & \text{if } N+k \equiv r (\text{mod} 2) \\ |2c_\alpha(1)-k-\frac{1}{2}|-\frac{1}{2}, & \text{otherwise} \end{cases}$$

In particular, $\alpha_1 \leq N-1$.

In [FQ], Foda and Quano derive the following additional finitized Rogers-Ramanujan identity:

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(kj+k-r)} \left[\begin{array}{c} 2N+k-r \\ N-kj \end{array} \right]_q = \sum q^{n_1^2+\cdots+n_{k-1}^2+n_r+\cdots+n_{k-1}} \times \tag{17}$$

$$\prod_{j=1}^{k-2} \left[\begin{array}{c} 2N-2(n_1+\cdots+n_{j-1})-n_j-n_{j+1}-b_{rj}^{(k)} \\ n_j-n_{j+1} \end{array} \right]_q \left[\begin{array}{c} N-(n_1+\cdots+n_{k-2}) \\ n_{k-1} \end{array} \right]_{q^2},$$

Partition π of 10 with all ranks in $[1,4]$	rank vector of π	the 2-color partition $(\alpha(\pi), c_{\alpha(\pi)})$ of 10
$(7,1,1,1)$	$[3]$	(10_2)
$(6,4)$	$[4,2]$	$(7_2,3_1)$
$(6,3,1)$	$[3,1]$	$(8_2,2_1)$
$(6,1,1,1)$	$[1]$	(10_1)
$(5,5)$	$[3,3]$	$(6_2,4_2)$
$(5,4,1)$	$[2,2]$	$(7_1,3_1)$
$(5,3,1,1)$	$[1,1]$	$(8_1,2_1)$
$(4,4,2)$	$[1,1]$	$(6_1,4_1)$

Figure 2: Example of the bijection of Theorem 3 when $M=7$, $r=1$, and $n=10$.

where the sum runs over $n_1 \geq \cdots \geq n_j = 0$ such that $n_1 + \cdots + n_{k-1} \leq N$ and $b_{ij}^{(k)}$ is the ij-entry of the $k \times (k-2)$ matrix

$$B^{(k)} = \begin{pmatrix} k-2 & k-3 & k-4 & \cdots & 1 \\ k-2 & k-3 & k-4 & \cdots & 1 \\ k-3 & k-4 & k-5 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 3 & 2 & 1 & \cdots & 1 \\ 2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

When $N \to \infty$, the identity reduces to (13) in the case of even M and $s=0$.

Foda and Quano prove that the left-hand side of (17) is the generating function for the set of partitions $F_n(M,r,u,v)$ in the special case that $M=2k$, $u=N+k-r$, and $v=N$. Combining this with Lemma 3 we get:

Corollary 4 *For $M=2k$, the left-hand side of (17) is the generating function for the set of $k-1$-color partitions (α, c_α) of n in $C_n(M,r)$ and the following additional constraint*

$$(iv) \quad (2N+k-r-1) - \alpha_1 \geq |2c_\alpha(1) - k + \frac{1}{2}| - \frac{1}{2}.$$

Partition π of 10 with all ranks in $[-1,3]$	rank vector of π	the 3-color partition $(\alpha(\pi), c_{\alpha(\pi)})$ of 10
$(6,2,1,1)$	$[2,0]$	$(9_2,1_1)$
$(6,1,1,1,1)$	$[1]$	(10_2)
$(5,4,1)$	$[2,2]$	$(7_2,3_2)$
$(5,3,2)$	$[2,0]$	$(7_2,3_1)$
$(5,3,1,1)$	$[1,1]$	$(8_2,2_2)$
$(5,2,2,1)$	$[1,-1]$	$(8_2,2_1)$
$(5,2,1,1,1)$	$[0,0]$	$(9_1,1_1)$
$(5,1,1,1,1,1)$	$[-1]$	(10_1)
$(4,4,2)$	$[1,1]$	$(6_2,4_2)$
$(4,4,1,1)$	$[0,2]$	$(7_1,3_2)$
$(4,3,3)$	$[1,0,0]$	$(6_2,3_1,1_1)$
$(4,3,2,1)$	$[0,0]$	$(7_1,3_1)$
$(4,3,1,1,1)$	$[-1,1]$	$(8_1,2_2)$
$(4,2,2,1,1)$	$[-1,-1]$	$(8_1,2_1)$
$(3,3,3,1)$	$[-1,0,0]$	$(6_1,3_1,1_1)$
$(3,3,2,2)$	$[-1-1]$	$(6_1,4_1)$
$(7,1,1,1)$	$[3]$	(10_3)
$(6,3,1)$	$[3,1]$	$(8_3,2_2)$
$(6,2,2)$	$[3,-1]$	$(8_3,2_1)$
$(5,5)$	$[3,3]$	$(6_3,4_3)$

Figure 3: Example of the bijection of Theorem 3 when $M=8$, $r=3$, and $n=10$.

Remark 3 *Clearly, our map in Theorem 3, when restricted to partitions in $F_n(M,r,u,v)$ with Durfee square of size d, gives a bijection with those $k-1$ color partitions of n in $C_n(M,r)$ which have exactly d parts and satisfy (iv) of of Lemma 3.*

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