Rational lecture hall polytopes and inflated Eulerian polynomials

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Abstract

For a sequence of positive integers $\mathbf{s} = (s_1, \ldots, s_n)$, we define the *rational lecture hall polytope* $\mathbf{R}_n^{(\mathbf{s})}$. We prove that its h^*-polynomial, $Q_n^{(\mathbf{s})}(x)$, has nonnegative integer coefficients that count certain statistics on $\mathbf{s}$-*inversion sequences*. The polynomial $Q_n^{(\mathbf{s})}(x)$ can be viewed as an *inflated* version of the $\mathbf{s}$-Eulerian polynomial, $A_n^{(\mathbf{s})}(x)$, associated with the *integral* lecture hall polytope, $\mathbf{P}_n^{(\mathbf{s})}$, introduced by Savage and Schuster. The result is applied in three ways: (1) in the theory of $\mathbf{s}$-lecture hall partitions, introduced by Bousquet-Mélou and Eriksson, the generating function, refined to include the size of the last part, now has an explicit description in terms of the inflated $\mathbf{s}$-Eulerian polynomial, $Q_n^{(\mathbf{s})}(x)$; (2) for special sequences, $\mathbf{s}$, we get an explicit formula for $Q_n^{(\mathbf{s})}(x)$ by computing the Ehrhart quasi-polynomial of $\mathbf{R}_n^{(\mathbf{s})}$; and (3) for many sequences, $\mathbf{s}$, the coefficients the inflated $\mathbf{s}$-Eulerian polynomial form a symmetric unimodal sequence, even when the coefficients of the (uninflated) $\mathbf{s}$-Eulerian polynomial, $A_n^{(\mathbf{s})}(x)$, do not.

Keywords: Lecture hall partitions, Eulerian polynomials, Ehrhart theory

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1 Introduction

For a sequence $\mathbf{s} = \{s_i\}_{i \geq 1}$ of positive integers, an $\mathbf{s}$-*lecture hall partition* is a finite integer sequence λ of length n satisfying

$$0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n}.$$

Let $\mathbf{L}_n^{(\mathbf{s})}$ denote the set of all $\mathbf{s}$-lecture hall partitions of length n. Lecture hall partitions were introduced by Bousquet-Mélou and Eriksson in a series of papers [1, 2, 3], where the problem of interest was to find the generating function

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} q^{|\lambda|}, \tag{1}$$

with $|\lambda| = \lambda_1 + \cdots + \lambda_n$. For several sequences $\mathbf{s}$, this generating function has a surprisingly nice form.

To gain a better understanding of this phenomenon, in [9], Savage and Schuster introduced the $\mathbf{s}$-*inversion sequences*, $\mathbf{I}_n^{(\mathbf{s})}$, and established a relationship between distributions of statistics on $\mathbf{L}_n^{(\mathbf{s})}$ and $\mathbf{I}_n^{(\mathbf{s})}$. In the the process, they defined the $\mathbf{s}$-*lecture hall polytope*, $\mathbf{P}_n^{(\mathbf{s})}$, and used it to prove a theorem relating the partition theory of $\mathbf{s}$-lecture hall partitions to their geometry. This context provides a broad generalization of previously unrelated results. We extend this here with a new variation on the Eulerian polynomials.

Recall that the Eulerian polynomials, $E_n(x)$, are defined by the relation

$$\sum_{t \geq 0} (t+1)^n x^t = \frac{E_n(x)}{(1-x)^{n+1}} \tag{2}$$

and they have the explicit combinatorial interpretation

$$E_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{des} \pi}, \tag{3}$$

where $\mathfrak{S}_n$ is the set of permutations of n elements and for $\pi \in \mathfrak{S}_n$, $\operatorname{des} \pi$ is the number of i in $\{1, 2, \ldots, n-1\}$ such that $\pi(i) > \pi(i+1)$. See Foata's history of the Eulerian numbers [7].

In [9], the integral polytope $\mathbf{P}_n^{(\mathbf{s})}$ gave rise to a new and natural generalization of the Eulerian polynomial, $A_n^{(\mathbf{s})}(x)$, called the $\mathbf{s}$-*Eulerian polynomial*. In the current paper, we extend the work in [9] by studying a *rational* version, $\mathbf{R}_n^{(\mathbf{s})}$, of the $\mathbf{s}$-lecture polytope. In a similar way, $\mathbf{R}_n^{(\mathbf{s})}$ will give rise to an *inflated* $\mathbf{s}$-*Eulerian polynomial*, $Q_n^{(\mathbf{s})}(x)$.

In Section 3, give a combinatorial interpretation of $Q_n^{(\mathbf{s})}(x)$ by showing that it models the distribution of certain statistics on $\mathbf{I}_n^{(\mathbf{s})}$.

In Section 4, to illustrate the significance of the main result, we apply it in two ways: (i) in the theory of **s**-lecture hall partitions, to show that the generating function, refined to include the size of the last part, now has an explicit description in terms of $Q_n^{(\mathbf{s})}(x)$; and (ii) for special sequences, **s**, to get an explicit formula for $Q_n^{(\mathbf{s})}(x)$. In Section 5, we consider symmetry and unimodality questions and observe that for several sequences, **s**, the coefficients of $Q_n^{(\mathbf{s})}(x)$ form a symmetric unimodal sequence, even when the coefficients of $A_n^{(\mathbf{s})}(x)$ do not. Concluding remarks follow in Section 6.

2 Background

2.1 Lecture Hall Polytopes and Ehrhart Theory

In [9], Savage and Schuster defined the *lecture hall polytope* $\mathbf{P}_n^{(\mathbf{s})}$ for a sequence $\mathbf{s} = \{s_i\}_{i\geq 1}$ of positive integers, by

$$\mathbf{P}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}. \tag{4}$$

$\mathbf{P}_n^{(\mathbf{s})}$ is a convex, simplicial polytope with the following $n+1$ vertices:

$$\{(0, 0, \ldots, 0, s_i, s_{i+1}, \ldots, s_n) \mid 1 \leq i \leq n+1\},$$

all with integer coordinates. The *t-th dilation of* $\mathbf{P}_n^{(\mathbf{s})}$ is the polytope

$$t\mathbf{P}_n^{(\mathbf{s})} \;=\; \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq t \right\}. \tag{5}$$

The set $\mathbb{Z}^n \subset \mathbb{R}^n$ denotes the n-dimensional integer lattice and its elements are called *lattice points*. Let $\mathbf{f}_n^{(\mathbf{s})}(t)$ be the number of lattice points in $t\mathbf{P}_n^{(\mathbf{s})}$:

$$\mathbf{f}_n^{(\mathbf{s})}(t) = |t\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n|.$$

In Ehrhart theory, for an n-dimensional polytope, $\mathbf{P}$, the function $f(t) = |t\mathbf{P} \cap \mathbb{Z}^n|$ is significant in that $\lim_{t\to\infty}(f(t)/t^n)$ is the *volume* of $\mathbf{P}$. Its generating function $\sum_{t\geq 0} f(t)x^t$ is the *Ehrhart series* of $\mathbf{P}$. The Ehrhart series is used to study the combinatorial structure of the set of integer points in $\mathbf{P}$. In our case, those integer points are **s**-lecture hall partitions and $\mathbf{f}_n^{(\mathbf{s})}(t)$ counts the number of **s**-lecture hall partitions of length n with the last part less than or equal to ts_n.

Since $\mathbf{P}_n^{(\mathbf{s})}$ is a convex polytope with integer vertices, it is known that $\mathbf{f}_n^{(\mathbf{s})}(t)$ is a *polynomial* function of t, called the *Ehrhart polynomial* of $\mathbf{P}_n^{(\mathbf{s})}$ [5, 6]. Thus, in view of (2), the Ehrhart series of $\mathbf{P}_n^{(\mathbf{s})}$ is a rational function of the form

$$\sum_{t\geq 0} \mathbf{f}_n^{(\mathbf{s})}(t)x^t = \frac{A_n^{(\mathbf{s})}(x)}{(1-x)^{n+1}}, \tag{6}$$

where $A_n^{(\mathbf{s})}(x)$ is a polynomial with integer coefficients. The sequence of coefficients is known as the $\mathbf{h}^*$-vector of $\mathbf{P}_n^{(\mathbf{s})}$ [12]. By a theorem of Stanley, since $\mathbf{P}_n^{(\mathbf{s})}$ is a convex integral polytope, all coefficients of $A_n^{(\mathbf{s})}(x)$ are *positive* integers and the goal, in general, is to characterize $A_n^{(\mathbf{s})}(x)$ by discovering what those coefficients count.

It was shown in [9] that when $\mathbf{s} = (1, 2, \ldots, n)$,

$$\mathbf{f}_n^{(\mathbf{s})}(t) = (t+1)^n.$$

Therefore, in view of (2), $A_n^{(1,2,\ldots,n)}(x)$ is the familiar Eulerian polynomial:

$$\frac{A_n^{(1,2,\ldots,n)}(x)}{(1-x)^{n+1}} \;=\; \sum_{t\geq 0}(t+1)^n x^t \;=\; \frac{E_n(x)}{(1-x)^{n+1}}. \tag{7}$$

For that reason, $A_n^{(\mathbf{s})}(x)$ is referred to as the $\mathbf{s}$-*Eulerian polynomial*

The following refinement of $\mathbf{f}_n^{(\mathbf{s})}(t)$ was defined in [9] to incorporate statistics significant in the theory of lecture hall partitions:

$$\mathbf{f}_n^{(\mathbf{s})}(t; u, q, z) \;=\; \sum_{\lambda \in t\mathbf{P}_n^{(\mathbf{s})} \cap Z^n} u^{|\lceil \lambda \rceil|} q^{|\lambda|} z^{|\,\epsilon^+(\lambda)\,|}, \tag{8}$$

where $|\lambda|$ is the weight of λ, and the other exponents are weights of the derived sequences

$$\lceil \lambda \rceil \;\triangleq\; \left(\left\lceil \frac{\lambda_1}{s_1} \right\rceil, \left\lceil \frac{\lambda_2}{s_2} \right\rceil, \ldots, \left\lceil \frac{\lambda_n}{s_n} \right\rceil \right),$$

$$\epsilon^+(\lambda) \;\triangleq\; \left(s_1 \left\lceil \frac{\lambda_1}{s_1} \right\rceil - \lambda_1,\; s_2 \left\lceil \frac{\lambda_2}{s_2} \right\rceil - \lambda_2,\; \ldots,\; s_n \left\lceil \frac{\lambda_n}{s_n} \right\rceil - \lambda_n \right).$$

2.2 Generalized Inversion Sequences

In order to characterize the polynomial $A_n^{(\mathbf{s})}(x)$, in [9], Savage and Schuster defined a generalization of the familiar inversion sequences associated with permutations, as well as several statistics on these inversion sequences. Given a sequence $\mathbf{s} = \{s_i\}_{i\geq 1}$ of positive integers, define $\mathbf{I}_n^{(\mathbf{s})}$ by

$$\mathbf{I}_n^{(\mathbf{s})} \;=\; \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \;|\; 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

We call a sequence $e \in \mathbf{I}_n^{(\mathbf{s})}$ an $\mathbf{s}$-*inversion sequence*. Let the ascent set of e be

$$\operatorname{Asc} e \;=\; \left\{ i \;|\; i = 0 \text{ and } e_1 > 0 \;\text{ or }\; 1 \leq i < n \text{ and } \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\}.$$

Then define the following statistics on e:

$$\begin{aligned}
\operatorname{asc} e &= \big|\operatorname{Asc} e\big|, \\
|e| &= \sum_{i=1}^{n} e_i, \\
\operatorname{lhp} e &= -|e| + \sum_{i \in \operatorname{Asc} e} \big(s_{i+1} + \ldots + s_n\big), \\
\operatorname{amaj} e &= \sum_{i \in \operatorname{Asc} e} (n-i).
\end{aligned}$$

2.3 Relationship between lecture hall polytopes and inversion sequences

In the same way that the Eulerian polynomial has the characterization (3) in terms of permutation statistics, in [9], it was shown that for any sequence $\mathbf{s}$ of positive integers, $A_n^{(\mathbf{s})}(x)$, defined by (6), is the ascent polynomial of the $\mathbf{s}$-inversion sequences, so that

$$\sum_{t \geq 0} \mathbf{f}_n^{(\mathbf{s})}(t)\, x^t \quad = \quad \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e}}{(1-x)^{n+1}}. \tag{9}$$

They actually proved the following refinement.

Theorem 1 ([9], Thm. 5) *For any sequence* $\mathbf{s}$ *of positive integers,*

$$\sum_{t \geq 0} \mathbf{f}_n^{(\mathbf{s})}(t; u, q, z)\, x^t \quad = \quad \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} u^{\operatorname{amaj} e}\, q^{\operatorname{lhp} e}\, z^{|\,e\,|}\, x^{\operatorname{asc} e}}{\prod_{i=0}^{n} (1 - x u^{n-i} q^{s_{i+1} + \cdots + s_n})}. \tag{10}$$

In Section 3, we extend this result to rational lecture hall polytopes and a variation of the ascent statistic on inversion sequences.

2.4 Barred inversion sequences

The proof of Theorem 1 was based on *barred inversion sequences*, derived from the *barred permutations* of Gessel and Stanley [8]. A *barred* $\mathbf{s}$*-inversion sequence* is an $e \in \mathbf{I}_n^{(\mathbf{s})}$ in which one or more vertical bars are inserted before and/or after elements e_i, with the stipulation that if i is an *ascent* of e, there is at least one bar in position i, the space immediately preceding e_{i+1}. Theorem 1 was proved by showing that both sides of the equation (10) enumerate barred $\mathbf{s}$-inversion sequences.

In the next section we will need a refinement. For $0 \leq k < s_n$, define a *barred*$_k$ $\mathbf{s}$-*inversion sequence* to be a barred $\mathbf{s}$-inversion sequence e with the added requirement that there must be a bar after e_n if $e_n < s_n - k$.

Observe that every barred$_k$ $\mathbf{s}$-inversion sequence must have at least 1 bar: the only $\mathbf{s}$-inversion sequence with no ascents is $e = (0, 0, \ldots, 0)$, but then $e_n = 0 < s_n - k$ for every $0 \le k < s_n$, so there must be a bar after e_n.

3 Rational lecture hall polytopes

3.1 Rational lecture hall polytopes and quasi-polynomials

The integral lecture hall polytope, $\mathbf{P}_n^{(\mathbf{s})}$, does provide information about the $\mathbf{s}$-lecture hall partitions, but it is limited by the requirement that its points λ satisfy $\lambda_n/s_n \le 1$. This means that as we move from $t\mathbf{P}_n^{(\mathbf{s})}$ to $(t+1)\mathbf{P}_n^{(\mathbf{s})}$, the bound on λ_n jumps by s_n, from $\lambda_n \le ts_n$ to $\lambda_n \le (t+1)s_n$. We will able to get more refined information about the $\mathbf{s}$-lecture hall partitions by modifyng our polytope so that its points satisfy $\lambda_n \le 1$.

To this end, given a sequence $\mathbf{s} = \{s_i\}_{i \ge 1}$ of positive integers, define the *rational lecture hall polytope* $\mathbf{R}_n^{(\mathbf{s})}$ by

$$\mathbf{R}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \cdots \le \frac{\lambda_n}{s_n} \le \frac{1}{s_n} \right\}.$$

$\mathbf{R}_n^{(\mathbf{s})}$ is now a convex simplicial polytope, whose vertices have rational (rather than integer) coordinates. The *t-th dilation of* $\mathbf{R}_n^{(\mathbf{s})}$ is the polytope

$$t\mathbf{R}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \cdots \le \frac{\lambda_n}{s_n} \le \frac{t}{s_n} \right\},$$

and we let $\mathbf{g}_n^{(\mathbf{s})}(t)$ be the number of lattice points in $t\mathbf{R}_n^{(\mathbf{s})}$:

$$\mathbf{g}_n^{(\mathbf{s})}(t) = |t\mathbf{R}_n^{(\mathbf{s})} \cap \mathbb{Z}^n|.$$

Now $\mathbf{g}_n^{(\mathbf{s})}(t)$ counts the number of $\mathbf{s}$-lecture hall partitions of length n with the last part less than or equal to t (rather than ts_n in the case of $\mathbf{f}_n^{(\mathbf{s})}(t)$.) It is evident that $\mathbf{P}_n^{(s)}$ is a dilation of $\mathbf{R}_n^{(s)}$, and that $\mathbf{f}_n^{(\mathbf{s})}(t) = \mathbf{g}_n^{(\mathbf{s})}(ts_n)$.

However, because $\mathbf{R}_n^{(\mathbf{s})}$ is a rational polytope, $\mathbf{g}_n^{(\mathbf{s})}(t)$ is not a polynomial in t, but rather a *quasi-polynomial* (see, e.g. [11], Section 4.4). A *quasi-polynomial* is a function, defined on the integers, of the form $h(n) = \sum_{i=0}^{d} c_i(n) n^i$, where each coefficient $c_i(n)$ is a periodic function with integral period.

As an example, recall from Section 2.1 that the integral polytope $\mathbf{P}_n^{(1,2,\ldots,n)}$ had Ehrhart polynomial $\mathbf{f}_n^{(1,2,\ldots,n)}(t) = (t+1)^n$. In contrast, for the rational polytope $\mathbf{R}_n^{(1,2,\ldots,n)}$, the Ehrhart quasi-polynomial $\mathbf{g}_n^{(1,2,\ldots,n)}(t)$ is given by the following result from [4].

Proposition 1 ([4], Thm. 2) *For integers i, j with $j \geq 0$ and $0 \leq i \leq n$,*

$$\mathbf{g}_n^{(1,2,\ldots,n)}(jn+i) \quad = \quad (j+1)^{n-i}(j+2)^i. \tag{11}$$

So $\mathbf{g}_n^{(1,2,\ldots,n)}(t)$ is actually described by n different polynomials, one for each value of t modulo n.

We will obtain new information about the $\mathbf{s}$-lecture hall partitions from the Ehrhart series of $\mathbf{R}_n^{(\mathbf{s})}$. This is known to have the form

$$\sum_{t \geq 0} \mathbf{g}_n^{\mathbf{s})}(t)x^t \;=\; \frac{Q_n^{(\mathbf{s})}(x)}{(1-x^{s_n})^{n+1}}, \tag{12}$$

where $Q_n^{(\mathbf{s})}(x)$ is a polynomial with positive integer coefficients. For general rational polytopes, very little is known about interpreting the polynomial appearing in the numerator of the rational function representaion of the Ehrhart series. In our main result in Section 3.3, we will give an explicit combinatorial characterization of $Q_n^{(\mathbf{s})}(x)$.

3.2 Key lemma for rational lecture hall polytopes

As we did with $\mathbf{f}_n^{(\mathbf{s})}(t)$ in (8), refine $\mathbf{g}_n^{(\mathbf{s})}(t)$ by

$$\mathbf{g}_n^{(\mathbf{s})}(t;u,q,z) \quad = \sum_{\lambda \in t\mathbf{R}_n^{(\mathbf{s})} \cap Z^n} u^{|\lceil\lambda\rceil|} q^{|\lambda|} z^{|\,\epsilon^+(\lambda)\,|}. \tag{13}$$

Our generalization of (10) to rational lecture hall polytopes will rely on the following lemma.

Lemma 1 *Let $\mathbf{s}$ be any sequence of positive integers. For integers $n \geq 0$ and $0 \leq k < s_n$,*

$$\sum_{j \geq 0} \mathbf{g}_n^{(\mathbf{s})}(js_n+k;u,q,z)x^j = \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} u^{\mathrm{amaj}\, e}\, q^{\mathrm{lhp}\, e}\, z^{|\,e\,|}\, x^{\mathrm{asc}\, e - \chi(e_n \geq s_n - k)}}{\prod_{i=0}^{n}(1 - xu^{n-i}q^{s_{i+1}+\cdots+s_n})}, \tag{14}$$

where χ evaluates to 1 given a true expression, and 0 otherwise.

Before proving the lemma, we illustrate with an example. Setting $\mathbf{s} = (1,2,\ldots,n)$ and $u = q = z = 1$ in Lemma 1 and using Proposition 1,we have:

Corollary 1 *For $0 \leq k < n$,*

$$\sum_{j \geq 0}(j+1)^{n-k}(j+2)^k\; x^j \quad = \quad \frac{\sum_{e \in \mathbf{I}_n^{(1,2,\ldots,n)}} x^{\mathrm{asc}\, e - \chi(e_n \geq n-k)}}{(1-x)^{n+1}}.$$

Corollary 2 *For* $0 \le k < n$,

$$\sum_{j\ge 0}(j+1)^{n-k}(j+2)^k\ x^j \quad = \quad \frac{\sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{des}\pi-\chi(\pi_n\le k)}}{(1-x)^{n+1}}.$$

Proof of Corollary 2 For $\pi \in \mathfrak{S}_n$, we define $e(\pi) = (e_1, e_2, \ldots e_n)$, by

$$e_i = |\{j \mid i > j \text{ and } \pi^{-1}(i) < \pi^{-1}(j)\}|.$$

It is shown in [9] that the mapping $\pi \to e = e(\pi^{-1})$ is a bijection between $\mathfrak{S}_n$ and $\mathbf{I}_n^{(1,2,\ldots,n)}$ such that i is a descent of π if and only if i is an ascent of e and $\pi_n \le k$ if and only if $e_n \ge n-k$. □

Proof of Lemma 1. Let e be a barred$_k$ **s**-inversion sequence with d_i bars in position i for $0 \le i \le n$. Assign to e the weight

$$w(e) = \frac{1}{x}\left(\frac{z}{q}\right)^{|\,e\,|}\prod_{i=0}^{n}(xu^{n-i}q^{s_{i+1}+\ldots+s_n})^{d_i}. \tag{15}$$

We show that in both sides of (14), the coefficient of x^j is the number of barred$_k$ **s**-inversion sequences with $j+1$ bars, each counted with the appropriate weight. For the right-hand-side, we will fix the inversion sequence, e, and sum over all $e \in \mathbf{I}_n^{(\mathbf{s})}$. For the left-hand-side, we will fix the number of bars, $j+1$, and sum over $j \ge 0$.

The first way is easier to see. A barred$_k$ **s**-inversion sequences $e \in I_n^{(\mathbf{s})}$ must have at least one bar following each ascent position, plus one bar after e_n if $e_n < s_n - k$. This minimally contributes $x^{\operatorname{asc} e+\chi(e_n<s_n-k)-1}\,u^{\operatorname{amaj} e}\,q^{\operatorname{lhp} e}\,z^{|\,e\,|}$ to $w(e)$. Note that $\chi(e_n < s_n - k) - 1 = -\chi(e_n \ge s_n - k.)$ But in each position i of e, any number $j \ge 0$ of additional bars may be placed, contributing a term in the sum below as an additional factor to $w(e)$:

$$\sum_{j=0}^{\infty}(xu^{n-i}q^{s_{i+1}+\cdots+s_n})^j \;=\; \frac{1}{1-xu^{n-i}q^{s_{i+1}+\cdots+s_n}}.$$

So, the right-hand side of (14) counts the number of barred$_k$ **s**-inversion sequences of $\mathbf{I}_n^{(\mathbf{s})}$, with their corresponding weight, and with the coefficient of x^j corresponding to the number of barred$_k$ **s**-inversion sequences with $j+1$ bars.

For the left-hand side of (14), we describe a weight-preserving bijection from **s**-lecture hall partitions in $(js_n+k)\mathbf{R}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ to barred$_k$ **s**-inversion sequences with $j+1$ bars. If $\lambda \in (js_n+k)\mathbf{R}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$, let

$$b = \lceil \lambda \rceil = \left(\left\lceil\frac{\lambda_1}{s_1}\right\rceil, \left\lceil\frac{\lambda_2}{s_2}\right\rceil, \ldots, \left\lceil\frac{\lambda_n}{s_n}\right\rceil\right).$$

Note that $b_n \leq j{+}1$. Let $e = \epsilon^+(\lambda) = (e_1, e_2, \ldots, e_n)$, so

$$e_i = s_i b_i - \lambda_i.$$

Then clearly $e \in I_n^{(\mathbf{s})}$. We "bar" e with j+1 bars by placing b_1 bars before e_1; $b_i - b_{i-1}$ bars before e_i for $2 \leq i \leq n$; and j+1 $- b_n$ bars after e_n. Since λ was an **s**-lecture hall partition, if $1 \leq i < n$ and there is no bar after e_i, then $b_i = b_{i+1}$ and therefore

$$\frac{e_i}{s_i} \;=\; b_i - \frac{\lambda_i}{s_i} \;\geq\; b_i - \frac{\lambda_{i+1}}{s_{i+1}} \;=\; b_{i+1} - \frac{\lambda_{i+1}}{s_{i+1}} \;=\; \frac{e_{i+1}}{s_{i+1}}.$$

Thus $i \notin \operatorname{Asc}(e)$. If there is no bar before e_1, then $b_1 = 0$, so $\lambda_1 = 0$ and thus $e_1 = 0$ and so in this case also, $0 \notin \operatorname{Asc} e$. To address our added stipulation, observe that if there are no bars after e_n, then $b_n = j + 1$. This means

$$e_n = s_n b_n - \lambda_n \geq s_n(j+1) - (js_n + k) \geq s_n - k.$$

Thus e is a barred$_k$ **s**-inversion sequence with $j + 1$ bars.

Before defining the inverse, note that this mapping preserves weights, as follows. For fixed j, the weight associated with $\lambda \in j\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ is

$$w(\lambda) \;=\; u^{\,|\lceil\lambda\rceil|}\, q^{\,|\lambda|}\, z^{\,|\,\epsilon^+(\lambda)\,|}\, x^j,$$

whereas, from (15), the weight of the barred$_k$ **s**-inversion sequence $e = \epsilon^+(\lambda)$, derived from λ, is

$$w(e) = \frac{1}{x}\left(\frac{z}{q}\right)^{|\,e\,|} \prod_{i=0}^{n} (x u^{n-i} q^{s_{i+1}+\ldots+s_n})^{d_i},$$

where $d_0 = b_1$, $d_n = j - b_n$, and for $1 \leq i \leq n-1$, $d_i = b_{i+1} - b_i$. Comparing the exponents of z, x, u, and q in $w(\lambda)$ and $w(e)$ we find that they agree: $|e| = |\epsilon^+(\lambda)|$ and

$$\begin{aligned}
\sum_{i=0}^{n} d_i &= -1 + b_1 + (j + 1 - b_n) + \sum_{i=1}^{n-1} (b_{i+1} - b_i) \;=\; j;\\
\sum_{i=0}^{n} (n-i) d_i &= n b_1 + \sum_{i=1}^{n-1} (n-i)(b_{i+1} - b_i) \;=\; |b|;\\
-|e| + \sum_{i=0}^{n} (s_i + \cdots + s_n)) d_i &= -|e| + \sum_{i=1}^{n} s_i b_i \;=\; |\lambda|.
\end{aligned}$$

Finally, to prove that the mapping from $(js_n + k)\mathbf{R}_n^{(\mathbf{s})} \cup \mathbb{Z}^n$ to barred$_k$ **s**-inversion sequences with $j+1$ bars is a bijection, we define its inverse. Let e be a barred$_k$ **s**-inversion sequence with $j + 1$ bars. For $1 \leq i \leq n$, let b_i be the total number of bars preceding e_i in any position. Then $b_1 \leq b_2 \leq \ldots \leq b_n$. Define $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_n)$ by

$$\lambda_i \;=\; s_i b_i - e_i.$$

We show $\lambda \in (js_n+k)\mathbf{R}_n^{(\mathbf{s})} \cup \mathbb{Z}^n$. First note that $\lambda_i \geq 0$, since $e_i < s_i$ and if $b_i = 0$, then, by definition of barred$_k$ $\mathbf{s}$-inversion sequences, no position $j < i$ is an ascent of e, so $e_1 = e_2 = \ldots = e_i = 0$. Secondly, we show that λ is an $\mathbf{s}$-lecture hall partition. Since e is an $\mathbf{s}$-inversion sequence, $\frac{e_j}{s_j} < 1$ for all j. Now, if $i \in \operatorname{Asc} e$, there is at least one bar between e_i and e_{i+1}, so $b_i < b_{i+1}$ and

$$\frac{\lambda_i}{s_i} = \frac{s_ib_i - e_i}{s_i} = b_i - \frac{e_i}{s_i} \leq b_i \leq b_{i+1} - 1 < b_{i+1} - \frac{e_{i+1}}{s_{i+1}} = \frac{\lambda_{i+1}}{s_{i+1}}.$$

On the other hand, if $i \notin \operatorname{Asc} e$, then

$$\frac{e_i}{s_i} \geq \frac{e_{i+1}}{s_{i+1}},$$

so

$$\frac{\lambda_i}{s_i} = \frac{s_ib_i - e_i}{s_i} = b_i - \frac{e_i}{s_i} \leq b_{i+1} - \frac{e_{i+1}}{s_{i+1}} = \frac{\lambda_{i+1}}{s_{i+1}}.$$

Finally, consider λ_n. There are only $j+1$ bars so $b_n \leq j+1$. If $b_n = j+1$, then by the added stipulation we have $e_n \geq s_n - k$. Thus,

$$\lambda_n = s_nb_n - e_n \leq s_n(j+1) - (s_n - k) = js_n + k.$$

Else, $b_n \leq j$, which gives

$$\lambda_n = s_nb_n - e_n \leq js_n - e_n < js_n + k.$$

This completes the proof. □

3.3 The Ehrhart series of rational lecture hall polytopes

Analogous to Theorem 1 for integral lecture hall polytopes, we can now give a complete characterization of the Ehrhart series of the rational lecture hall polytopes $\mathbf{R}_n^{(\mathbf{s})}$. As discussed at the send of Section 3.1, it is quite rare to be able to do so in the case of rational polytopes.

We now prove our main result about the Ehrhart series of rational lecture hall polytopes.

Theorem 2 *For any sequence* $\mathbf{s}$ *of positive integers,*

$$\sum_{t\geq 0} \mathbf{g}_n^{(s)}(t;u,q,z)x^t = \frac{\sum_{e\in \mathbf{I}_n^{(\mathbf{s})}} u^{\operatorname{amaj} e}\, q^{\operatorname{lhp} e}\, z^{|\,e\,|} \sum_{k=0}^{s_n-1} x^{k+s_n(\operatorname{asc} e - \chi(e_n \geq s_n - k))}}{\prod_{i=0}^{n}(1 - x^{s_n}u^{n-i}q^{s_{i+1}+\cdots+s_n})}. \quad (16)$$

Proof. Any nonnegative integer t can be written uniquely as $t = js_n + k$ for integers j, k satisfying $j \geq 0$ and $0 \leq k < s_n$. Use this to rewrite the sum and then apply Lemma 1:

$$\begin{aligned}
\sum_{t\geq 0} \mathbf{g}_n^{(s)}(t; u, q, z) x^t &= \sum_{j\geq 0} \sum_{k=0}^{s_n-1} g_n^{(s)}(js_n + k; u, q, z) x^{js_n+k} \\
&= \sum_{k=0}^{s_n-1} x^k \sum_{j\geq 0} g_n^{(s)}(js_n + k; u, q, z)(x^{s_n})^j \\
&= \sum_{k=0}^{s_n-1} x^k \frac{\sum_{e\in \mathbf{I}_n^{(\mathbf{s})}} u^{\operatorname{amaj} e} q^{\operatorname{lhp} e} z^{|e|} x^{s_n(\operatorname{asc} e - \chi(e_n \geq s_n - k))}}{\prod_{i=0}^{n}(1 - x^{s_n} u^{n-i} q^{s_{i+1}+\cdots+s_n})} \\
&= \frac{\sum_{e\in \mathbf{I}_n^{(\mathbf{s})}} u^{\operatorname{amaj} e} q^{\operatorname{lhp} e} z^{|e|} \sum_{k=0}^{s_n-1} x^{k+s_n(\operatorname{asc} e - \chi(e_n \geq s_n - k))}}{\prod_{i=0}^{n}(1 - x^{s_n} u^{n-i} q^{s_{i+1}+\cdots+s_n})}.
\end{aligned}$$

□

We show how to simplify this further in the next section.

3.4 The inflated s-Eulerian polynomial $Q_n^{(\mathbf{s})}(x)$

From (9), the **s**-Eulerian polynomial $A_n^{(\mathbf{s})}(x)$, defined by (6), satisfies

$$\sum_{t\geq 0} \mathbf{f}_n^{(\mathbf{s})}(t)\, x^t = \frac{A_n^{(\mathbf{s})}(x)}{(1-x)^{n+1}} = \frac{\sum_{e\in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e}}{(1-x)^{n+1}}.$$

Analogously, setting $u = q = z = 1$ in Theorem 2, the polynomial $Q_n^{(\mathbf{s})}(x)$, defined by (12), satisfies

$$\sum_{t\geq 0} \mathbf{g}_n^{(\mathbf{s})}(t)\, x^t = \frac{Q_n^{(\mathbf{s})}(x)}{(1-x^{s_n})^{n+1}} = \frac{\sum_{e\in \mathbf{I}_n^{(\mathbf{s})}} \sum_{k=0}^{s_n-1} x^{k+s_n(\operatorname{asc} e - \chi(e_n \geq s_n - k))}}{(1-x^{s_n})^{n+1}}. \tag{17}$$

We refer to $Q_n^{(\mathbf{s})}(x)$ as the *inflated* **s**-*Eulerian polynomial*, since it is an "inflated" version of $A_n^{(\mathbf{s})}(x)$ in which each monomial $x^{\operatorname{asc} e}$ is replaced by a polynomial as prescribed in (17). We now show that $Q_n^{(\mathbf{s})}(x)$ factors and Theorem 2 can be simplified.

Lemma 2 *For any sequence* **s** *of positive integers, and for* $e \in \mathbf{I}_n^{(\mathbf{s})}$,

$$\sum_{k=0}^{s_n-1} x^{k+s_n(\operatorname{asc} e - \chi(e_n \geq s_n - k))} = (1 + x + x^2 + \cdots + x^{s_n-1}) x^{s_n \operatorname{asc} e - e_n}.$$

Proof.

$$\begin{aligned}\sum_{k=0}^{s_n-1} x^{k+s_n(\operatorname{asc} e-\chi(e_n\ge s_n-k))} &= \sum_{j=1}^{s_n} x^{s_n-j+s_n(\operatorname{asc} e-\chi(e_n\ge j))}\\ &= \sum_{j=1}^{e_n} x^{s_n-j+s_n(\operatorname{asc} e-1)} + \sum_{j=e_n+1}^{s_n} x^{s_n-j+s_n \operatorname{asc} e}\\ &= x^{s_n \operatorname{asc} e-e_n}\Big(\sum_{j=1}^{e_n} x^{-j+e_n} + \sum_{j=e_n+1}^{s_n} x^{s_n-j+e_n}\Big)\\ &= x^{s_n \operatorname{asc} e-e_n}(x^{e_n-1}+x^{e_n-2}+\cdots 1+x^{s_n-1}+x^{s_n-2}+\cdots+x^{e_n})\\ &= x^{s_n \operatorname{asc} e-e_n}(1+x+x^2+\cdots+x^{s_n-1}).\end{aligned}$$

□

Applying Lemma 2 to (17) and to Theorem 2, respectively, gives the following corollaries.

Corollary 3 *For any sequence* $\mathbf{s}$ *of positive integers,*

$$Q_n^{(\mathbf{s})}(x) \;=\; \frac{1-x^{s_n}}{1-x}\sum_{e\in\mathbf{I}_n^{(\mathbf{s})}} x^{s_n \operatorname{asc} e-e_n}.$$

Corollary 4 *For any sequence* $\mathbf{s}$ *of positive integers,*

$$\sum_{t\ge 0}\mathbf{g}_n^{(s)}(t;u,q,z)x^t \;=\; \frac{\sum_{e\in\mathbf{I}_n^{(\mathbf{s})}} u^{\operatorname{amaj} e}\, q^{\operatorname{lhp} e}\, z^{|\,e\,|}\, x^{s_n \operatorname{asc} e-e_n}}{(1-x)\prod_{i=0}^{n-1}(1-x^{s_n}u^{n-i}q^{s_{i+1}+\cdots+s_n})}.$$

4 Applications

4.1 Enumeration of s-lecture hall partitions

We are now in a position to enumerate $\mathbf{s}$-lecture hall partitions, keeping track of the statistics important in partition theory and in Ehrhart theory.

Theorem 3 *For any sequence* $\mathbf{s}$ *of positive integers,*

$$\sum_{\lambda\in\mathbf{L}_n^{(\mathbf{s})}} u^{|\lceil\lambda\rceil|}q^{|\lambda|}z^{|\,\epsilon^+(\lambda)\,|}x^{\lambda_n} \;=\; \frac{\sum_{e\in\mathbf{I}_n^{(\mathbf{s})}} u^{\operatorname{amaj} e}\, q^{\operatorname{lhp} e}\, z^{|\,e\,|}\, x^{s_n \operatorname{asc} e-e_n}}{\prod_{i=0}^{n-1}(1-x^{s_n}u^{n-i}q^{s_{i+1}+\cdots+s_n})}.$$

Proof. Note that $\mathbf{g}_n^{(s)}(t)$, and its refinement $\mathbf{g}_n^{(s)}(t; u, q, z)$, count lecture hall partitions $\lambda \in \mathbf{L}_n^{(\mathbf{s})}$ with $\lambda_n \leq t$. What we want is to sum the number with $\lambda_n = t$. In order to do this, multiply the equation in Corollary 4 by $(1-x)$ to get the following.

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} u^{|\lceil \lambda \rceil|} q^{|\lambda|} z^{|\,\epsilon^+(\lambda)\,|} x^{\lambda_n} = (1-x) \sum_{t \geq 0} \mathbf{g}_n^{(s)}(t; u, q, z) x^t$$

$$= \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} u^{\mathrm{amaj}\, e}\, q^{\mathrm{lhp}\, e}\, z^{|\,e\,|}\, x^{s_n \mathrm{asc}\, e - e_n}}{\prod_{i=0}^{n-1} (1 - x^{s_n} u^{n-i} q^{s_{i+1} + \cdots + s_n})}.$$

□

In particular, setting $q = u = z = 1$, we have, for the first time, the generating function for $\mathbf{s}$ lecture hall partitions according to the size of their last part.

Corollary 5 *For any sequence* $\mathbf{s}$ *of positive integers,*

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} x^{\lambda_n} = \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{s_n \mathrm{asc}\, e - e_n}}{(1 - x^{s_n})^n}. \quad (18)$$

Previously, the best we knew was from [9] that

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} x^{\lceil \lambda_n \rceil} = \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\mathrm{asc}\, e}}{(1-x)^n}.$$

4.2 Computing $Q_n^{(\mathbf{s})}(x)$ for a special case

When $\mathbf{s} = (1, 2, \ldots, n)$, the $\mathbf{s}$-Eulerian polynomial $A_n^{(1,2,\ldots,n)}(x)$ is the usual Eulerian polynomial, $E_n(x)$, whose coefficient sequence is:

$n = 1:$ 1

$n = 2:$ 1, 1

$n = 3:$ 1, 4, 1

$n = 4:$ 1, 11, 11, 1

etc.

Then $Q_n^{(1,2,\ldots,n)}(x)$ is the inflated version of $E_n(x)$ and its coefficient sequence is:

$n = 1:$ 1

$n = 2:$ 1, 2, 1

$n = 3:$ 1, 2, 4, 4, 4, 2, 1

$n = 4:$ 1, 2, 4, 8, 11, 14, 16, 14, 11, 8, 4, 2, 1

etc.

Our results imply that the defining relation (2) for the Eulerian polynomials has the following analog for $Q_n^{(1,2,\ldots,n)}(x)$. We make use of the fact that from Proposition 1 we have an explicit formula for the Ehrhart quasi-polynomial $\mathbf{g}_n^{(1,2,\ldots,n)}(t)$.

Corollary 6 *The inflated* $(1, 2, \ldots, n)$*-Eulerian polynomials satisfy*

$$\frac{Q_n^{(1,2,\ldots,n)}(x)}{(1-x^n)^{n+1}} = \sum_{j\geq 0} [n]_{x(j+2)/(j+1)} (j+1)^n x^{nj},$$

where $[n]_w = 1 + w + w^2 + \cdots w^{n-1}$.

Proof. Combining Proposition 1 with (17), after writing t as $t = jn + i$ with $0 \leq i < n$,

$$\begin{aligned}\frac{Q_n^{(1,2,\ldots,n)}(x)}{(1-x^n)^{n+1}} &= \sum_{j\geq 0}\sum_{i=0}^{n-1}(j+1)^{n-i}(j+2)^i x^{nj+i} \\ &= \sum_{j\geq 0}(j+1)^n x^{nj}\sum_{i=0}^{n-1}\left(\frac{x(j+2)}{j+1}\right)^i \\ &= \sum_{j\geq 0}[n]_{x(j+2)/(j+1)}(j+1)^n x^{nj}.\end{aligned}$$

□

4.3 The inflated $1/k$-Eulerian polynomials

In [10], the **s**-Eulerian polynomials for the sequence

$$\mathbf{s} = (1,\ k+1,\ 2k+1,\ \ldots,\ (n-1)k+1)$$

were called the $1/k$-*Eulerian polynomials* and their properties were studied. For example, when $k = 2$, the coefficient sequences of the $1/k$-Eulerian polynomials are:

$n = 1:$ 1

$n = 2:$ 1, 2

$n = 3:$ 1, 10, 4

$n = 4:$ 1, 36, 60, 8

etc.

It can be checked that the inflated 1/2-Eulerian polynomials, $Q_n^{(1,3,5\ldots,2n-1)}(x)$, have the following coefficient sequences.

$n = 1:$ 1

$n = 2:$ 1, 2, 3, 2, 1

$n = 3:$ 1, 2, 4, 6, 9, 10, 11, 10, 9, 6, 4, 2, 1

$n = 4:$ 1, 2, 4, 8, 12, 18, 27, 36, 45, 54, 60, 66, 69, 66, 60, 54, 45, 36, 27, 18, 12, 8, 4, 2, 1

etc.

We can show an explicit formula for these polynomials, as we did in the special case $k = 1$ in the previous subsection. We take advantage of the fact that the Ehrhart quasi-polynomial of the $1/k$-rational lecture hall polytope was computed in [10].

Proposition 2 *([10], Thm. 5) Let* $\mathbf{s} = (1,\ k+1, \ldots, (n-1)k+1)$. *For integer* $t \geq 0$, *write* t *uniquely as* $t = js_n + i$, *where* $j \geq 0$ *and* $0 \leq i < s_n$. *Write* i *uniquely as* $i = d(n-1) + r$, *where* $d \geq 0$ *and* $0 \leq r < n - 1$. *Then*

$$\begin{aligned} \mathbf{g}_n^{(\mathbf{s})}(t) &= \mathbf{g}_n^{(\mathbf{s})}(j((n-1)k+1) + d(n-1) + r) \\ &= (-1)^j \sum_{p=0}^{j} \binom{\frac{1}{k} - 1}{j - p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+d+1)^{n-1-r}(kp+d+2)^r. \end{aligned}$$

With this, we can compute the Ehrhart series of the rational lecture hall polytope for this sequence $\mathbf{s}$. Observe that

$$\sum_{t \geq 0} \mathbf{g}_n^{(\mathbf{s})}(t)\ x^t \quad = \quad \sum_{j \geq 0} \sum_{d,r} \mathbf{g}_n^{(\mathbf{s})}(j((n-1)k+1) + d(n-1) + r)\ x^{js_n + d(n-1) + r},$$

where the inner sum is over all pairs (d, r) with $0 \leq d(n-1) + r < k(n-1) + 1 = s_n$.

Corollary 7 *Let* $\mathbf{s} = (1,\ k+1, \ldots, (n-1)k+1)$. *For any integer* $k \geq 1$,

$$\frac{Q_n^{(\mathbf{s})}(x)}{(1 - x^{s_n})^{n+1/k}} \quad = \quad \sum_{p \geq 0} \binom{p - 1 + 1/k}{p} (kp+1) \sum_{d,r} (kp+d+1)^{n-1-r}(kp+d+2)^r\ x^{ps_n + d(n-1) + r},$$

where the inner sum is over all pairs (d, r) *with* $0 \leq d(n-1) + r < k(n-1) + 1 = s_n$.

Proof. Apply Proposition 2 and (17). Observe that

$$\sum_{j \geq 0} \sum_{d,r} (-1)^j \sum_{p=0}^{j} \binom{\frac{1}{k} - 1}{j - p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+d+1)^{n-1-r}(kp+d+2)^r\ x^{js_n + d(n-1) + r}$$

can be rewritten, using the binomial theorem, as

$$(1-x^{s_n})^{1/k-1}\sum_{p\geq 0}\binom{p-1+1/k}{p}(kp+1)\sum_{d,r}(kp+d+1)^{n-1-r}(kp+d+2)^r\ x^{ps_n+d(n-1)+r},$$

and the result follows. $\square$

5 Symmetry and unimodality questions for $Q_n^{(\mathbf{s})}(x)$

We first give an example to show that $Q_n^{(\mathbf{s})}(x)$ does indeed contain more information about the s-lecture hall partitions than does $A_n^{(\mathbf{s})}(x)$. It can be checked that the (integral) s-lecture hall polytopes $\mathbf{P}_4^{(1,4,2,3)}$, $\mathbf{P}_4^{(2,4,1,3)}$, and $\mathbf{P}_4^{(1,2,4,3)}$ all give rise to the same s-Eulerian polynomial:

$$A_4^{(1,4,2,3)}(x) \ = \ A_4^{(2,4,1,3)}(x) \ = \ A_4^{(1,2,4,3)}(x) \ = \ 1+8x+13x^2+2x^3.$$

However, the rational s-lecture hall polytopes $\mathbf{R}_4^{(1,4,2,3)}$, $\mathbf{R}_4^{(2,4,1,3)}$, and $\mathbf{R}_4^{(1,2,4,3)}$ give rise to different inflated s-Eulerian polynomials:

$$\begin{aligned}
Q_4^{(1,4,2,3)}(x) &= 1+2x+6x^2+11x^3+16x^4+16x^5+11x^6+6x^7+2x^8+x^9\\
Q_4^{(2,4,1,3)}(x) &= 1+2x+3x^2+8x^3+13x^4+18x^5+13x^6+8x^7+3x^8+2^9+x^{10}\\
Q_4^{(1,2,4,3)}(x) &= 1+3x+7x^2+12x^3+16x^4+16x^5+11x^6+5x^7+x^8.
\end{aligned}$$

In Ehrhart theory and algebraic geometry, one studies the polynomials such as $A_n^{(\mathbf{s})}(x)$ and $Q_n^{(\mathbf{s})}(x)$ arising in the Ehrhart series of a polytope. The sequence of coefficients of the polynomial is referred to as the h^*-vector of the polytope and properties such as symmetry and unimodality of the h^*-vector are investigated.

As is evident from the examples in this section and in Section 4, the sequence of coefficients of $A_n^{(\mathbf{s})}(x)$ need not be symmetric. The same applies to $Q_n^{(\mathbf{s})}(x)$, as can be seen in the case $\mathbf{s}=(1,2,4,3)$ above. However, in all examples appearing in this paper, the h^*-vector is unimodal.

We make a few observations and conjectures concerning symmetry of inflated $\mathbf{s}$-Eulerian polynomials for special sequences $\mathbf{s}$.

In Section 4.3, we saw that the coefficient sequences of the 1/2-Eulerian polynomials are not always symmetric. However, at least for $n\leq 5$, the coefficient sequences of the *inflated* 1/2-Eulerian polynomials *are* symmetric. It is interesting that the refined information encoded in the inflated Eulerian polynomials seems to reveal symmetry that was not evident in the “uninflated” version.

Our experiments suggest the following:

Conjecture 1 *For every $k \geq 1$, and every $n \geq 0$, the coefficient sequences of the inflated $1/k$-Eulerian polynomials are symmetric and unimodal.*

Our experiments are limited in that to compute $Q_n^{(\mathbf{s})}(x)$, our program takes time proportional to $s_1 s_2 \cdots s_n s_n$.

Fix integer $\ell \geq 2$ and define the *ℓ-sequence* $\{a_n^{(\ell)}\}$ by $a_n^{(\ell)} = \ell a_{n-1}^{(\ell)} - a_{n-2}^{(\ell)}$, with initial conditions $a_1^{(\ell)} = 1$ and $a_2^{(\ell)} = \ell$. For example, $\{a_n^{(2)}\} = (1, 2, 3, 4, \ldots)$ and $\{a_n^{(3)}\} = (1, 3, 8, 21, 55 \ldots)$. Bousquet-Mèlou and Eriksson found in [2] that when $\mathbf{s}$ is an ℓ-sequence, the generating function for the $\mathbf{s}$-lecture hall partitions has a particularly simple form. Our experiments lead us to believe the following:

Conjecture 2 *If $\mathbf{s}$ is an ℓ-sequence, then for every $n \geq 1$, the coefficient sequence of the inflated $\mathbf{s}$-Eulerian polynomial $Q_n^{(\mathbf{s})}(x)$ is symmetric and unimodal.*

It is not the case that the h^*-vector of $\mathbf{R}_n^{(\mathbf{s})}$ is symmetric for every sequence defined by a second order linear recurrence. For the Fibonacci sequence, $\mathbf{s} = (1, 1, 2, 3, 5, 8, 13, \ldots)$, the coefficient sequences for $Q_n^{(\mathbf{s})}(x)$, $1 \leq n \leq 5$ are:

$n = 1:$ 1

$n = 2:$ 1

$n = 3:$ 1, 2, 1

$n = 4:$ 1, 2, 4, 4, 4, 2, 1

$n = 5:$ 1, 2, 4, 8, 12, 15, 18, 20, 18, 16, 13, 10, 6, 4, 2, 1.

In fact, the ℓ-sequences seem to be unique in the following sense.

Conjecture 3 *Let ℓ be a positive integer and let b be a nonzero integer satisfying $b > -\ell$. Let $\mathbf{s}$ be defined by the recurrence $s_n = \ell s_{n-1} + b s_{n-2}$ with initial conditions $s_1 = 1$, $s_2 = \ell$. Then unless $b = -1$, there exists a positive integer n, for which the coefficient sequence of $Q_n^{(\mathbf{s})}(x)$ is not symmetric.*

To the extent we have been able to test, symmetry fails for relatively small n.

Finally, is it true that for every sequence $\mathbf{s}$ of positive integers, the h^*-vector of the rational s-lecture hall polytope is unimodal? Within our limitations, we have tested many sequences $\mathbf{s}$, including ones that were not themselves monotone and found no counterexamples.

6 Concluding remarks

In this paper we have introduced rational s-lecture hall polytopes and inflated s-Eulerian polynomials. We proved a characterization of the h^*-vector of the rational s-lecture hall polytope in terms of statistics on s-inversion sequences. We applied the results to derive a generating function for s-lecture hall partitions which tracks the size of the last part. For special cases where a formula for the Ehrhart quasi-polynomials was known we could in derive a formula for the inflated s-Eulerian polynomials.

By implementing the combinatorial characterization of the inflated s-Eulerian polynomial, we explicitly computed them and observed properties such as symmetry and unimodality.

In ongoing work we continue to investigate the inflated s-Eulerian polynomials, including the symmetry and unimodality questions proposed in Section 5, in order to understand more about the $\mathbf{s}$-lecture hall partitions.

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