On the existence of symmetric chain decompositions in a quotient of the Boolean lattice

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Abstract

We highlight a question about binary necklaces, i.e., equivalence classes of binary strings under rotation. Is there a way to choose representatives of the n-bit necklaces so that the subposet of the Boolean lattice induced by those representatives has a symmetric chain decomposition? Alternatively, is the quotient of the Boolean lattice $\mathcal{B}_n$, under the action of the cyclic group Z_n, a symmetric chain order? The answer is known to be yes for all prime n and for composite $n \le 16$, but otherwise the question appears to be open. In this note we describe how it suffices to focus on subposets induced by periodic necklaces, substantially reducing the size of the problem. We mention a motivating application: determining whether minimum-region rotationally symmetric independent families of n curves exist for all n.

1 The problem

The Boolean lattice $\mathcal{B}_n$ is the collection of subsets of $[n] = \{1, 2, \ldots, n\}$, ordered by inclusion. An element S of $\mathcal{B}_n$ can be viewed as an n-bit string whose ith bit is 1 iff $i \in S$. Then $|S|$ is the cardinality of S, or the number of '1' bits in the corresponding string.

A *chain* in $\mathcal{B}_n$ is a sequence $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_t$ of elements of $\mathcal{B}_n$ such that $|S_i| = |S_{i-1} + 1|$. The chain is *symmetric* if $|S_1| + |S_t| = n$. A *symmetric chain decomposition* (SCD) of $\mathcal{B}_n$ is a partition of the elements of $\mathcal{B}_n$ into symmetric chains.

It is known that $\mathcal{B}_n$ has an SCD for every $n \ge 0$ and one construction, due to Greene and Kleitman [1], works as follows. Regard the elements of $\mathcal{B}_n$ as binary strings. View '1' bits

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as right parentheses and '0' bits as left parentheses and in each string, match parentheses in the usual way. Grow a chain by starting with a string x with no unmatched '1'. Change the first unmatched '0' in x to '1' to get its successor, y. Change the first unmatched '0' in y (if any) to '1' to get its successor. Continue until a string with no unmatched '0' is reached. Label the resulting chain by its first element.

For example, using this rule, the following complete list of chains for $\mathcal{B}_5$ is an SCD:

C_{00000} : 00000 → 10000 → 11000 → 11100 → 11110 → 11111

C_{01000} : 01000 → 01100 → 01110 → 01111

C_{01010} : 01010 → 01011

C_{01001} : 01001 → 01101

C_{00100} : 00100 → 10100 → 10110 → 10111

C_{00110} : 00110 → 00111

C_{00101} : 00101 → 10101

C_{00010} : 00010 → 10010 → 11010 → 11011

C_{00011} : 00011 → 10011

C_{00001} : 00001 → 10001 → 11001 → 11101

But consider a variation. For $x = x_1x_2\cdots x_n$, let σ denote the rotation of x defined by $\sigma(x) = x_2x_3\cdots x_nx_1$. Let $\sigma^1 = \sigma$, and $\sigma^i(x) = \sigma(\sigma^{i-1}(x))$, where $i > 1$. Define the relation $\triangle$ on the elements of $\mathcal{B}_n$ by $x\triangle y$ iff $y = \sigma^i(x)$ for some $i \geq 0$. Then $\triangle$ is an equivalence relation that partitions the elements of $\mathcal{B}_n$ into equivalence classes called *necklaces*.

The question is: Is there a way to choose a set R_n of necklace representatives, one from each necklace, so that the subposet of $\mathcal{B}_n$ induced by R_n has an SCD?

It turns out the answer is *yes* when n is prime. An explicit construction was found in [2] by introducing the idea of a *block code* to select necklace representatives, coupled with a variation of the Greene-Kleitman rule for building chains.

Define the *block code* $\beta(x)$ of a binary string x as follows. If x starts with 0 or ends with 1, then $\beta(x) = (\infty)$. Otherwise, x can be written in the form:

$$x = 1^{a_1}0^{b_1}1^{a_2}0^{b_2}\cdots 1^{a_t}0^{b_t}$$

for some $t > 0$, where $a_i > 0$, $b_i > 0$, $1 \leq i \leq t$, in which case,

$$\beta(x) = (a_1 + b_1, a_2 + b_2, \ldots, a_t + b_t).$$

As an example, the block codes of the string 1110101100010 and all of its rotations are shown below.

bit string	block code	bit string	block code
1110101100010	$(4,2,5,2)$	1100010111010	$(5,2,4,2)$
0111010110001	(∞)	0110001011101	(∞)
1011101011000	$(2,4,2,5)$	1011000101110	$(2,5,2,4)$
0101110101100	(∞)	0101100010111	(∞)
0010111010110	(∞)	1010110001011	(∞)
0001011101011	(∞)	1101011000101	(∞)
1000101110101	(∞)		

When n is prime, every n-bit string, other than 0^n and 1^n, has n distinct rotations. Furthermore, it is shown in [2] that when n is prime, no two different rotations of an n-bit string can have the same *finite* block code. Assuming that block codes are ordered lexicographically, in each necklace of n-bit strings (except 0^n, 1^n) the unique string with minimum block code can be chosen as the representative, *when n is prime.*

For n prime, let R_n be the set of n-bit strings that are the minimum-block-code representatives of their necklaces. Build chains as follows: Start with a string $x \in R_n$. If there is more than one unmatched '0' in x, change the first unmatched '0' to '1' to get its successor, y. If there is more than one unmatched '0' in y, change the first unmatched '0' in y to '1' to get its successor. Continue until a string with only one unmatched '0' is reached.

Note that a node x and its successor y have the *same* block code, so if x has the minimum block code among all of its rotations, then so does y. Thus every element of x's chain is the (minimum-block-code) representative of its necklace.

For example, using this rule to select the necklace representatives of $\mathcal{B}_7$ for R_7, the following complete list of chains for the subposet of $\mathcal{B}_7$ induced by R_7 is an SCD:

$C_{1000000}$: 1000000 → 1100000 → 1110000 → 1111000 → 1111100 → 1111110

$C_{1010000}$: 1010000 → 1011000 → 1011100 → 1011110

$C_{1010100}$: 1010100 → 1010110

$C_{1001000}$: 1001000 → 1101000 → 1101100 → 1101110

$C_{1001100}$: 1001100 → 1001110.

(We can include 0^7 and 1^7 by attaching them at the beginning and end of $C_{1000000}$.)

It is shown in [2] that when n is prime, this gives a symmetric chain decomposition of the subposet of $\mathcal{B}_n$ induced by R_n. This was a key element in [2] to show that rotationally symmetric Venn diagrams for n sets exist for all prime n.

When n is composite, symmetric Venn diagrams cannot exist for composite n, but the next best thing would be to show the existence of a rotationally symmetric independent family of curves with the minimum number of regions as discussed by Grünbaum in [4] and [5]. One approach is to pursue the SCD question for necklaces when n is composite.

When n is prime, every necklace of n-bit strings (other than 0^n and 1^n) has exactly n elements. This does not hold for composite n. When $n = 6$, for example, the necklace of 100000 has 6 elements, whereas the necklace of 100100 has only 3 elements and the necklace of 101010 has only 2.

Nevertheless, working by hand, for small composite n, it is not hard to find a set R_n of necklace representatives whose induced subposet has an SCD. When $n = 6$, for example, the following set of symmetric chains contain exactly one representative from each necklace of 6-bit strings.

$000000 \rightarrow 100000 \rightarrow 110000 \rightarrow 111000 \rightarrow 111100 \rightarrow 111110 \rightarrow 111111$

$101000 \rightarrow 101010 \rightarrow 101110$

$100100 \rightarrow 110100 \rightarrow 110110$

101100

Constructions for $n = 4, 6, 8, 9$ appear in the thesis of Weston [9]. In [6], Jiang was able to find constructions for $n = 12, 13, 15, 16$, by using a substantial simplification of the problem. We outline this approach below, but refer to [6] for details and proofs.

Define the block code $\beta(\eta)$ of a *necklace* η to be the minimum block code of any string in η. So, e.g., the block code of the necklace containing $x = 0101001011$ is $(2, 3, 2, 3)$. But note that this is the block code for *both* of the rotations 1011010100 and 1010010110 of x.

Now define $\mathcal{B}_n^{(\alpha)}$ to be the subposet of $\mathcal{B}_n$ induced by n-bit strings belonging to necklaces η with $\beta(\eta) = \alpha$. (If α is not the block code for any necklace, then $\mathcal{B}_n^{(\alpha)}$ is empty.) So strings in the same necklace are in the same $\mathcal{B}_n^{(\alpha)}$. Note that $\mathcal{B}_n^{(\alpha)}$ embeds symmetrically in $\mathcal{B}_n$, i.e., $\mathcal{B}_n^{(\alpha)}$ itself is symmetric about its middle levels and the middle levels of $\mathcal{B}_n^{(\alpha)}$ are contained in the middle levels of $\mathcal{B}_n$.

Thus, it suffices to focus on the posets $\mathcal{B}_n^{(\alpha)}$, for fixed α, and ask if there is a way to choose a set of necklace representatives $R_n^{(\alpha)}$ for strings in $\mathcal{B}_n^{(\alpha)}$ so that the subposet of $\mathcal{B}_n^{(\alpha)}$ induced by $R_n^{(\alpha)}$ has an SCD.

It is shown in [6] that this is always possible if α, regarded as a string, is aperiodic. (A string $x = x_1x_2 \ldots x_n$ is *periodic* if $\sigma^i(x) = x$ for some i with $0 < i < n$. So, e.g., block code $(2, 3, 2, 3)$ is periodic, but $(2, 2, 3, 3)$ is aperiodic.) Since no two distinct elements of a necklace can have the same (finite) *aperiodic* block code, the same rule for choosing necklace representatives and growing chains works as the one for prime n.

So, finally, only periodic block codes need be considered. Fortunately, there are not so many. When $n = 16$, there are only the 7 shown in the table below. One need only find a rule for choosing necklace representatives and for identifying initial elements of chains that would work with the Greene-Kleitman successor rule (or find a new rule compatible with the chosen representatives.) Unfortunately, because two different strings in the same necklace can have the same *periodic* block code, it is not at all obvious how to do this. Nevertheless, it was not difficult, using ad hoc techniques, to find SCDs for the necklaces of $\mathcal{B}_n^{(\alpha)}$ for all periodic block codes α for all $n \leq 16$. These are displayed in [6].

Periodic block code α	Number of necklaces with block code α
$(2,2,2,2,2,2,2,2)$	1
$(2,2,4,2,2,4)$	6
$(2,3,3,2,3,3)$	10
$(2,6,2,6)$	15
$(4,4,4,4)$	24
$(8,8)$	28
$(3,5,3,5)$	36

2 An application: independent families of curves

A collection of n curves in the plane is an *independent family* if, in the regions formed by the intersections of the interiors of the curves, every subset of $[n]$ is represented at least once. (For a Venn diagram, this is *exactly* once.) In [5], Grünbaum shows that an independent family of curves must have at least $2 + n(N_n - 2)$ regions, where N_n is the number of n-bit necklaces. He shows also that rotationally symmetric independent families of n curves exist for all n. But he asks if it is possible to find, for every n, a rotationally symmetric independent family of n curves with only $2 + n(N_n - 2)$ regions.

It turns out that solving the SCD question for necklaces for composite n will nearly settle Grünbaum's question in the same way that the SCD for necklaces for prime n settled the existence of rotationally symmetric Venn diagrams in [2]. "Nearly" because the SCD must have an additional "chain cover property". The SCDs in [6] and [9] have this property, so, as a result, they settled Grünbaum's question for $n \leq 16$. We refer to [6] and [9] for details and diagrams.

It is shown in [6] that if an SCD with the chain cover property can be found for n-bit necklaces with any given periodic block code, the resulting collection of chains for all n-bit necklaces will have the chain cover property and therefore produce a symmetric independent family of n curves with the minimum number of regions.

3 Concluding remarks

We note that in the case of prime n there is an existential proof that the quotient $\mathcal{N}_n$ of $\mathcal{B}_n$ under the action of Z_n has a symmetric chain decomposition. Stanley has shown [8] that any quotient of the Boolean lattice is a Peck poset (rank symmetric, rank unimodal, and strongly Sperner). When n is prime, $\mathcal{N}_n$ can be shown to have the *LYM property* and Griggs showed in [3] that a Peck poset with this property has an SCD.

Finally, for an overview of Venn diagrams, independent families of curves, and variations, the survey of Ruskey [7] is an excellent resource.

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