# Symmetrically Constrained Compositions

Matthias Beck [*]
Department of Mathematics
San Francisco State University
San Francisco, CA 94132
beck@math.sfsu.edu

Ira M. Gessel
Department of Mathematics
Brandeis University
Waltham, MA 02454–9110
gessel@brandeis.edu

Sunyoung Lee [†]
Computer Science
N. C. State University
Raleigh, NC 27695
slee7@unity.ncsu.edu

Carla D. Savage [‡]
Computer Science
N. C. State University
Raleigh, NC 27695
savage@csc.ncsu.edu

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*Dedicated to George Andrews on the occasion of his 70th birthday.*

**Abstract**

Given integers $a_1, a_2, \ldots, a_n$, with $a_1 + a_2 + \cdots + a_n \geq 1$, a symmetrically constrained composition $\lambda_1 + \lambda_2 + \cdots + \lambda_n = M$ of $M$ into $n$ nonnegative parts is one that satisfies each of the the $n!$ constraints $\left\{ \sum_{i=1}^{n} a_i \lambda_{\pi(i)} \geq 0 : \pi \in S_n \right\}$. We show how to compute the generating function of these compositions, combining methods from partition theory, permutation statistics, and lattice-point enumeration.

Keywords: symmetrically constrained composition, partition analysis, permutation statistics, generating function, lattice-point enumeration.

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# 1 Introduction

## 1.1 Constrained Compositions

This work was inspired by the "constrained compositions" introduced by Andrews, Paule, and Riese in [4]. We consider the problem of enumerating *symmetrically constrained compositions*, that is, compositions of an integer $M$ into $n$ nonnegative parts

$$M = \lambda_1 + \lambda_2 + \cdots + \lambda_n = |\lambda| \,,$$

where the sequence $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is constrained to satisfy a symmetric system of linear inequalities. For example, the compositions $\lambda_1 + \lambda_2 + \lambda_3$ of $M$ satisfying

$$\lambda_{\pi(1)} + \lambda_{\pi(2)} \geq \lambda_{\pi(3)} \tag{1}$$

for every permutation $\pi \in S_3$, are known as *integer-sided triangles* of perimeter $M$ [1, 2, 11, 14]. The number $\Delta_M$ of *incongruent* triangles of perimeter $M$ is given by

$$\sum_{M \geq 0} \Delta_M \, q^M \;=\; \sum_{\substack{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 \\ \lambda_2 + \lambda_3 \geq \lambda_1}} q^{|\lambda|} \;=\; \frac{1}{(1-q^2)(1-q^3)(1-q^4)} \,.$$

However, $3+2+1$ and $2+3+1$ are different *compositions* (i.e., ordered partitions) of 6 and counting the number $\Delta^*_M$ of *ordered* solutions to (1) gives

$$\sum_{M \geq 0} \Delta^*_M \, q^M \;=\; \sum_{\substack{\lambda_1 + \lambda_2 \geq \lambda_3 \\ \lambda_1 + \lambda_3 \geq \lambda_2 \\ \lambda_2 + \lambda_3 \geq \lambda_1}} q^{|\lambda|} \;=\; \frac{1 + 2q^2 + 2q^4 + q^6}{(1-q^2)(1-q^3)(1-q^4)} \;=\; \frac{1}{(1-q^2)^2(1-q)} \,. \tag{2}$$

One could generalize this example in several ways. For example, moving to $n$ dimensions, one could ask for the integer sequences $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying

$$\lambda_{\pi(1)} + \lambda_{\pi(2)} + \cdots + \lambda_{\pi(n-1)} \geq \lambda_{\pi(n)}$$

for all $n!$ permutations $\pi \in S_n$. Another generalization would be to study, given positive integers $k, l, m$, the integer sequences $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying

$$k\lambda_{\pi(1)} + l\lambda_{\pi(2)} \geq m\lambda_{\pi(3)} \qquad \text{for all } \pi \in S_n \,.$$

Another example, considered in [4], was inspired by a Putnam exam problem [12, Problem B3]: Enumerate all compositions of $M = \lambda_1 + \lambda_2$ into two parts satisfying

$$2\lambda_1 \geq \lambda_2 \qquad \text{and} \qquad 2\lambda_2 \geq \lambda_1 \,.$$

It is shown that

$$\sum_{\substack{2\lambda_1 \geq \lambda_2 \\ 2\lambda_2 \geq \lambda_1}} x^{\lambda_1} y^{\lambda_2} = \frac{1 + xy + x^2y^2}{(1 - xy^2)(1 - x^2y)}, \tag{3}$$

giving a complete parametrization of all solutions.

In [4], Andrews, Paule, and Riese demonstrate the suitability of the Omega package [3] for experimenting with problems of this sort and the power of Macmahon's partition analysis [2] to prove some elegant generalizations.

The goal of this paper is (1) to formulate a generalization of the symmetrically constrained compositions enumeration problem; (2) to show how this problem is connected to permutation statistics; (3) to show that the permutation statistics approach gives, for many cases, an effective computation method and, for certain cases, a way to derive compact formulas; and (4) to show that the insight provided by the geometry of lattice-point enumeration aids in the handling of the most general case.

### 1.2 The Symmetrically Constrained Compositions Enumeration Problem

Fix integers $a_1, a_2, \ldots, a_n$. We are interested in enumerating compositions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ that satisfy the $n!$ homogeneous linear constraints

$$a_1 \lambda_{\pi(1)} + a_2 \lambda_{\pi(2)} + \cdots + a_n \lambda_{\pi(n)} \geq 0 \qquad \text{for all } \pi \in S_n .$$

Specifically, we are interested in computing the generating functions

$$F(z_1, z_2, \ldots, z_n) := \sum_{\lambda} z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_n^{\lambda_n}$$

and

$$F(q) := F(q, q, \ldots, q) = \sum_{\lambda} q^{\lambda_1 + \lambda_2 + \cdots + \lambda_n},$$

by exploiting the symmetry of the constraints. Note that because of the symmetry, there is no loss of generality in assuming that

$$a_1 \leq a_2 \leq \cdots \leq a_n ,$$

which we will do from now on.

In Section 2, we show how to solve the enumeration problem when $\sum_{i=1}^n a_i = 1$. In certain special cases, we show that permutation statistics can be used to derive elegant formulas. We note that even this simple case is difficult for general purpose software like the Omega Package [3], Xin's improvement of Omega [15], and LattE macchiato [10, 13], designed to enumerate solutions to linear Diophantine equations and inequalities. In Section 3 we solve the general problem. We close this section with some notation and background on permutation statistics.

### 1.3 Permutation Statistics

Throughout the paper, the following notation is used: $[\,n\,]_q = (1-q^n)/(1-q)$; $[\,n\,]_q! = \prod_{i=1}^{n}[\,i\,]_q$; and $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$.

For a permutation $\pi = \pi(1)\pi(2)\cdots\pi(n)$ of $[n] := \{1,2,\ldots,n\}$, the *descent set of* $\pi$ is

$$D_\pi = \{\, j : \pi(j) > \pi(j+1)\,\}\,.$$

The statistic $\operatorname{des}(\pi) = |D(\pi)|$ is the number of descents of $\pi$ and the *major index* of $\pi$ is the sum of the descent positions: $\operatorname{maj}(\pi) = \sum_{i\in D(\pi)} i$. It is well known that

$$\sum_{\pi\in S_n} q^{\operatorname{maj}(\pi)} \;=\; \prod_{i=0}^{n}\frac{1-q^i}{1-q} = [\,n\,]_q! \tag{4}$$

(see, e.g., [14]). The joint distribution of $\operatorname{des}(\pi)$ and $\operatorname{maj}(\pi)$ over the set $S_n$ of all permutations of $[n]$ is given by Carlitz's $q$-Eulerian polynomial [6, 7]:

$$C_n(x,q) = \sum_{\pi\in S_n} x^{\operatorname{des}(\pi)} q^{\operatorname{maj}(\pi)} \;=\; \prod_{i=0}^{n}(1-xq^i)\sum_{j=1}^{\infty}[\,j\,]_q^n\, x^{j-1}.$$

Applying the definition of $[\,j\,]_q$ and the binomial theorem, we can rewrite this as

$$C_n(x,q) = \frac{(x;q)_{n+1}}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}\frac{(-q)^i}{1-q^i x}\,. \tag{5}$$

So, for example,

$$\begin{aligned} C_1(x,q) &= 1\\ C_2(x,q) &= 1+xq\\ C_3(x,q) &= 1+2xq+2xq^2+x^2q^3. \end{aligned} \tag{6}$$

If we take the limit as $x \to q^{-n}$ in (5) all terms except $i=n$ in the sum are canceled by $(q^{-n};q)_{n+1} = 0$ in the numerator, so

$$C_n(q^{-n},q) = \frac{(-q)^n}{(1-q)^n}\lim_{x\to q^{-n}}\frac{(x;q)_{n+1}}{1-q^n x} = \frac{(-q)^n}{(1-q)^n}\lim_{x\to q^{-n}}(x;q)_n = \frac{(-q)^n}{(1-q)^n}(q^{-n};q)_n\,. \tag{7}$$

Finally, for $i \le n-1$, let $S_n^{(i)}$ be the set of permutations of $[n]$ that have no descent in positions $\{n-i, n-i+1, ..., n-1\}$. Let

$$C_n^{(i)}(x,q) := \sum_{\pi\in S_n^{(i)}} x^{\operatorname{des}(\pi)} q^{\operatorname{maj}(\pi)}.$$

In [8], it is shown that

$$C_n^{(i)}(x,q) = \frac{C_n(x,q)}{(xq^{n-i};q)_i} - \sum_{k=1}^{i} \binom{n}{k} xq^{n-k} \frac{C_{n-k}(x,q)}{(xq^{n-i};q)_{i-k+1}}$$

so, in particular,

$$C_n^{(1)}(x,q) = \frac{C_n(x,q) - nxq^{n-1}C_{n-1}(x,q)}{1-xq^{n-1}}\,. \tag{8}$$

# 2 Symmetrically Constrained Compositions when $\sum a_i = 1$

## 2.1 The Main Theorem

**Theorem 1.** *Given integers $a_1 \leq a_2 \leq \cdots \leq a_n$ satisfying $\sum_{i=1}^n a_i = 1$, the generating function for those $\lambda \in \mathbb{Z}^n_{\geq 0}$ satisfying*

$$\sum_{j=1}^{n} a_j \lambda_{\pi(j)} \geq 0 \qquad \text{for all } \pi \in S_n$$

*is*

$$F(z_1, z_2, \ldots, z_n) = \sum_{\pi \in S_n} \frac{\prod_{j \in D_\pi} (z_1^{b_{1,j}} z_2^{b_{2,j}} \cdots z_n^{b_{n,j}})}{\prod_{j=1}^{n} (1 - z_1^{b_{1,j}} z_2^{b_{2,j}} \cdots z_n^{b_{n,j}})}$$

*where*

$$b_{i,j} = \begin{cases} 1 & \text{if } j = n, \\ -(a_1 + \cdots + a_j) & \text{if } n \geq i > j \geq 1, \\ 1 - (a_1 + \cdots + a_j) & \text{if } 1 \leq i \leq j < n. \end{cases}$$

*In particular, setting $z_1 = \cdots = z_n = q$ yields*

$$F(q) = \frac{\sum_{\pi \in S_n} \prod_{j \in D_\pi} q^{j - n\sum_{i=1}^{j} a_i}}{(1-q^n) \prod_{j=1}^{n-1} (1 - q^{j - n\sum_{i=1}^{j} a_i})}\,.$$

**Proof.** To simplify notation, let

$$F(z) = F(z_1, z_2, \ldots, z_n)\,.$$

For $b \in \mathbb{Z}^n$, let

$$z^b = z_1^{b_1} z_2^{b_2} \cdots z_n^{b_n}$$

and for $\pi \in S_n$, let

$$z_\pi = (z_{\pi(1)}, z_{\pi(2)}, \ldots, z_{\pi(n)})\,.$$

With

$$L := \left\{ \lambda \in \mathbb{Z}^n_{\geq 0} : \sum_{j=1}^{n} a_j \lambda_{\pi(j)} \geq 0 \text{ for all } \pi \in S_n \right\}$$

we have

$$F(z) = \sum_{\lambda \in L} z^{\lambda}.$$

Now we use the standard method of partitioning the elements of $L$ into classes $L_\pi$ indexed by permutations $\pi \in S_n$:

$$\begin{aligned} L_\pi = \Big\{ \lambda \in \mathbb{Z}^n \quad : \quad & \lambda_{\pi(1)} \geq \lambda_{\pi(2)} \geq \cdots \geq \lambda_{\pi(n)}, \\ & \sum_{i=1}^{n} a_i \lambda_{\sigma(i)} \geq 0 \ \text{ for all } \sigma \in S_n, \text{ and} \\ & \lambda_{\pi(i)} > \lambda_{\pi(i+1)} \text{ if } i \in D_\pi \Big\}. \end{aligned}$$

Since the last condition guarantees that no $\lambda$ is in more than one class, $L$ is the disjoint union

$$L = \bigcup_{\pi \in S_n} L_\pi \, .$$

Our goal now simplifies to computing

$$F_\pi(z) := \sum_{\lambda \in L_\pi} z^{\lambda},$$

because $F(z) = \sum_{\pi \in S_n} F_\pi(z)$. In $L_\pi$, since

$$\lambda_{\pi(1)} \geq \lambda_{\pi(2)} \geq \cdots \geq \lambda_{\pi(n)}$$

and since, by our assumption, $a_1 \leq a_2 \leq \cdots \leq a_n$, the $n!$ constraints

$$a_1 \lambda_{\sigma(1)} + a_2 \lambda_{\sigma(2)} + \cdots + a_n \lambda_{\sigma(n)} \geq 0 \qquad \text{for all } \sigma \in S_n$$

are all implied by the single constraint

$$a_1 \lambda_{\pi(1)} + a_2 \lambda_{\pi(2)} + \cdots + a_n \lambda_{\pi(n)} \geq 0 \, ,$$

so that we get the more compact description

$$L_\pi = \left\{ \lambda \in \mathbb{Z}^n : \begin{array}{l} \lambda_{\pi(1)} \geq \lambda_{\pi(2)} \geq \cdots \geq \lambda_{\pi(n)} \geq 0 \ \text{ and } \ \lambda_{\pi(j)} > \lambda_{\pi(j+1)} \text{ if } j \in D_\pi \\ a_1 \lambda_{\pi(1)} + a_2 \lambda_{\pi(2)} + \cdots + a_n \lambda_{\pi(n)} \geq 0 \end{array} \right\}.$$

But this means that all $L_\pi$ look similar, except for the strict inequalities determined by $D_\pi$. More precisely, if we let

$$\widetilde{L}_\pi := \{ \lambda \in L_{\mathrm{Id}} : \lambda_j > \lambda_{j+1} \text{ if } j \in D_\pi \}$$

and $G_\pi(z) := \sum_{\lambda \in \widetilde{L}_\pi} z^\lambda$, then

$$F_\pi(z) = G_\pi(z_\pi).$$

So it remains to find $G_\pi(z)$, the generating function for

$$\widetilde{L}_\pi = \left\{ \lambda \in \mathbb{Z}^n : \begin{array}{l} \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \;\text{ and }\; \lambda_j > \lambda_{j+1} \text{ if } j \in D_\pi \\ a_1\lambda_1 + a_2\lambda_2 + \cdots + a_n\lambda_n \geq 0 \end{array} \right\},$$

for a given $\pi \in S_n$.

The constraints of $\widetilde{L}_\pi$ are given by the system

$$\begin{bmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & & \ddots & & \\ & & & & 1 & -1 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \end{bmatrix} \lambda \;\geq\; \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n-1} \\ e_n \end{bmatrix}, \tag{9}$$

where

$$e_j = \begin{cases} 0 & \text{if } j \notin D_\pi, \\ 1 & \text{if } j \in D_\pi. \end{cases}$$

We make use of the following lemma, a well known result in lattice-point enumeration. This version was formulated in [9] for easy application to partition and composition enumeration problems.

**Lemma 1.** *Let $C = [c_{i,j}]$ be an $n \times n$ matrix of integers such that $C^{-1} = B = [b_{i,j}]$ exists and $b_{i,j}$ are all nonnegative integers. Let $e_1, \ldots, e_n$ be nonnegative integer constants. For each $1 \leq i \leq n$, let $c_i$ be the constraint*

$$c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \cdots + c_{i,n}\lambda_n \geq e_i.$$

*Let $S_C$ be the set of nonnegative integer sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying the constraints $c_i$ for all $i$, $1 \leq i \leq n$. Then the generating function for $S_C$ is:*

$$F_C(x_1, x_2, \ldots, x_n) = \sum_{\lambda \in S_C} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} = \frac{\prod_{j=1}^n (x_1^{b_{1,j}} x_2^{b_{2,j}} \cdots x_n^{b_{n,j}})^{e_j}}{\prod_{j=1}^n (1 - x_1^{b_{1,j}} x_2^{b_{2,j}} \cdots x_n^{b_{n,j}})}.$$

Now let $C$ be the matrix on the left side of (9). Then $\det(C) = a_1 + \cdots + a_n = 1$, so $C$ is invertible and $B = C^{-1}$ has all integer entries:

$$b_{i,j} = \begin{cases} 1 & \text{if } j = n, \\ -(a_1 + \cdots + a_j) & \text{if } n \geq i > j \geq 1, \\ 1 - (a_1 + \cdots + a_j) & \text{if } 1 \leq i \leq j < n. \end{cases}$$

If, in addition, $a_1 + \cdots + a_j \le 0$ for $1 \le j \le n-1$, the integer entries of $B = C^{-1}$ are all nonnegative and Lemma 1 gives the generating function $G_\pi(z)$ and the theorem follows.

To complete the proof, we show that if $a_1 + \cdots + a_n = 1$ and $a_1 \le a_2 \le \cdots \le a_n$, then for $1 \le j \le n-1$ we have $a_1 + \cdots + a_j \le 0$.

Let $j$ be the smallest index satisfying $1 \le j \le n-1$ and $a_1 + \cdots + a_j \le 0$, but $a_1 + \cdots + a_{j+1} > 0$. Then $a_{j+1} > -(a_1 + \cdots + a_j) \ge 0$. Thus

$$1 \le a_{j+1} \le \cdots \le a_n\,,$$

so

$$1 = a_1 + \cdots + a_n \ge a_1 + \cdots + a_{j+1} + n - j - 1 > n - j - 1\,.$$

So $j = n-1$ and therefore $a_1 + \cdots + a_j \le 0$ for $1 \le j \le n-1$. □

In Section 2.3 we derive an algorithm based on Theorem 1 for efficient computation of $F(q)$, given the $a_i$. In the next section, we give examples of how to combine Theorem 1 with results on permutation statistics to derive formulas for $F(q)$ in special cases.

## 2.2 Applications

**Example 1** Given positive integers $b$ and $n \ge 2$, let $L$ be the set of nonnegative integer sequences $\lambda$ satisfying

$$(nb - b + 1)\lambda_{\pi(n)} \ge b(\lambda_{\pi(1)} + \cdots + \lambda_{\pi(n-1)}) \qquad \text{for all } \pi \in S_n\,.$$

The case $n = 2$, $b = 1$ is the Putnam problem (3). Here $a = [-b, -b, \ldots, -b, nb - b + 1]$, so by Theorem 1,

$$F(q) \;=\; \frac{\sum_{\pi \in S_n} \prod_{j \in D_\pi} q^{j+jbn}}{(1-q^n)\prod_{j=1}^{n-1}(1-q^{j+jbn})} \;=\; \frac{\sum_{\pi \in S_n} (q^{1+bn})^{\mathrm{maj}(\pi)}}{(1-q^n)\prod_{j=1}^{n-1}(1-q^{j+jbn})}\,.$$

By (4), the numerator is just $[\,n\,]_{q^{1+bn}}!$ and simplifying gives

$$F(q) = \frac{1 - q^{n(nb+1)}}{(1-q^n)(1-q^{nb+1})^n}\,.$$

This generating function was discovered by Andrews, Paule, and Riese and a complete parametrization was proved in [4] using partition analysis.

**Example 2** Given positive integers $b$ and $n \ge 2$, let $L$ be the set of nonnegative integer sequences $\lambda$ satisfying

$$b(\lambda_{\pi(2)} + \cdots + \lambda_{\pi(n-1)}) \ge (nb - b - 1)\lambda_{\pi(1)} \qquad \text{for all } \pi \in S_n\,.$$

The case $n = 3$, $b = 1$ is the integer-sided triangle problem (2) and the case $n = 2$, $b = 2$ is the Putnam problem (3). Here $a = [-(nb - b - 1), b, b, \ldots, b]$, so by Theorem 1,

$$F(q) \;=\; \frac{\sum_{\pi\in S_n}\prod_{j\in D_\pi} q^{(bn-1)(n-j)}}{(1-q^n)\prod_{j=1}^{n-1}(1-q^{(bn-1)(n-j)})} \;=\; \frac{\sum_{\pi\in S_n}(q^{1-bn})^{\mathrm{maj}(\pi)}(q^{n(bn-1)})^{\mathrm{des}(\pi)}}{(1-q^n)\prod_{j=1}^{n-1}(1-q^{(bn-1)(n-j)})}\,.$$

By (7), the numerator is

$$C_n(q^{n(bn-1)}, q^{1-bn}) = \frac{(q^{n(bn-1)}; q^{1-bn})_n(-q)^{(1-bn)n}}{(1-q^{1-bn})^n}\,;$$

Simplifying further and dividing by the denominator gives

$$F(q) = \frac{1-q^{n(nb-1)}}{(1-q^n)(1-q^{nb-1})^n}\,.$$

This generating function was also originally proved by Andrews, Paule, and Riese in [4].

**Example 3** Given positive integers $b$ and $n \geq 2$, let $L$ be the set of nonnegative integer sequences $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying the constraints

$$(b+1)\lambda_{\pi(n)} \geq b\lambda_{\pi(1)} \qquad \text{for all } \pi \in S_n\,.$$

The case $n = 2$, $b = 1$ is the Putnam problem (3). Here $a = [-b, 0, 0, \ldots, 0, b+1]$ so by Theorem 1,

$$F(q) \;=\; \frac{\sum_{\pi\in S_n}\prod_{j\in D_\pi} q^{j+bn}}{(1-q^n)\prod_{j=1}^{n-1}(1-q^{j+bn})} \;=\; \frac{\sum_{\pi\in S_n} q^{\mathrm{maj}(\pi)}(q^{bn})^{\mathrm{des}(\pi)}}{(1-q^n)\prod_{j=1}^{n-1}(1-q^{j+bn})}\,.$$

By (5), the numerator is

$$C_n(q^{bn}, q) = \frac{(q^{bn}; q)_{n+1}}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}\frac{(-q^i)}{1-q^{bn+i}}\,.$$

Combining with the denominator and simplifying gives

$$F(q) = \frac{(1-q^{bn})(1-q^{bn+n})}{(1-q^n)(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}\frac{(-q^i)}{1-q^{bn+i}}\,.$$

**Example 4** Given positive integers $k \leq l$, and $n \geq 3$, let $m = k + l - 1$ and let $L$ be the set of nonnegative integer sequences $\lambda$ satisfying

$$k\lambda_{\pi(n-1)} + l\lambda_{\pi(n)} \geq m\lambda_{\pi(1)} \qquad \text{for all } \pi \in S_n\,.$$

The case $n = 3$ and $k = l = 1$ is the integer-sided triangles (2). Here $a = [-m, 0, 0, \ldots, 0, k, l]$, so by Theorem 1,

$$F(q) = \frac{\sum_{\pi\in S_n}\prod_{j\in D_\pi, j\neq n-1} q^{j+mn}\prod_{j\in D_\pi, j=n-1} q^{j+mn-nk}}{(1-q^n)(1-q^{nl-1})\prod_{j=1}^{n-2}(1-q^{j+bn})}\,.$$

Recall from (8) that $C_n^{(1)}$ is the joint distribution of des and maj over all permutations with no descent in position $n-1$. Then in $F(q)$, we can split the sum over $\pi \in S_n$ into two sums according to whether or not $i \in D_\pi$. We get that the numerator can be written as:

$$C_n^{(1)}(q^{nm}, q) + q^{-nk}(C_n(q^{nm}, q) - C_n^{(1)}(q^{nm}, q)) .$$

Using (8) and combining with the denominator gives (eventually)

$$F(q) = \frac{C_n(q^{nm}, q)(1 - q^{nl-1}) - C_{n-1}(q^{nm}, q)nq^{nm+n-1}(1 - q^{-nk})}{(1 - q^n)(1 - q^{nl-1})(q^{nm+1}; q)_{n-1}} .$$

Details appear in [8].

## 2.3 Efficient Enumeration of Symmetrically Constrained Compositions

We can compute the generating function $F(q)$ for compositions satisfying the $n!$ constraints

$$\sum_{i=1}^{n} a_i \lambda_{\pi(i)} \geq 0 \qquad \text{for all } \pi \in S_n$$

via Theorem 1. The denominator is given explicitly, but the numerator is a sum of $n!$ terms. However, regardless of the values of the $a_i$, the numerator of $F(q)$, when simplified, is a polynomial with at most $2^{n-1}$ terms (one for each possible descent set).

Let $u_1, u_2, \ldots$ be arbitrary and define polynomials $G_n$ by

$$G_n = \sum_{\pi \in S_n} \prod_{i \in D_\pi} u_i .$$

We can compute $G_n$ in the following way: Let

$$G_n^{(i)} = \sum_{\pi \in \mathfrak{S}_n^{(i)}} \prod_{i \in D_\pi} u_i ,$$

where $\mathfrak{S}_n^{(i)}$ is the set of all permutations $\pi \in S_n$ that end with $i$. Then $G_n = G_{n+1}^{(n+1)}$.

A permutation $\pi$ in $\mathfrak{S}_n^{(i)}$ can be obtained uniquely from some permutation $\bar{\pi}$ in $S_{n-1}$ by replacing each $j \geq i$ with $j+1$ and then appending $i$ at the end. The descent set of $\pi$ will be the same as the descent set of $\bar{\pi}$ if the last entry of $\bar{\pi}$ is less than $i$ and the descent set of $\pi$ will be $D_{\bar{\pi}} \cup \{n-1\}$ if the last entry of $\bar{\pi}$ is greater than or equal to $i$. Thus we have the recurrence

$$G_n^{(i)} = \sum_{j=1}^{i-1} G_{n-1}^{(j)} + u_{n-1} \sum_{j=i}^{n-1} G_{n-1}^{(j)}$$

with the initial condition $G_1^{(1)} = 1$. We can simplify this a bit to get **"Algorithm $G$"**:

$$G_n^{(i)} = G_n^{(i-1)} + (1 - u_{n-1}) G_{n-1}^{(i-1)} \quad \text{for } i > 1\,,$$

with $G_n^{(1)} = u_{n-1} \sum_{j=1}^{n-1} G_{n-1}^{(j)}$.

Now, to compute the numerator of $F(q)$ in Theorem 1 using Algorithm $G$, simply set $u_i = q^{i-n(a_1+\cdots+a_i)}$ for $1 \le i < n$ and compute $G_{n+1}^{(n+1)}$.

If we use dynamic programming to implement the recurrence of Algorithm $G$, (e.g. "option remember" in Maple), then to compute $G_n = G_{n+1}^{(n+1)}$, at most $O(n^2)$ polynomials are computed. However, we must consider the time required to compute them. In order to compute one of the $G_k^{(i)}$, essentially we only need to add two polynomials. It is fair to assume that the time is proportional to the number of terms in the polynomials times the logarithm of the coefficient magnitude. So, overall, the time (and number of terms) grows roughly like $2^n$ in the dimension $n$, but logarithmically in the coefficient size, which is considered polynomial time in fixed dimension. In practice, we found that we could compute $F(q)$ for arbitrary $a$ with $\sum a_i = 1$ within seconds for $n \le 11$, in about 10 seconds for $n = 12$ and in less than a minute up to $n = 15$, using a naive implementation in Maple on a tablet PC running Windows XP.

For comparison, there are existing software packages that, when given a collection of linear inequalities, produce the generating function for the integer points in the solution set. These packages include the Omega Package [3], Xin's speed-up of Omega [15], and LattE macchiato [10, 13]. We used these programs to compute symmetrically constrained compositions in $n$ dimensions, by giving as input the $n!$ inequalities. The computation became infeasible when $n \ge 4$ for the Omega package and Xin's program. LattE was able to handle examples for $n = 5$ in under 10 seconds and $n = 6$ in under an hour.

Thus exploiting the symmetry via Theorem 1 and Algorithm $G$ makes a huge difference in what we can compute.

# 3 The General Case

## 3.1 A General Version of the Main Theorem

We remove the requirement that $\sum a_i = 1$ and enumerate compositions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}_{\ge 0}^n$ that satisfy the $n!$ constraints

$$a_1 \lambda_{\pi(1)} + a_2 \lambda_{\pi(2)} + \cdots + a_n \lambda_{\pi(n)} \ge 0 \qquad \text{for all } \pi \in S_n \tag{10}$$

via the generating function $F(z) = \sum_\lambda z^\lambda$.

**Theorem 2.** *Given integers $a_1 \le a_2 \le \cdots \le a_n$, with $a_1+a_2+\cdots+a_j \le 0$ for $1 \le j \le n-1$ and $a_1+a_2+\cdots+a_n \ge 1$, define the vectors $A_1, A_2, \ldots, A_n \in \mathbb{Z}^n$ as the columns of the matrix*

$$\begin{bmatrix} a_2+\cdots+a_n & a_3+\cdots+a_n & a_4+\cdots+a_n & \cdots & a_n & 1 \\ -a_1 & a_3+\cdots+a_n & a_4+\cdots+a_n & & a_n & 1 \\ -a_1 & -a_1-a_2 & a_4+\cdots+a_n & & a_n & 1 \\ -a_1 & -a_1-a_2 & -a_1-a_2-a_3 & & a_n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a_1 & -a_1-a_2 & -a_1-a_2-a_3 & & a_n & 1 \\ -a_1 & -a_1-a_2 & -a_1-a_2-a_3 & \cdots & -a_1-\cdots-a_{n-1} & 1 \end{bmatrix}$$

*and let*

$$\mathcal{P} := \sum_{j=1}^{n} [0,1) A_j = \left\{ \sum_{i=1}^{n} c_i A_j : \ 0 \le c_i < 1 \right\}.$$

*Then*

$$F(z) = \sum_{p \in \mathcal{P} \cap \mathbb{Z}^n} \sum_{\pi \in S_n} \frac{z_\pi^p \prod_{i \in D_\pi,\, p_i = p_{i+1}} z_\pi^{A_i}}{\prod_{j=1}^{n} \left(1 - z_\pi^{A_i}\right)},$$

*where we take the product over all descent positions $i$ of $\pi$ for which the $i$th and the $(i+1)$st coordinate of $p$ are the same.*

If, for some $i$, $d$ divides every coordinate of $A_i$, we can replace $A_i$ by $A_i/d$ in Theorem 2 and thereby reduce the number of lattice points in $\mathcal{P}$ by a factor of $d$.

**Proof.** The start of our proof is similar to that of Theorem 1, except that we find it advantageous to view the compositions satisfying (10) as integer points in the cone

$$K := \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{\ge 0} : \sum_{j=1}^{n} a_j x_{\pi(j)} \ge 0 \ \text{ for all } \pi \in S_n \right\}.$$

From this point of view,

$$F(z) = \sum_{\lambda \in K \cap \mathbb{Z}^d} z^\lambda.$$

The setup now continues in analogy with the proof of Theorem 1. Like there, it suffices to study

$$\widetilde{K}_\pi := \left\{ x \in \mathbb{R}^n : \begin{array}{l} x_1 \ge x_2 \ge \cdots \ge x_n \ge 0 \ \text{ and } \ x_j > x_{j+1} \text{ if } j \in D_\pi \\ a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \ge 0 \end{array} \right\}$$

and the associated generating function $G_\pi(z) := \sum_{\lambda \in \widetilde{K}_\pi \cap \mathbb{Z}^d} z^\lambda$; then

$$F(z) = \sum_{\pi \in S_n} G_\pi(z_\pi).$$

First, we study the cone $K_{\mathrm{Id}}$. The constraints of $K_{\mathrm{Id}}$ are given by the system

$$\begin{bmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & & \ddots & & \\ & & & & 1 & -1 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \end{bmatrix} x \;\geq\; 0\,, \tag{11}$$

and the inverse of the matrix on the left of (11) is

$$\frac{1}{\sum_{j=1}^{n} a_j} \begin{bmatrix} a_2+\cdots+a_n & a_3+\cdots+a_n & a_4+\cdots+a_n & \cdots & a_n & 1 \\ -a_1 & a_3+\cdots+a_n & a_4+\cdots+a_n & & a_n & 1 \\ -a_1 & -a_1-a_2 & a_4+\cdots+a_n & & a_n & 1 \\ -a_1 & -a_1-a_2 & -a_1-a_2-a_3 & & a_n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a_1 & -a_1-a_2 & -a_1-a_2-a_3 & & a_n & 1 \\ -a_1 & -a_1-a_2 & -a_1-a_2-a_3 & \cdots & -a_1-\cdots-a_{n-1} & 1 \end{bmatrix}.$$

The conditions on $a_1, a_2, \ldots, a_n$ guarantee that the inverse exists and that $K_{\mathrm{Id}}$ is a cone in $\mathbb{R}^n_{\geq 0}$. Thus the columns $A_1, A_2, \ldots, A_n$ of this matrix form a set of generators of $K_{\mathrm{Id}}$ and by an easy tiling argument (see, e.g., [5, Chapter 3])

$$K_{\mathrm{Id}} \cap \mathbb{Z}_n = \left\{ p + \sum_{j=1}^{n} c_j A_j \,:\, p \in \mathcal{P},\ c \in \mathbb{Z}^n_{\geq 0} \right\}; \tag{12}$$

in other words,

$$G_{\mathrm{Id}}(z) = \frac{\sum_{p \in \mathcal{P} \cap \mathbb{Z}^n} z^p}{\prod_{j=1}^{n} \left(1 - z_\pi^{A_j}\right)}\,.$$

Before turning to $\widetilde{K}_\pi$, note that the generators $A_j$ have a special form: we have

$$A_{j,j} > A_{j+1,j} \text{ for } 1 \leq j < n \qquad \text{and} \qquad A_{i,j} = A_{i+1,j} \text{ for } j \neq i.$$

This follows because $\sum_{j=1}^{n} a_j \geq 1$ implies that

$$a_j + a_{j+1} + \cdots + a_n > -a_1 - a_2 - \cdots - a_{j-1}.$$

Thus, if $p \in K_{\mathrm{Id}} \cap \mathbb{Z}^n$ satisfies $p_j = p_{j+1}$, then for any $c \in \mathbb{Z}^n_{\geq 0}$, if

$$r = (p + A_j) + \sum_{i=1}^{n} c_i A_i$$

then

$$r_j > r_{j+1}.$$

Now, what about $\widetilde{K}_\pi$? It contains all points $y \in K_{\mathrm{Id}}$ except those $y$ with $y_i = y_{i+1}$ for some $i \in D_\pi$. By (12), every $y \in K_{\mathrm{Id}}$ has a unique representation as $y = p + \sum_{j=1}^n c_j A_j$ for some $c \in \mathbb{Z}^n_{\geq 0}$. Thus by the previous paragraph, $y_i = y_{i+1}$ iff both $p_i = p_{i+1}$ and $c_i = 0$. Now, in the same way as in (12),

$$\widetilde{K}_\pi \cap \mathbb{Z}^n = \left\{ p + \sum_{j=1}^n c_j A_j \,:\, p \in \mathcal{P},\ c \in \mathbb{Z}^n_{\geq 0},\ \text{and if } j \in D_\pi \text{ and } p_j = p_{j+1} \text{ then } c_j > 0 \right\}$$
$$= \left\{ p + \sum_{j=1}^n c_j A_j + \sum_{j \in D_\pi,\ p_j = p_{j+1}} A_j \,:\, p \in \mathcal{P},\ c \in \mathbb{Z}^n_{\geq 0} \right\}$$

Thus

$$G_\pi(z) = \sum_{p \in \mathcal{P} \cap \mathbb{Z}^n} \frac{z^p \prod_{j \in D_\pi,\ p_j = p_{j+1}} z_\pi^{A_i}}{\prod_{j=1}^n \left(1 - z_\pi^{A_i}\right)}.$$

□

In the special case of Theorem 1, $\sum_{j=1}^n a_j = 1$ and the origin is the only lattice point in $\mathcal{P}$.

### 3.2 Efficient Computation for the General Case

Give $a_1 \leq \cdots \leq a_n$ with $\sum_{j=1}^n a_j \geq 1$, once we find the generators $A_1, A_2, \ldots, A_n$, and the lattice points in $\mathcal{P}$, we can again use Algorithm $G$ to efficiently compute $F(q)$: By Theorem 2, setting $z = (q, q, \ldots, q)$,

$$F(q) = \sum_{p \in \mathcal{P} \cap \mathbb{Z}^n} q^{|p|} \frac{\sum_{\pi \in S_n} \prod_{i \in D_\pi, p_i = p_{i+1}} q^{|A_i|}}{\prod_{j=1}^n \left(1 - q^{|A_i|}\right)},$$

where $|x| = x_1 + \cdots + x_n$ for an $n$-dimensional vector $x$.

The denominator is easy. To find the numerator, for each point $p \in \mathcal{P} \cap \mathbb{Z}^n$, set

$$u_i = \begin{cases} q^{|A_i|} & \text{if } p_i = p_{i+1} \\ 1 & \text{otherwise} \end{cases}$$

and then compute $G_{n+1}^{(n+1)}$.

Now the running time also depends on $|\mathcal{P} \cap \mathbb{Z}^n|$. This can grow linearly with the magnitude of the entries (rather than the logarithm of the magnitude), even in fixed dimension. However, when $|\mathcal{P} \cap \mathbb{Z}^n|$ is of moderate size, this computation method can be quite effective.

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