LATTICE POINT GENERATING FUNCTIONS AND SYMMETRIC CONES

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ABSTRACT. We show that a recent identity of Beck–Gessel–Lee–Savage on the generating function of symmetrically contrained compositions of integers generalizes naturally to a family of convex polyhedral cones that are invariant under the action of a finite reflection group. We obtain general expressions for the multivariate generating functions of such cones, and work out their general form more specifically for all symmetry groups of type A (previously known) and types B and D (new). We obtain several applications of these expressions in type B, including identities involving permutation statistics and lecture hall partitions.

1. INTRODUCTION

Motivated by the "constrained compositions" introduced by Andrews–Paule–Riese [1], Beck–Gessel–Lee–Savage [3] enumerated **symmetrically constrained compositions**, i.e., compositions of an integer M into n nonnegative parts

$$M = \lambda_1 + \lambda_2 + \cdots + \lambda_n ,$$

where the sequence $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfies the symmetric system of linear inequalities

$$a_1 \lambda_{\pi(1)} + a_2 \lambda_{\pi(2)} + \cdots + a_n \lambda_{\pi(n)} \geq 0 \qquad \text{for all } \pi \in S_n .$$

Specifically, [3] discusses various approaches to compute, for a fixed set of parameters $a_1, a_2, \ldots, a_n$, the generating functions

$$F(z_1, z_2, \ldots, z_n) := \sum_{\lambda} z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_n^{\lambda_n}$$

and

$$F(q) := F(q, q, \ldots, q) = \sum_{\lambda} q^{\lambda_1 + \lambda_2 + \cdots + \lambda_n},$$

where both sums extend over all symmetrically constrained compositions λ. One viewpoint of [3] is geometric: The compositions $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are interpreted as integer lattice points in the cone

$$\{x \in \mathbf{R}^n \mid \forall\, \sigma \in W : (\sigma x, a) \geq 0\} , \tag{1}$$

where W is the image of the permutation representation of S_n, $a = (a_1, \ldots, a_n)$, and $(\ ,\ )$ is the standard inner product on $\mathbf{R}^n$. This viewpoint together with permutation statistics of S_n gave rise to explicit (and in some instances surprising) generating function formulas.

Our goal is to generalize the results in [3] to cones of the form (1) where W is another reflection group. In addition to obtaining general multivariate generating function identities, we obtain several applications of these results for hyperoctahedral groups. These applications are similar in spirit to the applications in the symmetric-group case found in [3].

The outline of our paper is as follows. The general setup for our approach is discussed in the next section, which also contains our central result, Theorem 2.8. Section 3 illustrates our approach by re-deriving the main result in [3]. Sections 4 and 5 consider cones constrained by reflection groups of type B and D, respectively. Further, Section 4 contains applications obtained through specializing our generating functions in the type-B case.

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2. General theory

Our goal in this section is to study integer points in cones that are constrained by the orbit of a single linear constraint under an appropriate group action on real space. This goal is realized in Theorem 2.8, where the multivariate generating function encoding the integer points in such a cone is expressed as a sum of simpler generating functions. Theorem 2.8 is an algebraic consequence of a geometric triangulation of the symmetric cone, which we obtain in Lemma 2.5. Proposition 2.6 makes the triangulation disjoint by using combinatorics of Coxeter groups as a tiebreaker for the walls separating the maximal cones in the triangulation. This is critical for our subsequent applications.

2.1. **Almost irreducible finite reflection groups, Coxeter groups, and descents.** In the following, we will consider finite reflection groups (i.e., finite subgroups of $\mathrm{O}(V)$ for some Euclidean space V that are generated by reflections—see, e.g., [10] for background) that act on the underlying Euclidean space in a restricted fashion. Namely, a finite reflection group $W \subset \mathrm{O}(V)$ acting on a Euclidean vector space V is called **almost irreducible** if V decomposes into W-invariant subspaces $V = V_1 \oplus V_2$ such that W acts irreducibly and nontrivially on V_1 and trivially on V_2, and that V_2 is 1-dimensional.

Example 2.1. S_n acts almost irreducibly on $\mathbf{R}^n$ by permutation of the components. The irreducible summand consists of all vectors with component sum 0, and the trivial summand consists of all vectors with equal components. This is the case considered in [3].

Example 2.2. Let V_1 be a Euclidean vector space and $W \subset \mathrm{O}(V_1)$ a nontrivial irreducible reflection group, i.e., a nontrivial reflection group such that V_1 does not contain any nontrivial proper W-invariant subspaces. Let W act trivially on $\mathbf{R}$ and set $V = V_1 \oplus \mathbf{R}$. Then W acts almost irreducibly on V.

A **Coxeter group** of rank r is a group admitting a presentation with generators $s_1, \ldots, s_r$ and relations $(s_j s_k)^{m_{jk}} = 1$ for $m_{jk} \in \{1, 2, 3, \ldots\} \cup \{\infty\}$ subject to the conditions that $m_{jk} = m_{kj}$ and $m_{jk} = 1 \iff j = k$. Here, a value of $m_{jk} = \infty$ is to be understood as the absence of the corresponding relation. Such generators are called **simple generators**. For each Coxeter group considered, we will suppose that simple generators have been fixed once and for all. We refer the reader to [5] or [10] for further information about Coxeter groups and their relation to reflection groups.

The **length** $l(\sigma)$ of an element $\sigma \in W$ of a Coxeter group W is the smallest integer such that there is a decomposition $\sigma = s_{j_1} \cdots s_{j_{l(\sigma)}}$ of σ as a product of $l(\sigma)$ not necessarily distinct simple generators. For any $\sigma \in W$, the **right descent set** of σ is

$$D_{\mathrm{r}}(\sigma) := \{j \in \{1, \ldots, r\} \mid l(\sigma s_j) < l(\sigma)\}\,. \tag{2}$$

Remark 2.3. Propositions 3.1, 4.1, and 5.1 review the connection between the definition of descent given here and definitions of descent for Coxeter groups of types A, B, and D in terms of the one-line notation.

Recall that if W is a finite reflection group, it is automatically a Coxeter group. Simple generators can be found as follows. Let $\mathcal{H}$ be the union of all reflection hyperplanes for W; denote by F the closure of a connected component in $V \setminus \mathcal{H}$. It is immediate that F is a convex polyhedral cone. Let $H_1, \ldots, H_r$ be the facet hyperplanes of F and let s_i be the reflection at H_i. Then $s_1, \ldots, s_r$ are simple generators of the Coxeter group W. See [6, V.3.2, Th. 1] for the proof of these statements.

A subset $F \subset V$ is a **fundamental domain** for W if F is the closure of an open set and each W-orbit intersects F in exactly one point. By [10, Section I.12], every such F is polyhedral, and is bounded by hyperplanes fixed by a set of simple reflections in W. Through the rest of this paper, when given a set of simple generators $s_1, \ldots, s_r$ of a reflection group W, we denote by F a fixed fundamental domain with bounding hyperplanes corresponding to $s_1, \ldots, s_r$.

2.2. **Triangulations of monoconditional cones.** Denote the value of a linear form $\varphi \in V^*$ on a vector $x \in V$ by $\langle x, \varphi \rangle$. Let $W \subset \mathrm{O}(V)$ be an almost irreducible reflection group. A **symmetric cone** $C \subset V$ is a convex polyhedral cone that is W-invariant. A symmetric cone is called **monoconditional** if there is a linear form $\varphi \in V^*$, such that $V_1, V_2 \not\subset \ker(\varphi)$ and

$$C = \{x \in V \mid \forall\, \sigma \in W : \langle \sigma x, \varphi \rangle \geq 0\}\,. \tag{3}$$

This generalizes (1).

Example 2.4. The positive orthant $\mathbf{R}^n_{\geq 0}$ is a monoconditional symmetric cone for the almost irreducible action of S_n on $\mathbf{R}^n$ by permutation of the components. A possible linear form defining it is the projection on the first component.

Recall that a convex polyhedral cone is **pointed** if it does not contain a line, and it is **simplicial** if it is the set of linear combinations with nonnegative coefficients

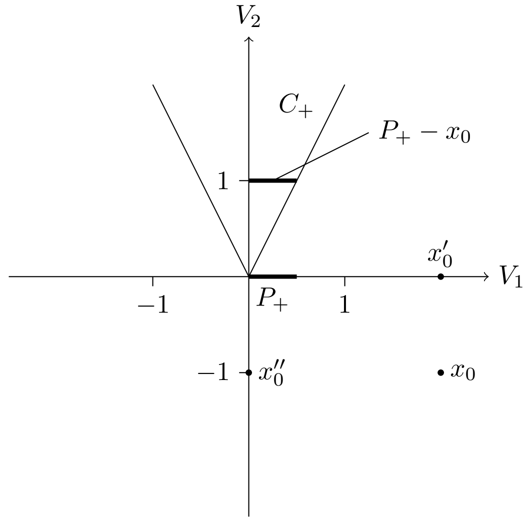

FIGURE 1. Notation used in the proof of Lemma 2.5. The cyclic group of order 2 acts almost irreducibly on $\mathbf{R}^2$ by sign change in the first component.

of a set of linearly independent vectors. A **triangulation** of a cone C is a finite collection T of simplicial cones such that C is the union of the elements of T and for any $\Delta_1, \Delta_2 \in T$, $\Delta_1 \cap \Delta_2$ is a face common to both Δ_1 and Δ_2.

Lemma 2.5. *Let $W \subset \mathrm{O}(V)$ be an almost irreducible reflection group. Let $C \subset V$ be a monoconditional symmetric cone. Then C is pointed. Let $F \subset V$ be a fundamental domain for the action of W on V. Then the cone $C_+ := C \cap F$ is simplicial. In particular, C admits the triangulation*

$$C = \bigcup_{\sigma \in W} \sigma\, C_+ \,.$$

Proof. Some of the notation used in this proof is shown in Figure 1 for convenience. Let $\varphi \in V^*$ be a linear form defining C as in (3); note that $(\ ,\ )$ is a W-invariant inner product on V. Let $x_\varphi \in V$ such that $\langle x, \varphi \rangle = -(x, x_\varphi)$ for all $x \in V$. Let x_0 be the unique element of $W x_\varphi \cap F$. Then

$$C_+ = \{ x \in F \mid \forall\, \sigma \in W : (\sigma x, x_0) \le 0 \} \,.$$

Let $V = V_1 \oplus V_2$ be the decomposition of V into the irreducible and trivial component. Let $x_0 = x_0' + x_0''$ with $x_0' \in V_1$ and $x_0'' \in V_2$. By definition $V_2 \not\subset \ker(\varphi)$, and so $x_\varphi \notin V_1$, thus $x_0 \notin V_1$, and hence $x_0'' \neq 0$. As φ is only determined up to multiplication by a positive scalar, suppose without loss of generality that $(x_0'', x_0'') = 1$. Note that every nonzero element of C_+ has a nonzero V_2-component; otherwise, for some $x \in C_+$ it would hold that $(\sigma x, x_0') \le (\sigma x, x_0'') = 0$ for all $\sigma \in W$. This would imply that $(x, \sigma x_0') \le 0$ for all $\sigma \in W$, which leads to a contradiction since $x \neq 0$ and $\sum_{\sigma \in W} \sigma x_0' = 0$ by the irreducibility of the action of W on V_1.

This implies that every nonzero element of C has a nonzero V_2-component, in particular, C is pointed.

Let

$$P_+ = \{x \in F \cap V_1 \mid \forall\, \sigma \in W : (\sigma x, x_0') \le 1\}\,.$$

Consider the reflections at the facet hyperplanes of F as simple generators of W. Let l denote the corresponding length function. Let $x \in F \cap V_1$ and $\sigma \in W$. Let H be a facet hyperplane of σF, such that $l(s\sigma) > l(\sigma)$ for the reflection s at H. We claim that in this situation

$$(s\sigma x, x_0') \le (\sigma x, x_0')\,. \tag{4}$$

Indeed, consider the decomposition $V_1 = (H \cap V_1) \oplus H^\perp$. According to this decomposition, write $x_0' = v_0 + w_0$ and $\sigma x = v_1 + w_1$; then $s\sigma x = v_1 - w_1$. We have $(\sigma x, x_0') = (v_1, v_0) + (w_1, w_0)$ and $(s\sigma x, x_0') = (v_1, v_0) - (w_1, w_0)$. Hence

$$(s\sigma x, x_0') = (\sigma x, x_0') - 2(w_1, w_0)\,. \tag{5}$$

Generally, if $\tau \in W$, then $l(\tau)$ equals the number of reflection hyperplanes between F and τF. As $l(s\sigma x) > l(\sigma x)$, this implies that x_0 and σx lie on the same side of H. Hence $(w_1, w_0) \ge 0$, and so (5) implies the claim (4).

By induction on $l(\sigma)$, (4) implies that

$$P_+ = \{x \in F \cap V_1 \mid (x, x_0') \le 1\}\,.$$

The cone C_+ is the cone over $P - x_0''$, thus $C_+ = \{x \in F \mid (x, x_0) \le 0\}$. The cone $F \cap V_1$ is a fundamental domain for the action of W on V_1. Hence $F \cap V_1$ is simplicial, as it is defined by the hyperplanes corresponding to the simple generators of W, and there are $\dim(V_1)$ many such hyperplanes. We have $F = (F \cap V_1) + V_2$, and so $\dim(F) = \dim(F \cap V_1) + 1$. The cone C_+ is defined in F by the single additional inequality $(x, x_0) \le 0$, thus C_+ is simplicial. ☐

Consider the situation of Lemma 2.5. Choose an order $H_1, \dots, H_{n-1}$ of the facet hyperplanes of F with the corresponding simple reflections $s_1, \dots, s_{n-1}$. For any subset $J \subset \{1, \dots, n-1\}$, let

$$C_J := C_+ \setminus \bigcup_{j \in J} H_j\,.$$

For example, $C_\emptyset = C_+$. If $J \ne \emptyset$ some of the facets of C_+ are removed.

Proposition 2.6. *In the situation of Lemma 2.5, C decomposes as a disjoint union*

$$C = \biguplus_{\sigma \in W} \sigma\, C_{D_{\mathrm{r}}(\sigma)}\,.$$

Proof. For $x \in C$, let

$$W(x) := \{\sigma \in W \mid x \in \sigma\, C_+\}\,.$$

Let $x_+ \in Wx \cap C_+$ be the (unique) W-conjugate of x contained in the fundamental domain C_+. Let $\tilde W := \{\sigma \in W \mid \sigma x_+ = x_+\}$ be the isotropy subgroup of x_+ in W. By Lemma 2.5, $C = \bigcup_{\sigma \in W} \sigma\, C_+$, and so the set $W(x)$ is nonempty. Fix any $\sigma \in W(x)$.

We claim that $W(x) = \sigma \tilde W$. Indeed, it is obvious that $W(x) \supseteq \sigma \tilde W$. For the opposite inclusion, let $\bar\sigma \in W(x)$. Then $\bar\sigma^{-1} x \in C_+$, so $\bar\sigma^{-1} = x_+$. By the same argument $\sigma^{-1} x = x_+$, so $\sigma^{-1}\bar\sigma \in \tilde W$. Hence $\bar\sigma = \sigma\sigma^{-1}\bar\sigma \in \sigma\tilde W$, proving the claim.

By [10, Theorem 1.12 (c)], the isotropy subgroup $\tilde W$ is generated by the reflections it contains. So $\tilde W$ is a parabolic subgroup of W. Hence $W(x) = \sigma \tilde W$ contains a unique element of minimal length [10, §1.10], denoted by σ_x.

Assume that $\sigma_x^{-1}x \in H_j$ for some $j \in D_{\mathrm{r}}(\sigma_x)$. Then $s_j\sigma_x^{-1}x = \sigma_x^{-1}x$, and so $x = \sigma_x s_j \sigma_x^{-1}x$. As $\sigma_x^{-1}x \in C_+$, this implies that $\sigma_x s_j \in W(x)$. On the other hand $l(\sigma_x s_j) < l(\sigma_x)$, a contradiction. Hence $\sigma_x^{-1}x \notin H_j$ for all $j \in D_{\mathrm{r}}(\sigma_x)$. Hence $\sigma_x^{-1}x \in C_{D_{\mathrm{r}}(\sigma_x)}$, and so $x \in \sigma_x C_{D_{\mathrm{r}}(\sigma_x)}$. This proves that $C = \bigcup_{\sigma\in W} \sigma\, C_{D_{\mathrm{r}}(\sigma)}$.

To prove disjointness, let $x \in \sigma\, C_{D_{\mathrm{r}}(\sigma)}$ for some $\sigma \in W$. We have to show that $\sigma = \sigma_x$. Clearly $\sigma \in W(x)$. It remains to show that σ has minimal length in $W(x)$. Assume that σ has not minimal length in $W(x)$. Then there is $j \in \{1, \dots, n-1\}$ such that $l(\sigma s_j) < l(\sigma)$ and $\sigma s_j \in W(x)$ [10, §1.10]. From $\sigma \in W(x)$ we conclude that $\sigma^{-1}x \in C_+$ and from $\sigma s_i \in W(x)$ that $s_j\sigma^{-1}x \in C_+$. Hence $\sigma^{-1}x \in H_j$. Since $l(\sigma s_j) < l(\sigma)$ we have $j \in D_{\mathrm{r}}(\sigma)$. Hence $\sigma^{-1}x \notin C_{D_{\mathrm{r}}(\sigma)}$, and so $x \notin \sigma\, C_{D_{\mathrm{r}}(\sigma)}$, a contradiction. □

Our proposition above is reminiscent of the theory of P-partitions [12, 13]. For a given finite poset P, one can produce a cone of P-partitions. The standard approach to studying a P-partition cone, originating in the work of Stanley referenced above, is to recognize that each such cone is a disjoint union of various chambers of the type A braid arrangement where each chamber has some of its facets removed. The removal of facets in a given chamber is controlled by the descent statistic for the permutation indexing that chamber. Thus, each P-partition cone admits a unimodular triangulation of form similar to Proposition 2.6.

2.3. **Generating functions for monoconditional cones.** Let $V_{\mathbf{C}}^* = V^* \otimes_{\mathbf{R}} \mathbf{C}$. Extend $\langle\ ,\ \rangle$ to $V \times V_{\mathbf{C}}^*$ by $\mathbf{C}$-linearity in the second argument. Let $\Gamma \subset V$ be a lattice and $S \subset V$. Suppose that there is a nonempty open subset $B \subset V_{\mathbf{C}}^*$ such that the series $\sum_{x\in S\cap\Gamma} e^{-\langle x,\varphi\rangle}$ converges for $\varphi \in B$ and has a meromorphic continuation to $V_{\mathbf{C}}^*$. We denote this continuation by f_S and call it the **generating function** of S with respect to Γ.

Example 2.7. If $C \subset V$ is a cone, let

$$C^\vee := \{\varphi \in V^* \mid \forall\, x \in C : \langle x, \varphi\rangle \geq 0\} \subset V^*$$

be its dual cone. The complexified dual of C is defined as

$$C_{\mathbf{C}}^\vee := \{\varphi \in V_{\mathbf{C}}^* \mid \forall\, x \in C : \Re(\langle x, \varphi\rangle) \geq 0\} = C^\vee + i\, V^*.$$

Let $C \subset V$ be a pointed cone, rational with respect to Γ. Then $\sum_{x\in C\cap\Gamma} e^{-\langle x,\varphi\rangle}$ converges on the interior of $C_{\mathbf{C}}^\vee$ and has a meromorphic continuation f_C to $V_{\mathbf{C}}^*$; see, e.g., [2, Chapter 13].

From now on, suppose that W is crystallographic, i.e., that there is a W-invariant lattice Γ in V. A full-dimensional simplicial cone $C \subset V$ is called **unimodular** (with respect to Γ) if it is generated by a basis of Γ. These generators are called **primitive**.

Theorem 2.8. *In the situation of Lemma 2.5, suppose that C_+ is unimodular with respect to Γ. Let $b_1, \dots, b_n$ be the primitive generators of C_+, enumerated in the unique way such that $b_j \notin H_j$ for $j \in \{1, \dots, n-1\}$. Then the generating function of C is*

$$f_C(\varphi) = \sum_{\sigma\in W} \frac{\prod_{j\in D_{\mathrm{r}}(\sigma)} e^{-\langle \sigma b_j, \varphi\rangle}}{(1 - e^{-\langle \sigma b_1,\varphi\rangle}) \cdots (1 - e^{-\langle \sigma b_n,\varphi\rangle})}\,.$$

In practice, Γ is often endowed with a distinguished basis. In this case, it is often more convenient to work with the following formulation.

Corollary 2.9. *In the situation of Theorem 2.8, let* $e_1, \dots, e_n$ *be a basis of* Γ*. Define coordinates* z_j *on* $V^*_{\mathbf{C}}$ *by* $z_j(\varphi) := e^{-\langle e_j, \varphi\rangle}$*. For* $a = a_1e_1 + \cdots + a_ne_n \in \Gamma$*, let* $z^a := z_1^{a_1} \cdots z_n^{a_n}$*. Then*

$$f_C = \sum_{\sigma\in W} \frac{\prod_{j\in D_{\mathrm{r}}(\sigma)} z^{\sigma b_j}}{(1 - z^{\sigma b_1}) \cdots (1 - z^{\sigma b_n})} .$$

Proof of Theorem 2.8. Since C is pointed (by Lemma 2.5), its generating series converges on a nonempty domain and the generating series of all W-conjugates of C_+ converge there. As C_+ is unimodular with primitive generators $b_1, \dots, b_n$, its generating function is

$$f_{C_+}(\varphi) = \frac{1}{(1 - e^{-\langle b_1,\varphi\rangle}) \cdots (1 - e^{-\langle b_n,\varphi\rangle})} .$$

With unimodularity it also follows that each generator of C_+ is only one lattice hyperplane away from the opposite facet. Hence

$$C_{\{j\}} \cap \Gamma = (C_+ \setminus H_j) \cap \Gamma = (C_+ \cap \Gamma) + b_j$$

for all $j \in \{1, \dots, n-1\}$. More generally, $C_J \cap \Gamma = (C_+ \cap \Gamma) + \sum_{j\in J} b_j$ for any $J \subset \{1, \dots, n-1\}$. Applying this observation to $J = D_{\mathrm{r}}(\sigma)$ for a $\sigma \in W$ and rephrasing it in terms of generating functions, one obtains

$$f_{C_{D_{\mathrm{r}}(\sigma)}}(\varphi) = \frac{\prod_{j\in D_{\mathrm{r}}(\sigma)} e^{-\langle b_j,\varphi\rangle}}{(1 - e^{-\langle b_1,\varphi\rangle}) \cdots (1 - e^{-\langle b_n,\varphi\rangle})} .$$

Hence for all $\sigma \in W$ it follows that

$$f_{\sigma C_{D_{\mathrm{r}}(\sigma)}}(\varphi) = \frac{\prod_{j\in D_{\mathrm{r}}(\sigma)} e^{-\langle \sigma b_j,\varphi\rangle}}{(1 - e^{-\langle \sigma b_1,\varphi\rangle}) \cdots (1 - e^{-\langle \sigma b_n,\varphi\rangle})} .$$

By Proposition 2.6, $f_C = \sum_{\sigma\in W} f_{\sigma C_{D_{\mathrm{r}}(\sigma)}}$, which proves the formula. □

3. Cones with the symmetry of a simplex

Theorem 2.8 specializes to more concrete identities once we fix a particular almost irreducible reflection group W. The case of W being the group of symmetries of a simplex has been treated in [3]. We include this case here to show how the result can be derived from Theorem 2.8.

Let S_n denote the group of permutations of the set $\{1, \dots, n\}$. For $\pi \in S_n$, we define the **descent set** of π as

$$D(\pi) := \{j \in \{1, \dots, n-1\} \mid \pi(j) > \pi(j+1)\} . \tag{6}$$

This is the standard definition used in the literature on permutations.

The group S_n acts on $\mathbf{R}^n$ by permutation of the components. For $\pi \in S_n$, let $\sigma_\pi \in \mathrm{O}(\mathbf{R}^n)$ denote the transformation by which π acts on $\mathbf{R}^n$. Let $W = \{\sigma_\pi \mid \pi \in S_n\} \subset \mathrm{O}(\mathbf{R}^n)$. Then W is the group of symmetries of the $(n-1)$-dimensional standard simplex. For $j = 1, \dots, n-1$, let $s_j \in W$ be the transposition of the jth and $(j+1)$st component in $\mathbf{R}^n$. Then $s_1, \dots, s_{n-1}$ are simple generators of W.

The following shows that the definitions of descent given in (2) and (6) agree.

Proposition 3.1 ([5, Proposition 1.5.3])**.** $D_{\mathrm{r}}(\sigma_\pi) = D(\pi)$ *for all* $\pi \in S_n$*.*

Our main result in this section is the following.

Proposition 3.2 ([3, Theorem 1])**.** *Fix integers $a_1 \le \cdots \le a_n$ such that $a_1 + \cdots + a_n = 1$. Let*

$$C := \{x \in \mathbf{R}^n \mid \forall \pi \in S_n : a_1 x_{\pi(1)} + \cdots + a_n x_{\pi(n)} \ge 0\}\,.$$

Let $\Sigma_j := a_1 + \cdots + a_j$ for $j \in \{1, \ldots, n-1\}$. The generating function of C with respect to $\mathbf{Z}^n$ is

$$f_C = \frac{1}{1 - z_1 \cdots z_n} \sum_{\pi \in S_n} \frac{\prod_{j \in D(\pi)} (z_1 \cdots z_n)^{-\Sigma_j} \prod_{i=1}^{j} z_{\pi(i)}}{\prod_{j=1}^{n-1} \left(1 - (z_1 \cdots z_n)^{-\Sigma_j} \prod_{i=1}^{j} z_{\pi(i)}\right)}\,.$$

Note that the condition on the a_i to be increasing is a normalization rather than a restriction.

Proof. The cone C is symmetric and monoconditional for W. Let $F = \{x \in \mathbf{R}^n \mid x_1 \ge \cdots \ge x_n\}$, a fundamental domain for W. Then our chosen simple generators $s_1, \ldots, s_{n-1}$ of W are the reflections at the facet hyperplanes of F. Let $x_0 = (-a_1, \ldots, -a_n) \in F$. By the proof of Lemma 2.5,

$$\begin{aligned} C_+ &= \{x \in F \mid (x, x_0) \le 0\} = \{x \in F \mid a_1 x_1 + \cdots + a_n x_n \ge 0\} \\ &= \{x \in \mathbf{R}^n \mid Ax \ge 0\}\,, \end{aligned}$$

where

$$A = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \\ a_1 & \cdots & \cdots & \cdots & a_n \end{pmatrix}.$$

The determinant of A is $a_1 + \cdots + a_n = 1$, i.e., A is unimodular and so is C_+. Let $b_1, \ldots, b_n$ be the primitive generators of C_+, enumerated in the unique way such that $b_j \notin H_j$ for $j \in \{1, \ldots, n-1\}$. Then by Corollary 2.9, the generating function of C is

$$f_C = \sum_{\sigma \in W} \frac{\prod_{j \in D_{\mathrm{r}}(\sigma)} z^{\sigma b_j}}{(1 - z^{\sigma b_1}) \cdots (1 - z^{\sigma b_n})}\,.$$

Proposition 3.2 follows once we describe $D_{\mathrm{r}}(\sigma)$, b_j, and the action of W explicitly. The inverse of A is

$$B := A^{-1} = \begin{pmatrix} \Sigma'_1 & \cdots & \Sigma'_{n-1} & 1 \\ -\Sigma_1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \Sigma'_{n-1} & \vdots \\ -\Sigma_1 & \cdots & -\Sigma_{n-1} & 1 \end{pmatrix},$$

where $\Sigma'_j := 1 - \Sigma_j$. Then b_j is the jth column vector of B. Let

$$b_{ij} := \begin{cases} 1 & \text{if } j = n, \\ 1 - \Sigma_j & \text{if } i \le j < n, \\ -\Sigma_j & \text{if } j < i \end{cases}$$

be the ith component of b_j, i.e., the (i,j)th component of B. As defined above, with $\pi \in S_n$ we associate $\sigma_\pi \in \mathrm{O}(n)$ by $\sigma_\pi e_i = e_{\pi(i)}$. Then $W = \{\sigma_\pi \mid \pi \in S_n\}$ and we have $D_{\mathrm{r}}(\sigma_\pi) = D(\pi)$. Hence

$$\begin{aligned} f_C &= \sum_{\sigma \in W} \frac{\prod_{j \in D_{\mathrm{r}}(\sigma)} z^{\sigma b_j}}{\prod_{j=1}^n (1 - z^{\sigma b_j})} \\ &= \sum_{\pi \in S_n} \frac{\prod_{j \in D_{\mathrm{r}}(\sigma_\pi)} z^{\sigma_\pi b_j}}{\prod_{j=1}^n (1 - z^{\sigma_\pi b_j})} \\ &= \sum_{\pi \in S_n} \frac{\prod_{j \in D(\pi)} \prod_{i=1}^n z_{\pi(i)}^{b_{ij}}}{\prod_{j=1}^n \left(1 - \prod_{i=1}^n z_{\pi(i)}^{b_{ij}}\right)} \\ &= \frac{1}{1 - z_1 \cdots z_n} \sum_{\pi \in S_n} \frac{\prod_{j \in D(\pi)} (z_1 \cdots z_n)^{-\Sigma_j} \prod_{i=1}^j z_{\pi(i)}}{\prod_{j=1}^{n-1} \left(1 - (z_1 \cdots z_n)^{-\Sigma_j} \prod_{i=1}^j z_{\pi(i)}\right)} . \qquad \square \end{aligned}$$

4. Cones with hyperoctahedral symmetry

We now consider the case of cones which are symmetric under the action of a hyperoctahedral group. Let $W \subset \mathrm{O}(n)$ be the hyperoctahedral group on the first $n-1$ components of $\mathbf{R}^n$. Let $s_1 \in W$ be the sign change in the first component and, for $j = 2, \ldots, n-1$, let $s_j \in W$ be the transposition of the $(j-1)$st and jth component in $\mathbf{R}^n$. Then $s_1, \ldots, s_{n-1}$ are simple generators of W.

For combinatorial (as opposed to geometric) arguments, it is often more convenient to use the following parameterization of the hyperoctahedral group: For $\pi \in S_{n-1}$ and $\varepsilon \in \{\pm 1\}^{n-1}$, define $\sigma_{\pi,\varepsilon} \in \mathrm{O}(n)$ by

$$\sigma_{\pi,\varepsilon} e_i = \varepsilon_i e_{\pi(i)} , \tag{7}$$

where we use the convention that $\pi(n) := n$ for $\pi \in S_{n-1}$ and $\varepsilon_n := 1$ for $\varepsilon \in \{\pm 1\}^{n-1}$. Then $W = \{\sigma_{\pi,\varepsilon} \mid \pi \in S_{n-1},\ \varepsilon \in \{\pm 1\}^{n-1}\}$. Let B_{n-1} denote the set $S_{n-1} \times \{\pm 1\}^{n-1}$, endowed with the group structure such that $\sigma : B_{n-1} \to W$ becomes an isomorphism of groups.

In terms of this parameterization, the right descent set of W can be expressed more explicitly. For $(\pi, \varepsilon) \in B_{n-1}$ let

$$D(\pi,\varepsilon) := \{ j \in \{1, \ldots, n-1\} \mid \varepsilon_{j-1}\pi(j-1) > \varepsilon_j \pi(j) \} \tag{8}$$

with the convention that $\varepsilon_0 \pi(0) := 0$. Then the following holds.

Proposition 4.1 ([5, Proposition 8.1.2])**.** *For all $(\pi,\varepsilon) \in B_{n-1}$, we have*

$$D_{\mathrm{r}}(\sigma_{\pi,\varepsilon}) = D(\pi,\varepsilon) .$$

Note that the descent set defined in (8) is translated by $+1$ with respect to definitions found in the literature on signed permutations. This is because to have a consistent setup in section 2, we always start the enumeration of the simple reflections with 1, whereas from a signed permutations perspective it is convenient to start this enumeration with 0.

We define the descent statistic on the hyperoctahedral group by setting the **descent number**

$$\mathrm{des}(\pi,\varepsilon) := |D(\pi,\varepsilon)|$$

for $(\pi,\varepsilon) \in B_{n-1}$. Similarly, the **major index** is

$$\operatorname{maj}(\pi,\varepsilon) := \sum_{j \in D(\pi,\varepsilon)} (j-1)$$

and the **comajor index** is

$$\operatorname{comaj}(\pi,\varepsilon) := \sum_{j \in D(\pi,\varepsilon)} (n-j)$$

for $(\pi,\varepsilon) \in B_{n-1}$. It follows that we have the relationship

$$\operatorname{comaj}(\pi,\varepsilon) = (n-1)\operatorname{des}(\pi,\varepsilon) - \operatorname{maj}(\pi,\varepsilon)\,. \tag{9}$$

4.1. **The multivariate generating function.** In this situation, Corollary 2.9 specializes as follows.

Proposition 4.2. *Fix integers $0 \le a_1 \le \cdots \le a_{n-1} \neq 0$. Let*

$$C := \{x \in \mathbf{R}^n \mid \forall\, \pi \in S_{n-1},\ \varepsilon \in \{\pm 1\}^{n-1} : \\ \varepsilon_1 a_1 x_{\pi(1)} + \cdots + \varepsilon_{n-1} a_{n-1} x_{\pi(n-1)} \le x_n\}\,.$$

The generating function of C with respect to $\mathbf{Z}^n$ is

$$f_C = \frac{1}{1-z_n} \sum_{\pi \in S_{n-1}} \sum_{\varepsilon \in \{\pm 1\}^{n-1}} \frac{\prod_{j \in D(\pi,\varepsilon)} \prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} z_n^{a_i}}{\prod_{j=1}^{n-1} \left(1 - \prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} z_n^{a_i}\right)}\,.$$

Note that the condition on the a_i to be nonnegative and increasing is a normalization rather than a restriction.

Proof. The cone C is symmetric and monoconditional for W. Let

$$F := \{x \in \mathbf{R}^n \mid 0 \le x_1 \le \cdots \le x_{n-1}\}\,,$$

a fundamental domain for W. The $s_1, \ldots, s_{n-1}$ defined previously are the simple generators of W corresponding to F. Let $x_0 := (a_1, \ldots, a_{n-1}, -1) \in F$. By the proof of Lemma 2.5,

$$\begin{aligned} C_+ &= \{x \in F \mid (x, x_0) \le 0\} = \{x \in F \mid a_1 x_1 + \cdots + a_{n-1} x_{n-1} \le x_n\} \\ &= \{x \in \mathbf{R}^n \mid Ax \ge 0\}\,, \end{aligned}$$

where

$$A = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots \, 0 & -1 & 1 & 0 \\ -a_1 & \cdots & \cdots & -a_{n-1} & 1 \end{pmatrix}.$$

The matrix A and hence C_+ is unimodular. Let $b_1, \ldots, b_n$ be the primitive generators of C_+, enumerated in the unique way such that $b_j \notin H_j$ for $j < n$. Then by Corollary 2.9, the generating function of C is

$$f_C = \sum_{\sigma \in W} \frac{\prod_{j \in D_{\mathrm{r}}(\sigma)} z^{\sigma b_j}}{(1 - z^{\sigma b_1}) \cdots (1 - z^{\sigma b_n})}\,.$$

The inverse of A is

$$B := A^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \\ \Sigma_1 & \cdots & \Sigma_{n-1} & 1 \end{pmatrix},$$

where $\Sigma_j := a_j + \cdots + a_{n-1}$. Then b_j is the jth column vector of B. Let

$$b_{ij} := \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } j \le i < n \text{ or } i = j = n, \\ \Sigma_j & \text{if } j < i = n \end{cases}$$

be the ith component of b_j, i.e., the (i,j)th component of B. By Proposition 4.1 and using our notation introduced at the beginning of this section,

$$\begin{aligned} f_C &= \sum_{\sigma \in W} \frac{\prod_{j \in D_{\mathrm{r}}(\sigma)} z^{\sigma b_j}}{\prod_{j=1}^{n}(1 - z^{\sigma b_j})} \\ &= \sum_{(\pi,\varepsilon) \in B_{n-1}} \frac{\prod_{j \in D_{\mathrm{r}}(\sigma_{\pi,\varepsilon})} z^{\sigma_{\pi,\varepsilon} b_j}}{\prod_{j=1}^{n}(1 - z^{\sigma_{\pi,\varepsilon} b_j})} \\ &= \sum_{(\pi,\varepsilon) \in B_{n-1}} \frac{\prod_{j \in D(\pi,\varepsilon)} \prod_{i=1}^{n} z_{\pi(i)}^{\varepsilon_i b_{ij}}}{\prod_{j=1}^{n} \left(1 - \prod_{i=1}^{n} z_{\pi(i)}^{\varepsilon_i b_{ij}}\right)} \\ &= \frac{1}{1 - z_n} \sum_{(\pi,\varepsilon) \in B_{n-1}} \frac{\prod_{j \in D(\pi,\varepsilon)} \prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} z_n^{a_i}}{\prod_{j=1}^{n-1} \left(1 - \prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} z_n^{a_i}\right)}. \end{aligned}$$

□

4.2. **Hyperoctahedral Eulerian polynomials.** In the remainder of section 4, we provide applications of Proposition 4.2 with connections to permutation statistics and Ehrhart theory. Our first application is well known, going back to [8] and [14]; the polyhedral perspective of the following identity was first established in [14], also using Ehrhart theory.

Corollary 4.3 ([8], [14])**.** *The hyperoctahedral Eulerian polynomials are given by*

$$\sum_{(\pi,\varepsilon) \in B_{n-1}} t^{\mathrm{des}(\pi,\varepsilon)} = (1-t)^n \sum_{k=0}^{\infty} (2k+1)^{n-1} t^k .$$

Proof. Let

$$P = [-1, 1]^{n-1} .$$

be the $(n-1)$-dimensional hypercube. Our strategy to prove Corollary 4.3 is to compute the Ehrhart series

$$\mathrm{Ehr}_P(t) := \sum_{k \ge 0} |kP \cap \mathbf{Z}^{n-1}| \cdot t^k$$

of P in two different ways and to conclude by comparing the results.

On the one hand, note that the cone C over P,

$$C = \{x \in \mathbf{R}^n \mid \forall j < n : |x_j| \le x_n\} ,$$

is the cone considered in Proposition 4.2 for $a_1 = \cdots = a_{n-2} = 0$, $a_{n-1} = 1$, so by Proposition 4.2 its generating function is

$$f_C = \frac{1}{1-z_n} \sum_{(\pi,\varepsilon)\in B_{n-1}} \frac{\prod_{j\in D(\pi,\varepsilon)} \left(z_n \prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} \right)}{\prod_{j=1}^{n-1} \left(1 - z_n \prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} \right)} .$$

Since $\mathrm{Ehr}_P(t)$ is obtained by evaluating f_C at $z_1 = \cdots = z_{n-1} = 1$, $z_n = t$, we obtain

$$\begin{aligned} \mathrm{Ehr}_P(t) &= \frac{1}{1-t} \sum_{(\pi,\varepsilon)\in B_{n-1}} \frac{\prod_{j\in D(\pi,\varepsilon)} t}{\prod_{j=1}^{n-1}(1-t)} \\ &= \frac{1}{(1-t)^n} \sum_{(\pi,\varepsilon)\in B_{n-1}} t^{\mathrm{des}(\pi,\varepsilon)} . \end{aligned}$$

On the other hand, by definition

$$\mathrm{Ehr}_P(t) = \sum_{k\geq 0} (2k+1)^{n-1} t^k .$$

Together, we obtain

$$\frac{1}{(1-t)^n} \sum_{(\pi,\varepsilon)\in B_{n-1}} t^{\mathrm{des}(\pi,\varepsilon)} = \sum_{k\geq 0} (2k+1)^{n-1} t^k$$

and Corollary 4.3 follows. □

4.3. **The distribution of the comajor index.** For $k \in \mathbf{N}$ and a variable t, let

$$[k]_t := 1 + t + t^2 + \cdots + t^{k-1} \qquad \text{and} \qquad [k]_t! := [1]_t [2]_t \cdots [k]_t .$$

We show here how to derive the distribution of the comajor index. This is likely well-known, as for example it follows from (10) below, but we could not find an explicit statement in the literature. It is worth comparing Corollary 4.4 with the distribution

$$\sum_{(\pi,\varepsilon)\in B_{n-1}} t^{\mathrm{maj}(\pi,\varepsilon)} = 2^{n-1} \, [n-1]_t! ,$$

which can also be derived from (10).

Corollary 4.4. *The distribution of the comajor index on the hyperoctahedral group is given by*

$$\sum_{(\pi,\varepsilon)\in B_{n-1}} t^{\mathrm{comaj}(\pi,\varepsilon)} = (1+t)^{n-1} \, [n-1]_t! .$$

Proof. Let

$$P = \{ x \in \mathbf{R}^{n-1} \mid |x_1| + \cdots + |x_{n-1}| \leq 1 \}$$

be the $(n-1)$-dimensional cross-polytope. Our strategy to prove Corollary 4.4 is to compute the Ehrhart series

$$\mathrm{Ehr}_P(t) := \sum_{k\geq 0} |kP \cap \mathbf{Z}^{n-1}| \cdot t^k$$

of P in two different ways and to conclude by comparing the results.

On the one hand, note that the cone C over P,

$$C = \{ x \in \mathbf{R}^n \mid |x_1| + \cdots + |x_{n-1}| \leq x_n \} ,$$

is the cone considered in Proposition 4.2 for $a_1 = \cdots = a_{n-1} = 1$, so by Proposition 4.2 its generating function is

$$f_C = \frac{1}{1-z_n} \sum_{(\pi,\varepsilon)\in B_{n-1}} \frac{\prod_{j\in D(\pi,\varepsilon)} \prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} z_n}{\prod_{j=1}^{n-1}\left(1-\prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} z_n\right)}.$$

Since $\mathrm{Ehr}_P(t)$ is obtained by evaluating f_C at $z_1 = \cdots = z_{n-1} = 1$, $z_n = t$, we obtain

$$\begin{aligned}\mathrm{Ehr}_P(t) &= \frac{1}{1-t} \sum_{(\pi,\varepsilon)\in B_{n-1}} \frac{\prod_{j\in D(\pi,\varepsilon)} \prod_{i=j}^{n-1} t}{\prod_{j=1}^{n-1}\left(1-\prod_{i=j}^{n-1} t\right)} \\ &= \frac{\sum_{(\pi,\varepsilon)\in B_{n-1}} \prod_{j\in D(\pi,\varepsilon)} t^{n-j}}{(1-t)\prod_{j=1}^{n-1}(1-t^{n-j})} \\ &= \frac{\sum_{(\pi,\varepsilon)\in B_{n-1}} t^{\mathrm{comaj}(\pi,\varepsilon)}}{(1-t)\prod_{j=1}^{n-1}(1-t^{n-j})}.\end{aligned}$$

On the other hand, it is known [4, Theorem 2.7] that

$$\mathrm{Ehr}_P(t) = \frac{(1+t)^{n-1}}{(1-t)^n}.$$

Together, we obtain

$$\frac{\sum_{(\pi,\varepsilon)\in B_{n-1}} t^{\mathrm{comaj}(\pi,\varepsilon)}}{(1-t)\prod_{j=1}^{n-1}(1-t^{n-j})} = \frac{(1+t)^{n-1}}{(1-t)^n},$$

so

$$\begin{aligned}\sum_{(\pi,\varepsilon)\in B_{n-1}} t^{\mathrm{comaj}(\pi,\varepsilon)} &= \frac{(1+t)^{n-1}(1-t)\prod_{j=1}^{n-1}(1-t^{n-j})}{(1-t)^n} \\ &= \frac{(1+t)^{n-1}\prod_{j=1}^{n-1}(1-t^{n-j})}{(1-t)^{n-1}} \\ &= (1+t)^{n-1}\prod_{j=1}^{n-1}\frac{1-t^{n-j}}{1-t} \\ &= (1+t)^{n-1}\,[n-1]_t!\,.\end{aligned}$$

□

Remark 4.5. The distributions for the descent and comajor index statistics on B_n arise from studying simple choices of the a_i from the set of 0/1-vectors. It would be interesting to determine the structure of the multivariate generating functions (or their specializations) when other 0/1-vectors are used; due to the hyperoctahedral symmetry of our cones, this amounts to studying the case

$$(a_1, a_2, \ldots, a_{n-1}) = (0, \ldots, 0, 1, \ldots, 1).$$

The resulting cones interpolate naturally between cones over hypercubes and cones over crosspolytopes. While we were not able to treat this family of polytopes using the methods exposed in this article, the following section shows how to do so for a different interpolation.

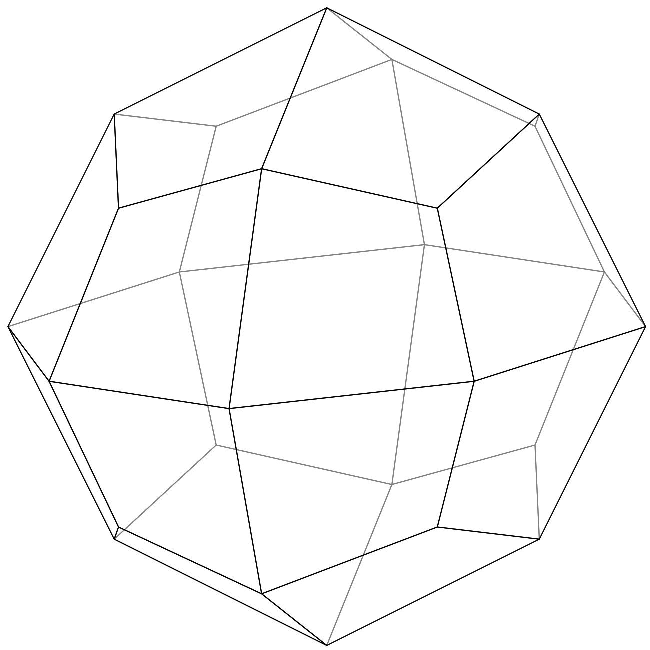

FIGURE 2. The rational polytope $P = \{x \in \mathbf{R}^3 \mid |x_1| + |x_2| + |x_3| + \max\{|x_1|, |x_2|, |x_3|\} \le 1\}$. Its Ehrhart series can be computed by Corollary 4.6.

4.4. **Almost constant coefficients.** In this section, we show how to use Proposition 4.2 to obtain a closed form expression for the Ehrhart series of a family of rational polytopes interpolating between the hypercube and the cross polytope (considered in subsections 4.2 and 4.3, respectively). These are the polytopes P such that the cone over P is of the form considered in Proposition 4.2 such that a_1 through a_{n-2} coincide. An example of such a polytope is shown in Figure 2.

Corollary 4.6. *Let $b, c \ge 0$, not both 0. Let*

$$P = \{x \in \mathbf{R}^{n-1} \mid c \cdot (|x_1| + \cdots + |x_{n-1}|) + b \cdot \max\{|x_1|, \ldots, |x_{n-1}|\} \le 1\}.$$

Then the Ehrhart series of P is

$$\mathrm{Ehr}_P(t) = \begin{cases} [c]_t (1+t^c)^{n-1}/(1-t^c)^n & \text{if } b = 0, \\ [b]_t \sum_{k \ge 0} \left([k+1]_{t^c} + t^c [k]_{t^c}\right)^{n-1} t^{bk} & \text{if } b \ge 1. \end{cases}$$

Proof. In Proposition 4.2, we set $a_1 = \cdots = a_{n-2} = c$ and $a_{n-1} = c + b$. Then $\mathrm{Ehr}_P(t) = f_C(t) := f_C(1, \ldots, 1, t)$, the generating function of C evaluated at $z_1, \ldots, z_{n-1} = 1$, $z_n = t$.

For $b > 0$, the generating function $f_C(t)$ becomes

$$\begin{aligned} f_C(t) &= \frac{\sum_{(\pi,\varepsilon) \in B_{n-1}} \prod_{j \in D(\pi,\varepsilon)} t^{c(n-j)+b}}{(1-t) \prod_{j=1}^{n-1} (1 - t^{c(n-j)+b})} \\ &= \frac{\sum_{(\pi,\varepsilon) \in B_{n-1}} (t^c)^{\mathrm{comaj}(\pi,\varepsilon)} (t^b)^{\mathrm{des}(\pi,\varepsilon)}}{(1-t) \prod_{j=1}^{n-1} (1 - (t^c)^j t^b)}. \end{aligned}$$

For $c \geq 1$, if $b = 0$ then $f_C(t) = f_P(t^c)[c]_t$ where P is the $(n-1)$-dimensional crosspolytope discussed in subsection 4.3; this can be seen by direct computation using Proposition 4.2, and is also a consequence of the fact that crosspolytopes are reflexive [4, Chapter 4]. Otherwise, to simplify, we make use of a result of Chow and Gessel [9, eq. (26)] to compute the joint distribution of descent and comajor index over B_n, namely

$$\sum_{k\geq 0}([k+1]_q + [k]_q)^n x^k = \frac{\sum_{(\pi,\varepsilon)\in B_n} x^{\mathrm{des}(\pi,\varepsilon)} q^{\mathrm{maj}(\pi,\varepsilon)}}{\prod_{j=0}^{n}(1-xq^i)}. \tag{10}$$

Observe that for $(\pi,\varepsilon) \in B_n$, using (9),

$$q^{\mathrm{comaj}(\pi,\varepsilon)} = (q^n)^{\mathrm{des}(\pi,\varepsilon)}(1/q)^{\mathrm{maj}(\pi,\varepsilon)}.$$

Thus, substituting into (10), we get

$$\begin{aligned}\sum_{(\pi,\varepsilon)\in B_n} x^{\mathrm{des}(\pi,\varepsilon)} q^{\mathrm{comaj}(\pi,\varepsilon)} &= \sum_{(\pi,\varepsilon)\in B_n} (xq^n)^{\mathrm{des}(\pi,\varepsilon)}(1/q)^{\mathrm{maj}(\pi,\varepsilon)} \\ &= \prod_{i=0}^{n}(1-xq^{n-i})\sum_{k\geq 0}([k+1]_{1/q} + [k]_{1/q})^n (xq^n)^k \\ &= \prod_{i=0}^{n}(1-xq^{i})\sum_{k\geq 0}([k+1]_{q} + q[k]_{q})^n (x)^k.\end{aligned}$$

To get the numerator of $f_C(t)$, we set $n = n-1$, $x = t^b$ and $q = t^c$ in the last line above and get:

$$\begin{aligned}f_C(t) =& \frac{\prod_{i=0}^{n-1}(1-t^{ci+b})\sum_{k\geq 0}([k+1]_{t^c} + t^c[k]_{t^c})^{n-1}t^{bk}}{(1-t)\prod_{j=1}^{n-1}(1-t^{cj+b})} \\ =& [b]_t\sum_{k\geq 0}([k+1]_{t^c} + t^c[k]_{t^c})^{n-1}t^{bk}.\end{aligned}$$

□

4.5. **Coefficients in arithmetic progression and lecture hall partitions.** If we further generalize the results of the previous subsections to allow a_i to be a linear function of i, then $f_C(t)$ can be expressed in terms of lecture hall partitions.

Lecture hall partitions, introduced by Bousquet-Mélou and Eriksson [7], are elements of the set

$$L_n = \{\lambda \in \mathbf{Z}^n \mid 0 \leq \tfrac{\lambda_1}{1} \leq \tfrac{\lambda_2}{2} \leq \cdots \leq \tfrac{\lambda_n}{n}\}.$$

The following relationship between statistics on lecture hall partitions and statistics on signed permutations follows from work of Pensyl and Savage [11]. Define the lecture hall polytope $P_{n,2}$ by

$$P_{n,2} = \left\{\lambda \in \mathbb{R}^n \;\middle|\; 0 \leq \frac{\lambda_1}{2} \leq \frac{\lambda_2}{4} \leq \cdots \leq \frac{\lambda_n}{2n} \leq 1\right\}.$$

For $\lambda \in L_n$, $(\pi,\varepsilon) \in B_n$, and $\lceil x\rceil = \inf([x,\infty)\cap \mathbf{Z})$, set

$$\mathrm{stat}_1(\lambda) = \sum_{i=1}^{n}\left\lceil\frac{\lambda_i}{2i}\right\rceil, \quad \mathrm{stat}_2(\lambda) = \sum_{i=1}^{n}2i\left\lceil\frac{\lambda_i}{2i}\right\rceil,$$

$$\mathrm{cobin}(\pi,\varepsilon) = \sum_{j\in D(\pi,\varepsilon)}(j+\cdots+n).$$

Setting $k = 2$, $w = 1$, and $z = y$ in [11, Theorem 6], and using the bijection of [11, Theorem 3], the following holds. For a positive integer n, using the notation

$$f_{n,2}(t; q, y) \;=\; \sum_{\lambda \in tP_{n,2} \,\cap \mathbb{Z}^n} q^{\mathrm{stat}_1(\lambda)} y^{\mathrm{stat}_2(\lambda)}\,, \tag{11}$$

we have that

$$\sum_{t \geq 0} f_{n,2}(t; q, y)\, x^t \;=\; \frac{\sum_{(\pi,\varepsilon)\in B_n} q^{\mathrm{comaj}(\pi,\varepsilon)} x^{\mathrm{des}(\pi,\varepsilon)} (y^2)^{\mathrm{cobin}(\pi,\varepsilon)}}{(1-x)\prod_{i=0}^{n-1}(1 - xq^{n-i}y^{2((i+1)+\cdots+n)})}\,. \tag{12}$$

Thus,

$$\begin{aligned}
\sum_{\lambda\in L_n} x^{\lceil \lambda_n/(2n)\rceil} q^{\mathrm{stat}_1(\lambda)} y^{\mathrm{stat}_2(\lambda)} &= \sum_{t\geq 0} x^t \sum_{\lambda \in L_n : \lceil \lambda_n/(2n)\rceil = t} q^{\mathrm{stat}_1(\lambda)} y^{\mathrm{stat}_2(\lambda)} \\
&= 1 + \sum_{t \geq 1} x^t \left(f_{n,2}(t; q, y) - f_{n,2}(t-1; q, y)\right) \\
&= (1-x) \sum_{t\geq 0} x^t f_{n,2}(t; q, y) \\
&= \frac{\sum_{(\pi,\varepsilon)\in B_n} q^{\mathrm{comaj}(\pi,\varepsilon)} x^{\mathrm{des}(\pi,\varepsilon)} (y^2)^{\mathrm{cobin}(\pi,\varepsilon)}}{\prod_{i=0}^{n-1}(1 - xq^{n-i}y^{2((i+1)+\cdots+n)})}\,,
\end{aligned}$$

from which it follows that

$$\sum_{\lambda\in L_n} x^{\lceil \frac{\lambda_n}{2n}\rceil} q^{\mathrm{stat}_1(\lambda)} y^{\mathrm{stat}_2(\lambda)} = \frac{\sum_{(\pi,\varepsilon)\in B_n} q^{\mathrm{comaj}(\pi,\varepsilon)} x^{\mathrm{des}(\pi,\varepsilon)} (y^2)^{\mathrm{cobin}(\pi,\varepsilon)}}{\prod_{i=0}^{n-1}(1 - xq^{n-i}y^{2((i+1)+\cdots+n)})}\,. \tag{13}$$

We can apply Proposition 4.2, with $z_1 = \cdots = z_{n-1} = 1$, $z_n = t$ and appropriate choices of a_i, to establish a surprising connection between lecture hall partitions and type B symmetrically constrained cones.

Corollary 4.7. *Let $d \geq 0$, $c \geq -2d$, and $b \geq 0$, not all 0. Define $a_1, \ldots, a_{n-1}$ by*

$$a_i = 2di + c \quad (i = 1, \ldots, n-2), \quad a_{n-1} = 2d(n-1) + c + b\,.$$

Then

$$f_C(t) = \frac{1}{1-t} \sum_{\lambda \in L_{n-1}} t^{\sum_{i=1}^{n-1} a_i \lceil \frac{\lambda_i}{2i} \rceil}\,.$$

Proof. Observe that for $1 \leq i \leq n-2$, substituting the values of a_i into Proposition 4.2 with $z_1 = \cdots = z_{n-1} = 1$, $z_n = t$, and then using (13), we obtain

$$\begin{aligned}
f_C(t) &= \frac{\sum_{(\pi,\varepsilon)\in B_{n-1}} \prod_{j \in D(\pi,\varepsilon)} t^{a_j + \cdots + a_{n-1}}}{(1-t)\prod_{j=1}^{n-1}(1 - t^{a_j+\cdots+a_{n-1}})} \\
&= \frac{\sum_{(\pi,\varepsilon)\in B_{n-1}} \prod_{j \in D(\pi,\varepsilon)} t^{b + (n-j)c + 2d(j+\cdots+(n-1))}}{(1-t)\prod_{i=1}^{n-1}(1 - t^{(b + c(n-i) + 2d(i+\cdots+(n-1))})} \\
&= \frac{\sum_{(\pi,\varepsilon)\in B_{n-1}} (t^c)^{\mathrm{comaj}(\pi,\varepsilon)} (t^b)^{\mathrm{des}(\pi,\varepsilon)} (t^{2d})^{\mathrm{cobin}(\pi,\varepsilon)}}{(1-t)\prod_{i=1}^{n-1}(1 - t^{(b + c(n-i) + 2d(i+\cdots+(n-1))})} \\
&= \frac{\sum_{(\pi,\varepsilon)\in B_{n-1}} (t^c)^{\mathrm{comaj}(\pi,\varepsilon)} (t^b)^{\mathrm{des}(\pi,\varepsilon)} (t^{2d})^{\mathrm{cobin}(\pi,\varepsilon)}}{(1-t)\prod_{i=1}^{n-1}(1 - t^{(b + c(n-i) + 2d(i+\cdots+(n-1))})}
\end{aligned}$$

$$= \frac{1}{1-t} \sum_{\lambda \in L_{n-1}} t^{b\left\lceil \frac{\lambda_{n-1}}{2n-2} \right\rceil + c\,\mathrm{stat}_1(\lambda) + d\,\mathrm{stat}_2(\lambda)}$$

$$= \frac{1}{1-t} \sum_{\lambda \in L_{n-1}} t^{\sum_{i=1}^{n-1} a_i \left\lceil \frac{\lambda_i}{2i} \right\rceil} . \qquad \square$$

As an example, let $n = 3$, $a_1 = 2$ and $a_2 = 4$. Then

$$C = \{x \in \mathbf{R}^3 \mid \forall \pi \in S_2,\ \varepsilon \in \{\pm 1\}^2 : 2\varepsilon_1 x_{\pi(1)} + 4\varepsilon_2 x_{\pi(2)} \le x_3\},$$

and from Proposition 4.2,

$$\sum_{x \in C} t^{x_3} = \frac{1 + 3t^3 + 3t^6 + t^{10}}{(1-t)(1-t^6)(1-t^4)}$$
$$= 1 + t + t^2 + t^3 + 5t^4 + 5t^5 + 9t^6 + 9t^7 + 13t^8 + 13t^9 + \cdots .$$

On the other hand, checking the corollary, we have

$$\frac{1}{1-t} \sum_{\lambda \in L_2} t^{2\lceil \frac{\lambda_1}{2} \rceil + 4\lceil \frac{\lambda_2}{4} \rceil} = \frac{1 + 4t^4 + 4t^6 + 4t^8 + 8t^{10} + 8t^{12} + 8t^{14} + \cdots}{1-t}$$
$$= 1 + t + t^2 + t^3 + 5t^4 + 5t^5 + 9t^6 + 9t^7 + 13t^8 + 13t^9 + \cdots .$$

Remark 4.8. It would be interesting to find a direct correspondence between lecture hall partitions and the points in the cone considered in Corollary 4.7.

5. Cones with symmetry of type D

In this section, we consider the case of monoconditional cones with symmetry given by a Coxeter group of type D. Unsurprisingly, much of the setup in this section is similar to the hyperoctahedral case; the most notable new feature is that we consider lattice point enumeration with respect to a sublattice of the standard integer lattice.

Throughout this section, let W be the finite reflection group of type D_{n-1} on the first $n-1$ components of $\mathbf{R}^n$. Specifically, let $s_1 \in \mathrm{O}(n)$ be the reflection at the hyperplane $\{x \in \mathbf{R}^n \mid x_1 + x_2 = 0\}$. For $j = 2, \ldots, n-1$, let $s_j \in \mathrm{O}(n)$ be the transposition of the $(j-1)$st and jth component in $\mathbf{R}^n$. Then $s_1, \ldots, s_{n-1}$ are the simple generators of W.

We next describe $D_{\mathrm{r}}(\sigma)$ and the action of W explicitly. Let

$$E_{n-1} := \{\varepsilon \in \{\pm 1\}^{n-1} \mid \varepsilon_1 \cdots \varepsilon_{n-1} = 1\}.$$

For $\pi \in S_{n-1}$ and $\varepsilon \in E_{n-1}$, define $\sigma_{\pi,\varepsilon} \in \mathrm{O}(n)$ by $\sigma_{\pi,\varepsilon} e_i = \varepsilon_i e_{\pi(i)}$ for $i < n$ and $\sigma_{\pi,\varepsilon} e_n = e_n$. Then $W = \{\sigma_{\pi,\varepsilon} \mid \pi \in S_{n-1}, \varepsilon \in E_{n-1}\}$. We use the convention that $\pi(n) := n$ for $\pi \in S_{n-1}$ and $\varepsilon_n := 1$ for $\varepsilon \in E_{n-1}$.

For $\pi \in S_{n-1}$ and $\varepsilon \in E_{n-1}$ let

$$D(\pi, \varepsilon) := \{j \in \{1, \ldots, n-1\} \mid \varepsilon_{j-1}\pi(j-1) > \varepsilon_j \pi(j)\}$$

with the convention that $\varepsilon_0 \pi(0) := -\varepsilon_2 \pi(2)$.

Proposition 5.1 ([5, Proposition 8.2.2])**.** *For all $\sigma_{\pi,\varepsilon} \in W$ we have*

$$D_{\mathrm{r}}(\sigma_{\pi,\varepsilon}) = D(\pi, \varepsilon).$$

For a proposition P, we use the symbol

$$[P] := \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

Proposition 5.2. *Fix integers $a_1, \ldots, a_{n-1}$ such that $|a_1| \le a_2 \le \cdots \le a_{n-1} \ne 0$. Let*

$$C := \{x \in \mathbf{R}^n \mid \forall \pi \in S_{n-1},\ \varepsilon \in E_{n-1} : \\ \varepsilon_1 a_1 x_{\pi(1)} + \cdots + \varepsilon_{n-1} a_{n-1} x_{\pi(n-1)} \le x_n\}\,.$$

Let

$$\Gamma := \{x \in \mathbf{Z}^n \mid x_1 \equiv \cdots \equiv x_{n-1} \mod (2)\}.$$

The generating function of C with respect to Γ is

$$f_C = \frac{1}{1 - z_n} \sum_{\pi \in S_{n-1}} \sum_{\varepsilon \in E_{n-1}} \frac{\prod_{j \in D(\pi,\varepsilon)} \left(z_{\pi(1)}^{-\varepsilon_1} z_n^{-a_1}\right)^{[j=2]} \left(\prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} z_n^{a_i}\right)^{1+[j\ge 3]}}{\prod_{j=1}^{n-1} \left(1 - \left(z_{\pi(1)}^{-\varepsilon_1} z_n^{-a_1}\right)^{[j=2]} \left(\prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} z_n^{a_i}\right)^{1+[j\ge 3]}\right)},$$

where $z_1, \ldots, z_n$ are the coordinates corresponding to the standard lattice $\mathbf{Z}^n \subset \mathbf{R}^n$.

Note that the conditions on the a_i are normalizations rather than restrictions.

Proof. The cone C is symmetric and monoconditional for W. Let

$$F := \{x \in \mathbf{R}^n \mid |x_1| \le x_2 \le \cdots \le x_{n-1}\}\,,$$

a fundamental domain for W. Our simple generators $s_1, \ldots, s_{n-1}$ defined at the beginning of this section are the simple generators of W corresponding to the facets of F. Let $x_0 := (a_1, \ldots, a_n, -1) \in F$. By the proof of Lemma 2.5,

$$\begin{aligned} C_+ &= \{x \in F \mid (x, x_0) \le 0\} = \{x \in F \mid a_1 x_1 + \cdots + a_{n-1} x_{n-1} \le x_n\} \\ &= \{x \in \mathbf{R}^n \mid Ax \ge 0\}\,, \end{aligned}$$

where

$$A := \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 & 0 \\ -a_1 & \cdots & \cdots & \cdots & -a_{n-1} & 1 \end{pmatrix}.$$

The inverse of A is

$$A^{-1} = \begin{pmatrix} 1/2 & -1/2 & 0 & \cdots & \cdots & 0 \\ 1/2 & 1/2 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & 1 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1/2 & 1/2 & 1 & \cdots & 1 & 0 \\ \Sigma_1/2 & \Sigma_2'/2 & \Sigma_3 & \cdots & \Sigma_{n-1} & 1 \end{pmatrix},$$

where $\Sigma_j := a_j + \cdots + a_{n-1}$ and $\Sigma_2' := \Sigma_2 - a_1$. Hence the Γ-primitive generators of C_+ are the column vectors $b_1, \ldots, b_n$ of the matrix

$$B := \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & 2 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 2 & \cdots & 2 & 0 \\ \Sigma_1 & \Sigma_2' & 2\Sigma_3 & \cdots & 2\Sigma_{n-1} & 1 \end{pmatrix}.$$

As $\det(B) = 2^{n-2} = |\mathbf{Z}^n/\Gamma|$ it follows that C_+ is unimodular. Note that $b_1, \ldots, b_n$ are enumerated in the unique way such that $b_j \notin H_j$ for $j < n$, where H_j is the reflection hyperplane for the reflection s_j. Hence by Corollary 2.9 the generating function of C is

$$f_C = \sum_{\sigma \in W} \frac{\prod_{j \in D_{\mathrm{r}}(\sigma)} z^{\sigma b_j}}{(1 - z^{\sigma b_1}) \cdots (1 - z^{\sigma b_n})}.$$

Let

$$b_{ij} := \begin{cases} -1 & \text{if } i = 1,\ j = 2, \\ 1 & \text{if } j \leq i < n \text{ or } i = j = n, \\ 0 & \text{if } i < j \geq 2, \\ \Sigma_1 & \text{if } i = n,\ j = 1, \\ \Sigma_2' & \text{if } i = n,\ j = 2, \\ 2\Sigma_j & \text{if } i = n,\ 2 < j < n \end{cases}$$

be the ith component of b_j, i.e., the (i,j)th component of B. Thus

$$\begin{aligned} f_C &= \sum_{\sigma \in W} \frac{\prod_{j \in D_{\mathrm{r}}(\sigma)} z^{\sigma b_j}}{\prod_{j=1}^{n} (1 - z^{\sigma b_j})} \\ &= \sum_{\pi \in S_{n-1}} \sum_{\varepsilon \in E_{n-1}} \frac{\prod_{j \in D_{\mathrm{r}}(\sigma_{\pi,\varepsilon})} z^{\sigma_{\pi,\varepsilon} b_j}}{\prod_{j=1}^{n} (1 - z^{\sigma_{\pi,\varepsilon} b_j})} \\ &= \sum_{\pi \in S_{n-1}} \sum_{\varepsilon \in E_{n-1}} \frac{\prod_{j \in D(\pi,\varepsilon)} \prod_{i=1}^{n} z_{\pi(i)}^{\varepsilon_i b_{ij}}}{\prod_{j=1}^{n} \left(1 - \prod_{i=1}^{n} z_{\pi(i)}^{\varepsilon_i b_{ij}}\right)} \\ &= \frac{1}{1 - z_n} \sum_{\pi \in S_{n-1}} \sum_{\varepsilon \in E_{n-1}} \frac{\prod_{j \in D(\pi,\varepsilon)} \left(z_{\pi(1)}^{-\varepsilon_1} z_n^{-a_1}\right)^{[j=2]} \left(\prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} z_n^{a_i}\right)^{1+[j \geq 3]}}{\prod_{j=1}^{n-1} \left(1 - \left(z_{\pi(1)}^{-\varepsilon_1} z_n^{-a_1}\right)^{[j=2]} \left(\prod_{i=j}^{n-1} z_{\pi(i)}^{\varepsilon_i} z_n^{a_i}\right)^{1+[j \geq 3]}\right)}. \qquad \square \end{aligned}$$

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