LECTURE HALL THEOREMS, q-SERIES AND TRUNCATED OBJECTS

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Abstract. We show here that the refined theorems for both lecture hall partitions and anti-lecture hall compositions can be obtained as straightforward consequences of two q-Chu Vandermonde identities, once an appropriate recurrence is derived. We use this approach to get new lecture hall-type theorems for truncated objects. The *truncated lecture hall partitions* are sequences $(\lambda_1, \dots, \lambda_k)$ such that

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \dots \geq \frac{\lambda_k}{n-k+1} \geq 0$$

and we show that their generating function is :

$$\sum_{m=0}^{k} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m+1}{2}} \frac{(-q^{n-m+1};q)_m}{(q^{2n-m+1};q)_m}.$$

From this, we are able to give a combinatorial characterization of truncated lecture hall partitions and new finite versions of refinements of Euler's theorem. The *truncated anti-lecture hall compositions* are sequences $(\lambda_1, \dots, \lambda_k)$ such that

$$\frac{\lambda_1}{n-k+1} \geq \frac{\lambda_2}{n-k+2} \geq \dots \geq \frac{\lambda_k}{n} \geq 0.$$

We show that their generating function is :

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1};q)_k}{(q^{2(n-k+1)};q)_k},$$

giving a finite version of a well-known partition identity. We give two different multivariate refinements of these new results : the q-calculus approach gives (u,v,q)-refinements, while a completely different approach gives odd/even (x,y)-refinements.

1. Introduction

For a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of nonnegative integers, define the *weight* of λ to be $|\lambda| = \lambda_1 + \cdots + \lambda_n$ and call each λ_i a *part* of λ. If λ has all parts nonnegative, we call it a *composition* and if, in addition, λ is a non-increasing sequence, we call it a *partition*.

In [4], inspired by work of Eriksson and Eriksson on Coxeter groups [9], Bousquet-Mélou and Eriksson considered *lecture hall partitions*, specifically, the set L_n of partitions, λ, into n nonnegative parts satisfying

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_{n-1}}{2} \geq \frac{\lambda_n}{1} \geq 0,$$

and proved the following surprising result.

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The Lecture Hall Theorem [4]**:**

$$L_n(q) \triangleq \sum_{\lambda \in L_n} q^{|\lambda|} = \frac{1}{(q;q^2)_n} \tag{1}$$

with $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$.

In [4], Bousquet-Mélou and Eriksson gave two proofs of the Lecture Hall Theorem: one based on Coxeter groups and one combinatorial. Subsequently, Andrews gave a proof based on partition analysis [2]. A refinement and generalizations of the identity (1) were given by Bousquet-Mélou and Eriksson in [5]. They showed that an elegant mapping between certain partitions in L_n and partitions in L_{n-1} gives a functional equation which easily implies the result. The first bijective proof of the Lecture Hall Theorem was given by Yee [13] and is close to [4]. Others followed [3, 11].

An involution in [4] gave the following nice refinement of (1).

The Odd/Even Lecture Hall Theorem [4]**:** Given $\lambda = (\lambda_1, \lambda_2, \ldots)$, define $\lambda_o = (\lambda_1, \lambda_3, \ldots)$ and $\lambda_e = (\lambda_2, \lambda_4, \ldots)$. Then

$$L_n(x,y) \triangleq \sum_{\lambda \in L_n} x^{|\lambda_o|} y^{|\lambda_e|} = \prod_{i=1}^{n} \frac{1}{1-x^i y^{i-1}}. \tag{2}$$

A different approach of Bousquet-Mélou and Eriksson in [6] led to the following.

The Refined Lecture Hall Theorem [6]**:**

$$L_n(u,v,q) \triangleq \sum_{\lambda \in L_n} q^{|\lambda|} u^{|\lceil \lambda \rceil|} v^{o(\lceil \lambda \rceil)} = \frac{(-uvq;q)_n}{(u^2q^{n+1};q)_n}, \tag{3}$$

where for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, $o(\lambda)$ is the number of odd parts of λ and $\lceil \lambda \rceil$ is the partition $(\lceil \lambda_1/n \rceil, \lceil \lambda_2/(n-1) \rceil, \ldots, \lceil \lambda_{n-1}/2 \rceil, \lceil \lambda_n/1 \rceil)$.

Setting $u = v = 1$ in (3) gives (1). Yee gave a beautiful bijective proof [14] of this theorem.

In [7], we considered a new twist on these results by studying the set A_n of *compositions* into at most n parts satisfying

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \geq \frac{\lambda_n}{n} \geq 0.$$

We referred to these as *anti-lecture hall compositions* and showed the following with a bijective proof along the lines of Yee's proof of (3) in [14]

The Refined Anti-Lecture Hall Theorem [7]**:**

$$A_n(u,v,q) \triangleq \sum_{\lambda \in A_n} q^{|\lambda|} u^{|\lfloor \lambda \rfloor|} v^{o(\lfloor \lambda \rfloor)} = \frac{(-uvq;q)_n}{(u^2q^2;q)_n}, \tag{4}$$

where $\lfloor \lambda \rfloor = (\lfloor \lambda_1/1 \rfloor, \lfloor \lambda_2/2 \rfloor, \ldots, \lfloor \lambda_n/n \rfloor)$ and $o(\lambda)$ denotes the number of odd parts of a composition λ.

Setting $u = v = 1$ in (4) gives the following analog of (1):

The Anti-Lecture Hall Theorem [7]**:**

$$A_n(q) \triangleq \sum_{\lambda \in A_n} q^{|\lambda|} = \frac{(-q;q)_n}{(q^2;q)_n}. \tag{5}$$

At the time of [7] we did not have a q-series proof of (4) and we were not able to prove an odd/even refinement of (5) analogous to (2), although we had a conjecture as to its form.

In this paper we extend all of these results. Our starting point is a proof of The Refined Lecture Hall Theorem from [6], where Bousquet-Mélou and Eriksson gave a two-step proof of (3) using basic q-series identities. They explained separately the numerator and the denominator using elementary techniques, deriving a recurrence to obtain the denominator and noting that the recurrence could be solved using a special case of the q-analog of the Chu-Vandermonde summation ([1], 3.3.10).

In Section 2.1, we pursue their approach, but proceed directly to a recurrence for $L_n(u, v, q)$ which can be solved in a straightforward way using the identity q-Chu Vandermonde II [10]:

$$\frac{a^n (c/a;q)_n}{(c;q)_n} = \sum_{m=0}^{n} \frac{(a;q)_m (q^{-n};q)_m}{(c;q)_m (q;q)_m} q^m.$$

With this modified approach, we give, in Section 2.2, a new proof of the Refined Anti-Lecture Hall Theorem. We show that it can be obtained using another q-Chu Vandermonde I [10]:

$$\frac{(c/a;q)_n}{(c;q)_n} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(a;q)_m}{(c;q)_m} (-c/a)^m q^{\binom{m}{2}},$$

where $\begin{bmatrix} n \\ m \end{bmatrix}_q = (q^{n-m+1};q)_m/(q;q)_m$ is the classical Gaussian polynomial, the generating function for partitions into m nonnegative parts of size at most $n-m$.

From this point on, all of our results are new. We conjectured them thanks to the Maple implementation of the generating function developed in [8]. In Section 3, we show how the q-series techniques of Section 2 can be extended to get new identities for the enumeration of *truncated objects*. For $n \geq k$, let $L_{n,k}$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ into k nonnegative parts satisfying

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_k}{n-k+1} \geq 0.$$

We refer to these as *truncated lecture hall partitions*. (Note that the case $n = k$ corresponds to the ordinary lecture hall partitions.) These objects were introduced by Eriksen in [11], where he gave a recurrence for their generating function in one variable, but no closed-form solution.

We show in Section 3.1 how to compute the three-variable generating function:

$$L_{n,k}(u, v, q) \triangleq \sum_{\lambda \in L_{n,k}} q^{|\lambda|} u^{|\lceil \lambda \rceil|} v^{o(\lfloor \lambda \rfloor)}$$

where $\lceil \lambda \rceil = (\lceil \lambda_1/n \rceil, \lceil \lambda_2/(n-1) \rceil, \ldots, \lceil \lambda_{k-1}/(n-k+2) \rceil, \lceil \lambda_k/(n-k+1) \rceil)$.

The Refined Truncated Lecture Hall Theorem:

(6)
$$L_{n,k}(u, v, q) = \sum_{m=0}^{k} (uv)^m q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-(u/v) q^{n-m+1};q)_m}{(u^2 q^{2n-m+1};q)_m}.$$

In Section 3.2 we study *truncated anti-lecture hall compositions*, defined for each $n \geq k-1$ as the set $A_{n,k}$ of compositions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ into k nonnegative parts satisfying

$$\frac{\lambda_1}{n-k+1} \geq \frac{\lambda_2}{n-k+2} \geq \cdots \geq \frac{\lambda_k}{n} \geq 0.$$

(When $n = k$, these are the ordinary anti-lecture hall compositions.) If we let $\lfloor \lambda \rfloor$ denote the partition $(\lfloor \lambda_1/(n-k+1) \rfloor, \lfloor \lambda_2/(n-k+2) \rfloor, \ldots, \lfloor \lambda_{k-1}/(n-1) \rfloor, \lfloor \lambda_n/n \rfloor)$, we get the following, for $n \geq k$:

The Refined Truncated Anti-Lecture Hall Theorem:

$$A_{n,k}(u,v,q) \triangleq \sum_{\lambda \in A_{n,k}} q^{|\lambda|} u^{|\lfloor \lambda \rfloor|} v^{o(\lfloor \lambda \rfloor)} = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uvq^{n-k+1};q)_k}{(u^2 q^{2(n-k+1)};q)_k}. \tag{7}$$

We next consider odd/even refinements of the generating functions for truncated objects, analogous to (2) for lecture hall partitions. In Section 4.1, we show the following for truncated lecture hall partitions.

The Odd/Even Truncated Lecture Hall Theorem:

$$L_{n,k}(x,y) \triangleq \sum_{\lambda \in L_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|} = \sum_{m=0}^{k} \frac{\left(x^{\lfloor m/2 \rfloor +1} y^{\lfloor m/2 \rfloor}\right)^{\lceil m/2 \rceil} \begin{bmatrix} n - \lceil m/2 \rceil \\ \lfloor m/2 \rfloor \end{bmatrix}_{xy}}{(x;xy)_{\lceil m/2 \rceil} (x^n y^{n-1}; (xy)^{-1})_{\lfloor m/2 \rfloor}}. \tag{8}$$

Similarly, in Section 4.2 we find the odd/even generating function for truncated anti-lecture hall compositions. Let $n \geq k-1$.

The Odd/Even Truncated Anti-Lecture Hall Theorem:

$$A_{n,k}(x,y) \triangleq \sum_{\lambda \in A_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|} = \frac{\begin{bmatrix} n \\ \lfloor k/2 \rfloor \end{bmatrix}_{xy}}{(x;xy)_{\lceil k/2 \rceil} (x^{n-k+1} y^{n-k+2}; xy)_{\lfloor k/2 \rfloor}}. \tag{9}$$

In particular, setting $n = k$ in (9) gives for the first time the odd/even generating function for anti-lecture hall compositions:

$$A_n(x,y) = A_{n,n}(x,y) = \frac{\begin{bmatrix} n \\ \lfloor n/2 \rfloor \end{bmatrix}_{xy}}{(x;xy)_{\lceil n/2 \rceil} (xy^2; xy)_{\lfloor n/2 \rfloor}}. \tag{10}$$

One of the many interesting things about lecture hall partitions is that the Lecture Hall Theorem gives them a simple interpretation in terms of partitions into odd parts: *the number of partitions of N in L_n is equal to the number of partitions of N into odd parts less than $2n$.* A similar interpretation of truncated lecture hall partitions is not so evident from their generating function in (6) or in (8). However, in Section 5 we show the following correspondence between truncated lecture hall partitions and partitions into odd parts with certain restrictions.

Characterization of Truncated Lecture Hall Partitions:

> The number of truncated lecture hall partitions of N in $L_{n,k}$ is equal to the number of partitions of N into odd parts less than $2n$, with the following constraint on the parts: at most $\lfloor k/2 \rfloor$ parts can be chosen from the set
>
> $$\{2\lceil k/2 \rceil + 1, 2\lceil k/2 \rceil + 3, \ldots, 2(n - \lfloor k/2 \rfloor) - 1\}.$$

As $n \to \infty$, the set L_n of lecture hall partitions approaches the set of partitions into distinct parts. Similarly, the right-hand side of (1) approaches the set of partitions into odd parts. In this sense, the Lecture Hall Theorem is viewed as a *finite version* of Euler's Theorem, which states: *The number of partitions of an integer N into distinct parts is equal to the number of partitions of N into odd parts.* In Section 5.1, we show how our results on truncated lecture hall partitions lead to finite versions of certain refinements of Euler's Theorem that are implied by Sylvester's bijection [12].

We show in Section 5.2 how truncated anti-lecture hall theorems can be viewed as finite versions of another well-known identity: *The number of partitions of N into k parts is equal to the number of partitions of N with no part larger than k.*

2. The Refined Lecture Hall Theorems

We will make use of the q-multinomial coefficient, defined for $n = n_0 + n_1 + \cdots + n_t$ by

$$\begin{bmatrix} n \\ n_0, n_1, \ldots, n_t \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n_0}(q;q)_{n_1}\cdots(q;q)_{n_t}},$$

from which it follows that

$$\begin{bmatrix} n \\ n_0, n_1 \ldots n_t \end{bmatrix}_q = \begin{bmatrix} n \\ n_0 \end{bmatrix}_q \begin{bmatrix} n-n_0 \\ n_1, \ldots, n_t \end{bmatrix}_q. \tag{11}$$

2.1. Lecture Hall Partitions.

We review the proof of the Refined Lecture Hall Theorem (3) of Bousquet-Mélou and Eriksson. The basic ideas of this proof come from [6]. Given a lecture hall partition $\lambda \in L_n$, denote by $\lceil \lambda \rceil$ the sequence $(\lceil \lambda_1/n \rceil, \lceil \lambda_2/(n-1) \rceil, \ldots, \lceil \lambda_n/1 \rceil)$. We can write $\lambda_i = (n-i+1)\mu_i - r_i$, with $0 \le r_i \le n-i$ for $1 \le i \le n$. Then $(\mu_1, \ldots, \mu_n) = \lceil \lambda \rceil$.

Proposition 1. [6] *A partition λ is in L_n if and only if*

(1) $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_n \ge 0$ *and*

(2) $r_i \le r_{i+1}$ *whenever* $\mu_i = \mu_{i+1}$.

The first condition implies that $\lceil \lambda \rceil$ is a partition into n nonnegative parts.

Let P_n be the set of partitions into n nonnegative parts. Let μ be a partition in P_n. In [6] the authors compute the generating function of the lecture hall partitions λ having $\lceil \lambda \rceil = \mu$. Let this generating function be $L_\mu(q)$.

Proposition 2. [6] *For $\mu \in P_n$,*

$$L_\mu(q) \triangleq \sum_{\substack{\lambda \in L_n \\ \lceil \lambda \rceil = \mu}} q^{|\lambda|} = q^{\sum_{i=1}^n (n-i+1)\mu_i} \begin{bmatrix} n \\ m_0, m_1, \ldots, m_{\mu_1} \end{bmatrix}_{1/q}.$$

where m_i is the multiplicity of the part i in μ.

Proof. We sketch here the main ideas of the proof. As $|\lambda| = \sum_{i=1}^n ((n-i+1)\mu_i - r_i)$,

$$L_\mu(q) = q^{\sum_{i=1}^n (n-i+1)\mu_i} \sum_{(r_1,\ldots,r_n)} q^{-\sum_{i=1}^n r_i}.$$

For $0 \le i \le \mu_1$, let

$$\ell_i = \begin{cases} 0 & \text{if } i = 0 \\ \sum_{j=\mu_1-i+1}^{\mu_1} m_j & \text{otherwise.} \end{cases}$$

Then the condition (2) in Proposition 1 implies that $(r_{\ell_{i+1}}, r_{\ell_{i+1}-1}, \ldots, r_{\ell_i+1})$ is a partition into $\ell_{i+1} - \ell_i = m_{\mu_1-i}$ nonnegative parts and that these parts are less than or equal to $n - \ell_{i+1}$. Therefore their generating function is well-known to be a Gaussian polynomial:

$$\sum_{(r_{\ell_{i+1}},\ldots,r_{\ell_i+1})} q^{(r_{\ell_{i+1}}+\ldots+r_{\ell_i+1})} = \begin{bmatrix} n-\ell_{i+1}+m_{\mu_1-i} \\ m_{\mu_1-i} \end{bmatrix}_q = \begin{bmatrix} n-\ell_i \\ m_{\mu_1-i} \end{bmatrix}_q . \tag{12}$$

Hence the result follows from the computation below, which uses (12) for the second equality and repeated application of (11) for the third. b

$$\begin{aligned} \sum_{(r_1,\ldots,r_n)} q^{-\sum_{i=1}^n r_i} &= \prod_{i=0}^{\mu_1} \sum_{(r_{\ell_{i+1}},\ldots,r_{\ell_i+1})} (1/q)^{(r_{\ell_{i+1}}+\ldots+r_{\ell_i+1})} \\ &= \prod_{i=0}^{\mu_1} \begin{bmatrix} n-\ell_i \\ m_{\mu_1-i} \end{bmatrix}_{1/q} \\ &= \begin{bmatrix} n \\ m_0, m_1, \ldots, m_{\mu_1} \end{bmatrix}_{1/q} . \end{aligned}$$

□

At this point we take a different route than [6], proceeding directly to enumeration of the partitions μ with all parts positive. Let $P_{n,m}$ be the set of partitions into n nonnegative parts, m of which are positive. Given μ in $P_{m,n}$, we define $\tilde{\mu}$ in P_m by $\tilde{\mu}_i = \mu_i - 1$, $1 \le i \le m$. Then

Proposition 3. *For $\mu \in P_{m,n}$,*

$$L_\mu(q) = q^{(n-m)|\tilde{\mu}|+\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q L_{\tilde{\mu}}(q).$$

Proof. Using Proposition 2, we know that if m_i is the multiplicity of i in μ, then

$$L_\mu(q) = q^{\sum_{i=1}^m (n-i+1)(\tilde{\mu}_i+1)} \begin{bmatrix} n \\ m_0, m_1, \ldots, m_{\mu_1} \end{bmatrix}_{1/q},$$

and that

$$L_{\tilde{\mu}}(q) = q^{\sum_{i=1}^m (m-i+1)\tilde{\mu}_i} \begin{bmatrix} m \\ m_1, \ldots, m_{\mu_1} \end{bmatrix}_{1/q}.$$

Since $m_0 = n - m$, using (11), we can write the first equation as

$$L_\mu(q) = q^{(n-m)|\tilde{\mu}|+m(n+1)-m(m+1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_{1/q} q^{\sum_{i=1}^m (m-i+1)\tilde{\mu}_i} \begin{bmatrix} m \\ m_1, \ldots, m_{\mu_1} \end{bmatrix}_{1/q}.$$

The result follows from the second equation and the identity

$$\begin{bmatrix} n \\ m \end{bmatrix}_{1/q} = q^{-m(n-m)} \begin{bmatrix} n \\ m \end{bmatrix}_q .$$

□

An easy consequence of Proposition 3 is that

$$u^{|\mu|}v^{o(\mu)}L_\mu(q) = (uv)^m u^{|\tilde\mu|}v^{-o(\tilde\mu)}q^{(n-m)|\tilde\mu|+\binom{m+1}{2}}\begin{bmatrix} n \\ m\end{bmatrix}_q L_{\tilde\mu}(q), \tag{13}$$

as $|\mu| = |\tilde\mu| + m$ and $o(\mu) = m - o(\tilde\mu)$, where $o(\lambda)$ denotes the number of odd parts of λ.

In order to prove the Refined Lecture Hall Theorem (3), the generating function we are looking for, can be rewritten as:

$$L_n(u,v,q) = \sum_{\mu\in P_n} u^{|\mu|}v^{o(\mu)}L_\mu(q) = \sum_{m=0}^{n}\sum_{\mu\in P_{n,m}} u^{|\mu|}v^{o(\mu)}L_\mu(q). \tag{14}$$

Then we can prove the following recurrence for $L_n(u,v,q)$.

Proposition 4. *$L_0(u,v,q) = 1$ and for $n > 0$,*

$$L_n(u,v,q) = \sum_{m=0}^{n}\begin{bmatrix} n \\ m\end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} L_m(uq^{n-m}, 1/v, q).$$

Proof.

$$\begin{aligned}
L_n(u,v,q) &= \sum_{m=0}^{n}\sum_{\mu\in P_{n,m}} u^{|\mu|}v^{o(\mu)}L_\mu(q)\\
&= \sum_{m=0}^{n}\sum_{\tilde\mu\in P_m} (uv)^m u^{|\tilde\mu|}v^{-o(\tilde\mu)}q^{(n-m)|\tilde\mu|+\binom{m+1}{2}}\begin{bmatrix} n \\ m\end{bmatrix}_q L_{\tilde\mu}(q) \qquad \text{(using (13))}\\
&= \sum_{m=0}^{n}\begin{bmatrix} n \\ m\end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}}\sum_{\tilde\mu\in P_m}(uq^{n-m})^{|\tilde\mu|}v^{-o(\tilde\mu)}L_{\tilde\mu}(q)
\end{aligned}$$

If we now apply the first equality of (14) to $L_m(uq^{n-m}, 1/v, q)$, we get the last sum in the last line above and the proposition is proved. □

Proof of The Refined Lecture Hall Theorem (3): We solve the recurrence of Proposition 4 using the identity q-Chu Vandermonde II:

$$\frac{a^n(c/a;q)_n}{(c;q)_n} = \sum_{m=0}^{n}\frac{(a;q)_m(q^{-n};q)_m}{(c;q)_m(q;q)_m}q^m. \tag{15}$$

If we set $a = -vq^{-n}/u$, $c = q^{-2n}/u^2$ in (15), we get that

$$(-vq^{-n}/u)^n\frac{(-q^{-n}/uv;q)_n}{(q^{-2n}/u^2;q)_n} = \sum_{m=0}^{n}\frac{(-vq^{-n}/u;q)_m(q^{-n};q)_m}{(q^{-2n}/u^2;q)_m(q;q)_m}q^m.$$

Now on every factor above of the form $(bq^{-t};q)_m$ we use the identity

$$(bq^{-t};q)_m = (-bq^{-t+(m-1)/2})^m(q^{t-m+1}/b;q)_m$$

and get

$$\frac{(-uvq;q)_n}{(u^2q^{n+1};q)_n} = \sum_{m=0}^{n}(uv)^m q^{\binom{m+1}{2}}\begin{bmatrix} n \\ m\end{bmatrix}_q\frac{(-(u/v)q^{n-m+1};q)_m}{(u^2q^{2n-m+1};q)_m}.$$

This shows that

$$L_n(u,v,q) = \frac{(-uvq;q)_n}{(u^2q^{n+1};q)_n}$$

is the solution to the recurrence of Proposition 4. □

2.2. **Anti-Lecture Hall Compositions.** A q-series proof of the Refined Anti-Lecture Hall Theorem (4) will follow the same approach as in the previous subsection. Given an anti-lecture hall composition $\lambda \in A_n$, define the floor of λ as $\lfloor \lambda \rfloor = (\lfloor \lambda_1/1 \rfloor, \lfloor \lambda_2/2 \rfloor, \ldots, \lfloor \lambda_n/n \rfloor)$. Then write $\lambda_i = i\mu_i + r_i$, with $0 \leq r_i \leq i-1$ for $1 \leq i \leq n$. Note that $(\mu_1, \ldots, \mu_n) = \lfloor \lambda \rfloor$ and that $\lambda \in A_n$ if and only if
(i) $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$ and
(ii) $r_i \geq r_{i+1}$ whenever $\mu_i = \mu_{i+1}$.

The condition (i) implies that $\lfloor \lambda \rfloor$ is a partition in P_n. We fix $\mu \in P_n$ and compute the generating function A_μ of the anti-lecture hall compositions λ having $\lfloor \lambda \rfloor = \mu$.

Proposition 5. *For $\mu \in P_n$,*

$$A_\mu(q) \triangleq \sum_{\substack{\lambda \in A_n \\ \lfloor \lambda \rfloor = \mu}} q^{|\lambda|} = q^{\sum_{i=1}^n i\mu_i} \begin{bmatrix} n \\ m_0, m_1, \ldots, m_{\mu_1} \end{bmatrix}_q.$$

where m_i is the multiplicity of the part i in μ.

Proof. As $\lambda_i = i\mu_i + r_i$

$$A_\mu(q) = q^{\sum_{i=1}^n i\mu_i} \sum_{(r_1,\ldots,r_n)} q^{\sum_{i=1}^n r_i}.$$

For $0 \leq i \leq \mu_1+1$, let $\ell_i = n - \sum_{j=0}^{i-1} m_j$. Then the condition (ii) implies that $(r_{\ell_{i+1}+1}, r_{\ell_{i+1}+2}, \ldots, r_{\ell i})$ is a partition into $\ell_i - \ell_{i+1} = m_i$ nonnegative parts and that these parts are less than or equal to ℓ_{i+1}. Therefore, since $\ell_0 = n$,

$$\sum_{(r_1,\ldots,r_n)} q^{\sum_{i=1}^n r_i} = \prod_{i=0}^{\mu_1} \begin{bmatrix} \ell_{i+1} + m_i \\ m_i \end{bmatrix}_q = \begin{bmatrix} n \\ m_0, m_1, \ldots, m_{\mu_1} \end{bmatrix}_q.$$

□

As before, if $\mu \in P_{n,m}$ then we can get $\tilde{\mu}$ in P_m by $\tilde{\mu}_i = \mu_i - 1$, $1 \leq i \leq m$.

Proposition 6. *For $\mu \in P_{n,m}$,*

$$A_\mu(q) = q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q A_{\tilde{\mu}}(q).$$

Proof. From Proposition 5 we know that

$$A_\mu(q) = q^{\sum_{i=1}^m i(\tilde{\mu}_i+1)} \begin{bmatrix} n \\ m_0, m_1, \ldots, m_{\mu_1} \end{bmatrix}_q,$$

and that

$$A_{\tilde{\mu}}(q) = q^{\sum_{i=1}^m i\tilde{\mu}_i} \begin{bmatrix} m \\ m_1, \ldots, m_{\mu_1} \end{bmatrix}_q.$$

Since $m = n - m_0$, we get the result. □

If $\mu \in P_{n,m}$, then $|\mu| = |\tilde{\mu}| + m$ and $o(\mu) = m - o(\tilde{\mu})$, so we can conclude from Proposition 6 that:

$$u^{|\mu|} v^{o(\mu)} A_\mu(q) = (uv)^m u^{|\tilde{\mu}|} v^{-o(\tilde{\mu})} q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q A_{\tilde{\mu}}(q). \qquad (16)$$

In order to prove the Refined Anti-Lecture Hall Theorem (4), the generating function we are looking for, $A_n(u,v,q) \triangleq \sum_{\lambda\in A_n} u^{|\lfloor\lambda\rfloor|} v^{o(\lfloor\lambda\rfloor)} q^{|\lambda|}$ can be rewritten as

$$A_n(u,v,q) = \sum_{\mu\in P_n} u^{|\mu|} v^{o(u)} A_\mu(q) = \sum_{m=0}^{n} \sum_{\mu\in P_{n,m}} u^{|\mu|} v^{o(\mu)} A_\mu(q). \tag{17}$$

Then we can prove the following recurrence for $A_n(u,v,q)$.

Proposition 7. *$A_0(u,v,q) = 1$ and for $n > 0$*

$$A_n(u,v,q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} A_m(u,1/v,q).$$

Proof.

$$\begin{aligned} A_n(u,v,q) &= \sum_{m=0}^{n} \sum_{\mu\in P_{n,m}} u^{|\mu|} v^{o(\mu)} A_\mu(q) \\ &= \sum_{m=0}^{n} \sum_{\tilde{\mu}\in P_m} (uv)^m u^{|\tilde{\mu}|} v^{-o(\tilde{\mu})} q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q A_{\tilde{\mu}}(q) \qquad \text{(using (16))} \\ &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} A_m(u,1/v,q), \end{aligned}$$

where the last step follows from first equality of (17). □

Proof of the Refined Anti-Lecture Hall Theorem (4). We solve the recurrence of Proposition 7 using the identity q-Chu Vandermonde I:

$$\frac{(c/a;q)_n}{(c;q)_n} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(a;q)_m}{(c;q)_m} (-c/a)^m q^{\binom{m}{2}}. \tag{18}$$

Setting $a = -uq/v$ and $c = u^2q^2$ in (18) gives immediately

$$\frac{(-uvq;q)_n}{(u^2q^2;q)_n} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} \frac{(-uq/v;q)_m}{(u^2q^2;q)_m}.$$

Using the recurrence of Proposition 7, we conclude that

$$A_n(u,v,q) = \frac{(-uvq;q)_n}{(u^2q^2;q)_n}.$$

□

3. Refined Lecture Hall Theorems for Truncated Objects

In this section we apply the techniques used in Section 2 to derive the new refined truncated lecture hall theorems.

3.1. **Truncated Lecture Hall Partitions.** Recall that a truncated lecture hall partition in $L_{n,k}$ is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_k}{n-k+1} \geq 0.$$

Given a lecture hall partition $\lambda \in L_{n,k}$, we write $\lambda_i = (n-i+1)\mu_i - r_i$, with $0 \leq r_i \leq n-i$ for $1 \leq i \leq k$. Let $\lceil \lambda \rceil = (\lceil \lambda_1/n \rceil, \ldots, \lceil \lambda_k/(n-k+1) \rceil)$. Then $\mu = \lceil \lambda \rceil$. Note that λ has k positive parts if and only if μ has k positive parts.

Theorem 1. (The Refined Truncated Lecture Hall Theorem)

$$L_{n,k}(u,v,q) = \sum_{m=0}^{k} (uv)^m q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-(u/v)q^{n-m+1};q)_m}{(u^2q^{2n-m+1};q)_m}. \tag{19}$$

Proof. Let $\bar{L}_{n,k}$ be the set of lecture hall partitions in L_n with k positive parts. Then

$$\bar{L}_{n,k}(u,v,q) \triangleq \sum_{\lambda \in \bar{L}_{n,k}} u^{|\lceil\lambda\rceil|} v^{o(\lceil\lambda\rceil)} q^{|\lambda|} = \sum_{\mu \in P_{n,k}} u^{|\mu|} v^{o(\mu)} L_\mu.$$

It follows from the proof of Proposition 4 (ignoring the outer sum there) that

$$\bar{L}_{n,k}(u,v,q) = (uv)^k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q L_k(uq^{n-k}, 1/v, q)$$

and then applying the Refined Lecture Hall Theorem (3) to L_k gives

$$\bar{L}_{n,k}(u,v,q) = (uv)^k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-(u/v)q^{n-k+1};q)_k}{(u^2q^{2n-k+1};q)_k}. \tag{20}$$

The result follows since $L_{n,k}(u,v,q) = \sum_{m=0}^{k} \bar{L}_{n,m}(u,v,q)$. □

3.2. **Truncated Anti-Lecture Hall Compositions.** Let $n \geq k$. A truncated anti-lecture hall composition in $A_{n,k}$ is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that

$$\frac{\lambda_1}{n-k+1} \geq \frac{\lambda_2}{n-k+2} \geq \ldots \geq \frac{\lambda_k}{n} \geq 0.$$

We write $\lambda_i = (n-k+i)\mu_i + r_i$, with $0 \leq r_i \leq n-k+i-1$ for $1 \leq i \leq k$ and define $\lfloor \lambda \rfloor = (\mu_1, \ldots, \mu_k)$. As before, $\lambda \in A_{n,k}$ if and only if
(i) $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k \geq 0$ and
(ii) $r_i \geq r_{i+1}$ whenever $\mu_i = \mu_{i+1}$.

For $\mu \in P_k$, we compute the generating function $A_{\mu,k}(q)$ of those $\lambda \in A_{n,k}$ having $\lfloor \lambda \rfloor = \mu$.

Proposition 8.

$$A_{\mu,k}(q) \triangleq \sum_{\substack{\lambda \in A_{n,k} \\ \lfloor\lambda\rfloor = \mu}} q^{|\lambda|} = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)|\mu|} A_\mu(q),$$

where $A_\mu(q)$ is the generating function of $\lambda \in A_k$ having $\lfloor \lambda \rfloor = \mu$.

Proof.

$$A_{\mu,k}(q) = q^{\sum_{i=1}^{k}(n-k+i)\mu_i} \sum_{(r_1,\ldots,r_k)} q^{\sum_{i=1}^{k} r_i}.$$

For $0 \le i \le \mu_1+1$, let $\ell_i = n-\sum_{j=0}^{i-1} m_j$. Then the condition (ii) implies that $(r_{\ell_{i+1}+1}, r_{\ell_{i+1}+2}, \ldots, r_{\ell i})$ is a partition into m_i nonnegative parts and that these parts are less than or equal to ℓ_{i+1}. Therefore, since $\ell_i - \ell_{i+1} = m_i$ and $\ell_{\mu_1+1} = n-k$,

$$\sum_{(r_1,\ldots,r_k)} q^{\sum_{i=1}^k r_i} = \prod_{i=0}^{\mu_1} \begin{bmatrix} \ell_{i+1}+m_i \\ m_i \end{bmatrix}_q = \begin{bmatrix} n \\ m_0, m_1, \ldots, m_{\mu_1}, n-k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ m_0, m_1, \ldots, m_{\mu_1} \end{bmatrix}_q .$$

By Proposition 5,

$$A_\mu(q) = q^{\sum_{i=1}^k i\mu_i} \begin{bmatrix} k \\ m_0, m_1, \ldots, m_{\mu_1} \end{bmatrix}_q ,$$

and the result follows. □

Theorem 2. (The Refined Truncated Anti-Lecture Hall Theorem).

$$A_{n,k}(u,v,q) \triangleq \sum_{\lambda \in A_{n,k}} q^{|\lambda|} u^{|\lfloor \lambda \rfloor|} v^{o(\lfloor \lambda \rfloor)} = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uvq^{n-k+1};q)_k}{(u^2q^{2(n-k+1)};q)_k}. \tag{21}$$

Proof. Applying first Proposition 8, then the definition of $A_k(u,v,q)$, and finally the Refined Anti-Lecture Hall Theorem 4, we get

$$\begin{aligned} A_{n,k}(u,v,q) &= \sum_{\mu \in P_k} u^{|\mu|} v^{o(\mu)} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)|\mu|} A_\mu(q) \\ &= \begin{bmatrix} n \\ k \end{bmatrix}_q A_k(uq^{n-k}, v, q) \\ &= \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uvq^{n-k+1};q)_k}{(u^2q^{2(n-k+1)};q)_k}. \end{aligned}$$

□

4. Odd/Even Generating Functions for Truncated Objects

In this section we adapt the technique introduced in [4] to get the two-variable generating functions of the truncated objects. In [4], Bousquet-Mélou and Eriksson introduce a bijection Υ from $\mathcal{R}_{n-1} \times [0, n-1]$ to $\mathcal{R}_n$ where $\mathcal{R}_n$ is the set of the reduced Lecture Hall partitions. See [4] for further details. We restate here that bijection as a bijection Υ_n: $L_{n-1} \times \mathbb{N} \to L_n$. For $\lambda \in L_{n-1}$ and $s \in \mathbb{N}$, $\Upsilon_n(\lambda, s) = \mu$, where

$$\begin{aligned} \mu_1 &\leftarrow \left\lceil \frac{n\lambda_1}{n-1} \right\rceil + s \\ \mu_{2\ell} &\leftarrow \lambda_{2\ell-1}, \quad 1 \le \ell \le n/2; \\ \mu_{2\ell+1} &\leftarrow \begin{cases} \left\lceil \frac{(n-2\ell)\lambda_{2\ell+1}}{n-2\ell-1} \right\rceil + \left\lfloor \frac{(n-2\ell)\lambda_{2\ell-1}}{n-2\ell+1} \right\rfloor - \lambda_{2\ell}, & 1 \le \ell < (n-1)/2 \\ \left\lceil \frac{(n-2\ell)\lambda_{2\ell-1}}{n-2\ell+1} \right\rceil - \lambda_{2\ell}, & \ell = (n-1)/2. \end{cases} \end{aligned}$$

It is proved in [4] that $\mu \in L_n$, that $|\mu_e| = |\lambda_o|$ and $|\mu_o| = 2|\lambda_o| - |\lambda_e| + s$ and that Υ_n is a bijection. This implies that:

$$\begin{aligned} L_n(x,y) &\triangleq \sum_{\mu \in L_n} x^{|\mu_o|} y^{|\mu_e|} = \sum_{\lambda \in L_{n-1}} \sum_{s=0}^{\infty} x^{2|\lambda_o| - |\lambda_e| + s} y^{|\lambda_o|} \\ &= \frac{1}{1-x} \sum_{\lambda \in L_{n-1}} (x^2 y)^{|\lambda_o|} (1/x)^{|\lambda_e|} = \frac{L_{n-1}(x^2 y, x^{-1})}{1-x}, \end{aligned}$$

giving the recurrence

$$L_n(x,y) = \frac{L_{n-1}(x^2 y, x^{-1})}{1-x}. \tag{22}$$

As $L_0(x,y) = 1$, this gives $L_n = 1/(x;xy)_n$, the Odd/Even Lecture Hall Theorem (2).

4.1. **Truncated Lecture Hall Partitions.** Recall that $\bar{L}_{n,k}$ for $k \leq n$ is the set of partitions in $L_{n,k}$ with k positive parts. Let

$$L_{n,k}(x,y) \triangleq \sum_{\lambda \in L_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|}; \quad \bar{L}_{n,k}(x,y) \triangleq \sum_{\lambda \in \bar{L}_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|}.$$

Note that $\bar{L}_{n,k}(x,y) = L_{n,k}(x,y) - L_{n,k-1}(x,y)$.

For $n \geq k \geq 1$, define a variation on the function Υ,

$$\Upsilon_{n,k} : \bar{L}_{n-1,k-1} \times \mathbb{N} \to \bar{L}_{n,k},$$

by $\Upsilon_{n,k}(\lambda, s) = (\mu_1, \mu_2, \ldots, \mu_k)$, where

$$\begin{aligned} \mu_1 &\leftarrow \left\lceil \frac{n\lambda_1}{n-1} \right\rceil + s \\ \mu_{2\ell} &\leftarrow \lambda_{2\ell-1}, \quad 1 \leq \ell \leq k/2; \\ \mu_{2\ell+1} &\leftarrow \left\lceil \frac{(n-2\ell)\lambda_{2\ell+1}}{n-2\ell-1} \right\rceil + \left\lfloor \frac{(n-2\ell)\lambda_{2\ell-1}}{n-2\ell+1} \right\rfloor - \lambda_{2\ell}, \quad 1 \leq \ell \leq (k-2)/2; \end{aligned}$$

and, if k is odd with $k = 2t+1$,

$$\mu_{2t+1} \leftarrow \left\lfloor \frac{(n-2t)\lambda_{2t-1}}{n-2t+1} \right\rfloor - \lambda_{2t} + 1.$$

As was true for the function Υ_n, $\Upsilon_{n,k}$ is one-to-one. It is straightforward to check when k is odd that the image of $\bar{L}_{n-1,k-1} \times \mathbb{N}$ under $\Upsilon_{n,k}$ is $\bar{L}_{n,k}$, and that $|\mu_e| = |\lambda_o|$ and $|\mu_o| = 2|\lambda_o| - |\lambda_e| + s + 1$. This gives a recurrence for $\bar{L}_{n,k}(x,y)$ when k is odd.

Proposition 9.

$$\bar{L}_{n,2k+1}(x,y) = \frac{x}{1-x} \bar{L}_{n-1,2k}(x^2 y, x^{-1}). \tag{23}$$

Getting a recurrence for the even case will be harder. For $i > 0$, let $\bar{L}_{n,k,i}$ be the set of partitions in $\bar{L}_{n,k}$ whose k^{th} part is equal to i. Let $\bar{L}_{n,k,0} = \bar{L}_{n,k-1}$. We can check that when k is even,

$\Upsilon_{n,k}$ gives a bijection between $\bar{L}_{n-1,k-1,i} \times \mathbb{N}$ and $\bar{L}_{n,k,i}$. Furthermore, when $\lambda \in \bar{L}_{n-1,2k-1,i}$ and $\mu = \Upsilon_{n,2k}(\lambda, s)$, then $|\mu_e| = |\lambda_o|$ and

$$|\mu_o| = 2|\lambda_o| - |\lambda_e| + s - \left\lfloor \frac{(n-2k)i}{n-2k+1} \right\rfloor = 2|\lambda_o| - |\lambda_e| + s + l - i,$$

with $l = \lfloor i/(n-2k+1) \rfloor$. This implies that:

Proposition 10. *For $i > 0$,*

$$\bar{L}_{n,2k,i}(x,y) = \frac{x^{l-i}}{1-x} \bar{L}_{n-1,2k-1,i}(x^2y, 1/x), \tag{24}$$

with $l = \lfloor i/(n-2k+1) \rfloor$.

Now we will decompose the set $\bar{L}_{n,2k+1,i}$. In what follows, in order to compress the notation, we will sometimes write a function $f(x,y)$ as f, when the arguments are (x,y).

Proposition 11. *Let $i = l(n-2k) + r$, where $1 \le r \le n-2k$.*

If $r \ge 2$, then

$$\bar{L}_{n,2k+1,i} = xL_{n,2k+1,i-1} - \frac{x}{1-x} \bar{L}_{n-1,2k-1,i+l}(x^2y, 1/x).$$

Otherwise, $r = 1$ and

$$\bar{L}_{n,2k+1,i} = xL_{n,2k+1,i-1} - \frac{x}{1-x} \bar{L}_{n-1,2k-1,i+l-1}(x^2y, 1/x) - \frac{x}{1-x} \bar{L}_{n-1,2k-1,i+l}(x^2y, 1/x).$$

Proof. For $i = l(n-2k) + r$,

$$\bar{L}_{n,2k+1,i} = \begin{cases} x\bar{L}_{n,2k+1,i-1} - x^i \bar{L}_{n,2k,i+l} & \text{if } r \ge 2 \text{ or } i = 1 \\ x\bar{L}_{n,2k+1,i-1} - x^i \bar{L}_{n,2k,i+l-1} - x^i \bar{L}_{n,2k,i+l} & \text{if } r = 1 \text{ and } i \ne 1 \end{cases} \tag{25}$$

which can be seen as follows. Adding one to the $(2k+1)^{th}$ part of a partition in $\bar{L}_{n,2k+1,i-1}$ gives a partition in $\bar{L}_{n,2k+1,i}$, except in the following two cases (i) if the $2k^{th}$ part was equal to $i + l$ and (ii) if $r = 1$ and the $2k^{th}$ part was equal to $i + l - 1$.

Now using Proposition 10, for $i = l(n-2k) + r$ with $1 \le r \le n - 2k$,

$$\bar{L}_{n,2k,i+l} = \frac{x^{1-i}}{1-x} \bar{L}_{n-1,2k-1,i+l}(x^2y, 1/x),$$

and for $i = l(n-2k) + 1$,

$$\bar{L}_{n,2k,i+l-1} = \frac{x^{1-i}}{1-x} \bar{L}_{n-1,2k-1,i+l-1}(x^2y, 1/x).$$

Combining this with the recurrence (25), we get the result. □

We will combine the previous results to get a recurrence for $\bar{L}_{n,2k}(x,y)$.

Proposition 12.

$$\bar{L}_{n,2k} = \begin{cases} \bar{L}_{n-1,2k}(x^2y, 1/x) + \frac{1}{1-x} \bar{L}_{n-1,2k-1}(x^2y, 1/x) & \text{if } n > 2k \\ \frac{1}{1-x} \bar{L}_{n-1,2k-1}(x^2y, 1/x) & \text{if } n = 2k \end{cases} \tag{26}$$

Proof. The $n = 2k$ case is (22). Assume $n > 2k$. From Proposition 11, for $i = l(n-2k) + r$ with $2 \le r \le n - 2k$, we have

$$\bar{L}_{n,2k+1,i} = x\bar{L}_{n,2k+1,i-1} - \frac{x}{1-x}\bar{L}_{n-1,2k-1,i+l}(x^2y, 1/x).$$

and for $i = l(n-2k) + 1$,

$$\bar{L}_{n,2k+1,i} = x\bar{L}_{n,2k+1,i-1} - \frac{x}{1-x}\bar{L}_{n-1,2k-1,i+l-1}(x^2y, 1/x) - \frac{x}{1-x}\bar{L}_{n-1,2k-1,i+l}(x^2y, 1/x).$$

We sum

$$\bar{L}_{n,2k+1} = \sum_{i=1}^{\infty} \bar{L}_{n,2k+1,i} = x\bar{L}_{n,2k} + x\sum_{i=1}^{\infty} \bar{L}_{n,2k+1,i} - \frac{x}{1-x}\sum_{i=1}^{\infty} \bar{L}_{n-1,2k-1,i}(x^2y, 1/x).$$

Therefore

$$\bar{L}_{n,2k+1} = x\bar{L}_{n,2k} + x\bar{L}_{n,2k+1} - \frac{x}{1-x}\bar{L}_{n-1,2k-1}(x^2y, 1/x).$$

Using Proposition 9, that

$$(1-x)\bar{L}_{n,2k+1} = x\bar{L}_{n-1,2k}(x^2y, 1/x),$$

we get the result. □

Now we can compute the generating function we were looking for.

Theorem 3. *For $n \ge k \ge 0$,*

$$\bar{L}_{n,k}(x, y) = \frac{(x^{\lfloor k/2 \rfloor + 1} y^{\lfloor k/2 \rfloor})^{\lceil k/2 \rceil} \begin{bmatrix} n - \lceil k/2 \rceil \\ \lfloor k/2 \rfloor \end{bmatrix}_{xy}}{(x; xy)_{\lceil k/2 \rceil} (x^n y^{n-1}; (xy)^{-1})_{\lfloor k/2 \rfloor}}.$$

Proof. For $k = 0$ the result holds, as $\bar{L}_{n,0} = 1$. Let $n \ge m > 0$ and assume inductively that the theorem is true for $(n, m-1)$, $(n-1, m-1)$, and, if $n > m$, for $(n-1, m)$.

For the odd case $m = 2k + 1$, by Proposition 9,

$$\bar{L}_{n,2k+1} = \frac{x}{1-x}\bar{L}_{n-1,2k}(x^2y, x^{-1}).$$

By the induction hypothesis,

$$\bar{L}_{n-1,2k} = \frac{(x^{k+1}y^k)^k \begin{bmatrix} n-1-k \\ k \end{bmatrix}_{xy}}{(x; xy)_k (x^{n-1}y^{n-2}; (xy)^{-1})_k},$$

and substituting in the previous equation gives the result.

For the even case $m = 2k$, we have by Proposition 12,

$$\bar{L}_{n,2k} = \begin{cases} \bar{L}_{n-1,2k}(x^2y, 1/x) + \frac{1}{1-x}\bar{L}_{n-1,2k-1}(x^2y, 1/x) & \text{if } n > 2k \\ \frac{1}{1-x}\bar{L}_{n-1,2k-1}(x^2y, 1/x) & \text{if } n = 2k. \end{cases} \tag{27}$$

By the induction hypothesis,

$$\bar{L}_{n-1,2k-1} = \frac{(x^k y^{k-1})^k \begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}_{xy}}{(x; xy)_k (x^{n-1}y^{n-2}; (xy)^{-1})_{k-1}}.$$

When $n = 2k$, substituting this in the previous equation gives the result. When $n > 2k$, by the induction hypothesis, we also have

$$\bar{L}_{n-1,2k} = \frac{(x^{k+1}y^k)^k \begin{bmatrix} n-1-k \\ k \end{bmatrix}_{xy}}{(x;xy)_k(x^{n-1}y^{n-2};(xy)^{-1})_k}.$$

Then let

$$\bar{L}_{n,2k} = \bar{L}_{n-1,2k}(x^2y, 1/x) + \frac{1}{1-x}\bar{L}_{n-1,2k-1}(x^2y, 1/x) = \frac{N_{n,k}}{D_{n,k}}$$

with

$$D_{n,k} = (x;xy)_{k+1}(x^n y^{n-1};(xy)^{-1})_k.$$

Then

$$N_{n,k} = (x^{k+2}y^{k+1})^k \begin{bmatrix} n-1-k \\ k \end{bmatrix}_{xy} (1-x) + (x^{k+1}y^k)^k \begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}_{xy} (1 - x^{n-k+1}y^{n-k}).$$

We use these identities to simplify the numerator.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q; \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \tag{28}$$

So,

$$\begin{aligned} N_{n,k} &= (x^{k+1}y^k)^k \left(\begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}_{xy} + x^k y^k \begin{bmatrix} n-1-k \\ k \end{bmatrix}_{xy} \right) \\ &\quad -(x^{k+1}y^k)^k x^{k+1}y^k \left(\begin{bmatrix} n-1-k \\ k \end{bmatrix}_{xy} + x^{n-2k}y^{n-2k} \begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}_{xy} \right) \\ &= (x^{k+1}y^k)^k(1 - x^{k+1}y^k) \begin{bmatrix} n-k \\ k \end{bmatrix}_{xy}. \end{aligned}$$

□

The Odd/Even Truncated Lecture Hall Theorem (8) is an immediate consequence of Theorem 3.

4.2. **Truncated Anti-Lecture Hall Compositions.** Recall that for $n \geq k-1$, $A_{n,k}$ is the set of compositions such that:

$$\frac{\lambda_1}{n-k+1} \geq \ldots \geq \frac{\lambda_k}{n} \geq 0.$$

(If $n = k-1$, λ_1 need only satisfy $\lambda_1 \geq 0$.) Our goal is to compute the generating function

$$A_{n,k}(x, y) = \sum_{\lambda \in A_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|}.$$

We will again adapt the mapping of [4]. For $n \geq k \geq 1$, define

$$\Theta_{n,k} : A_{n,k-1} \times \mathbb{N} \to A_{n,k},$$

by $\Theta_{n,k}(\lambda, s) = (\mu_1, \mu_2, \ldots, \mu_k) = \mu$, where

$$\begin{aligned}
\mu_1 &\leftarrow \left\lceil \frac{(n-k+1)\lambda_1}{n-k+2} \right\rceil + s \\
\mu_{2\ell} &\leftarrow \lambda_{2\ell-1}, \quad 1 \le \ell \le k/2; \\
\mu_{2\ell+1} &\leftarrow \left\lceil \frac{(n+2\ell-k+1)\lambda_{2\ell+1}}{n+2\ell-k+2} \right\rceil + \left\lfloor \frac{(n+2\ell-k+1)\lambda_{2\ell-1}}{n+2\ell-k} \right\rfloor - \lambda_{2\ell}, \quad 1 \le \ell \le (k-2)/2.
\end{aligned}$$

and if k is odd with $k = 2t+1$

$$\mu_{2t+1} \leftarrow \left\lfloor \frac{n\lambda_{2t-1}}{n-1} \right\rfloor - \lambda_{2t}.$$

Similar to the mapping $\Upsilon_{n,k}$ of the previous section, it can be checked that $\Theta_{n,k}$ is one-to-one and $\Theta_{n,k}(A_{n,k-1} \times \mathbb{N}) \subseteq A_{n,k}$. Furthermore, when k is odd, $\Theta_{n,k}$ is onto $A_{n,k}$ and if $\mu = \Theta_{n,k}(\lambda, s)$, then $|\mu_e| = |\lambda_o|$ and $|\mu_o| = 2|\lambda_o| - |\lambda_e| + s$. This implies:

Proposition 13. *For $n \ge 2k$,*

$$A_{n,2k+1}(x,y) = \frac{1}{1-x} A_{n,2k}(x^2 y, x^{-1}). \tag{29}$$

Again the recurrence for even k is more difficult. For $i \ge 0$, let $A_{n,k,i}$ be the set of compositions in $A_{n,k}$ with k^{th} part equal to i. It can be checked that $\Theta_{n,2k}$ maps $A_{n,2k-1,i} \times \mathbb{N}$ bijectively to $A_{n,2k,i}$. If $\mu = \Theta_{n,2k}(\lambda, s)$, it is not too hard to see that $\mu \in A_{n,2k+1}$, $|\mu_e| = |\lambda_o|$ and $|\mu_o| = 2|\lambda_o| - |\lambda_e| + s - i - l$ with $l = \lfloor i/n \rfloor$. This implies:

Proposition 14. *For $n \ge 2k$,*

$$A_{n,2k,i}(x,y) = \frac{x^{-i-l}}{1-x} A_{n,2k-1,i}(x^2 y, x^{-1}). \tag{30}$$

with $l = \lfloor i/n \rfloor$.

Proposition 15. *For $n \ge 2k$,*

$$(1-x)A_{n,2k+1} = A_{n-1,2k} - \frac{x}{1-x} A_{n-1,2k-1}(x^2 y, 1/x), \tag{31}$$

Proof. Let $i = ln + r$ with $0 \le r \le n-1$. It follows from the definitions that $A_{n,k,0} = A_{n-1,k-1}$. Consider the case $r = 0$ and $l > 0$. If $\lambda \in A_{n,2k+1,i-1}$, then $\lambda_{2k}/(n-1) \ge (i-1)/n$, so $\lambda_{2k}/(n-1) \ge i/n$. Therefore adding 1 to the last part of λ gives a composition in $A_{n,2k+1,i}$. Thus, $A_{n,2k+1,i} = xA_{n,2k+1,i-1}$. For $r > 0$, we have

$$A_{n,2k+1,i} = xA_{n,2k+1,i-1} - x^i A_{n-1,2k,i-l-1}.$$

To see this, adding one to the $(2k+1)^{th}$ part of a composition in $A_{n,2k+1,i-1}$ gives a composition in $A_{n,2k+1,i}$ unless the $(2k)^{th}$ part was equal to $i-l-1$.

By Proposition 14, we get that

$$A_{n-1,2k,i-l-1} = \frac{x^{1-i}}{1-x} A_{n-1,2k-1,i-l-1}(x^2 y, 1/x).$$

We apply this and get if $r \ge 1$,

$$A_{n,2k+1,i} = xA_{n,2k+1,i-1} - \frac{x}{1-x} A_{n-1,2k-1,i-l-1}(x^2 y, 1/x);$$

if $r = 0$ and $l > 0$

$$A_{n,2k+1,i} = xA_{n,2k+1,i-1};$$

otherwise, $i = 0$ and $A_{n,k,0} = A_{n-1,k-1}$. Now we sum over all i:

$$\sum_{i=1}^{\infty} A_{n,2k+1,i} = x\sum_{i=1}^{\infty} A_{n,2k+1,i-1} - \frac{x}{1-x}\sum_{r=1}^{n-1}\sum_{l=0}^{\infty} A_{n-1,2k-1,ln+r-l-1}(x^2y, 1/x).$$

This gives

$$A_{n,2k+1} - A_{n-1,2k} = xA_{n,2k+1} - \frac{x}{1-x}A_{n-1,2k-1}(x^2y, 1/x),$$

and therefore the result. □

Now we get a recurrence for the number of even parts.

Proposition 16.

$$A_{n,2k} = \begin{cases} \frac{1}{1-x}A_{2k-1,2k-1}(y,x) & \text{if } n = 2k-1 \\ A_{n-1,2k}(1/y, xy^2) + \frac{y}{1-y}A_{n-1,2k-1} & \text{if } n \geq 2k. \end{cases}$$

Proof. For the case $n = 2k - 1$. These are the objects such that

$$\frac{\lambda_2}{1} \geq \cdots \geq \frac{\lambda_{2k}}{2k-1} \geq 0.$$

and $\lambda_1 \geq 0$. Then

$$A_{2k-1,k}(x, y) = \frac{1}{1-x}A_{2k-1,2k-1}(y, x).$$

For $n \geq 2k$, we use the previous Proposition 15. From Proposition 13, we know that $(1 - x)A_{n,2k+1} = A_{n,2k}(x^2y, 1/x)$. Therefore,

$$A_{n,2k}(x^2y, 1/x) = A_{n-1,2k} - \frac{x}{1-x}A_{n-1,2k-1}(x^2y, 1/x).$$

We make the substitutions $x = 1/y$ and $y = xy^2$ and get the result. □

Remark. Note that, using Proposition 16, we get:

$$A_{2k,2k} = \frac{1}{1-y}(A_{2k-1,2k-1} - yA_{2k-1,2k-1}(xy^2, 1/y)).$$

Theorem 4. (The Odd/Even Truncated Anti-Lecture Hall Theorem) *For $n \geq k - 1$ and $n, k \geq 0$,*

$$A_{n,k}(x, y) = \frac{\begin{bmatrix} n \\ \lfloor k/2 \rfloor \end{bmatrix}_{xy}}{(x; xy)_{\lceil k/2 \rceil}(x^{n-k+1}y^{n-k+2}; xy)_{\lfloor k/2 \rfloor}}.$$

Proof. We will prove the result by induction. We know that $A_{n,0} = 1$. Let $n + 1 \geq m > 0$ and assume inductively that the theorem is true for $(n, m-1)$, $(n-1, m-1)$, and, if $n \geq m$, for $(n-1, m)$. For the odd case $m = 2k + 1$, by the induction hypothesis,

$$A_{n,2k} = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{xy}}{(x; xy)_k(x^{n-2k+1}y^{n-2k+2}; xy)_k}.$$

Apply Proposition 13,

$$A_{n,2k+1} = \frac{1}{1-x} A_{n,2k}(x^2y, 1/x),$$

and get the result.

For the even case $m = 2k$, if $n = 2k-1$, by the induction hypothesis, using Proposition 16:

$$A_{2k-1,2k}(x,y) = \frac{1}{1-x} A_{2k-1,2k-1}(y,x) = \frac{\begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}_{xy}}{(y;xy)_k (x;xy)_k}.$$

Now for $n \geq 2k$, by the induction hypothesis, using Proposition 16:

$$A_{n-1,2k}(1/y, xy^2) = \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{xy}}{(1/y;xy)_k (x^{n-2k+1}y^{n-2k+2};xy)_k}; \quad A_{n-1,2k-1} = \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{xy}}{(x;xy)_k (x^{n-2k+1}y^{n-2k+2};xy)_{k-1}}.$$

$$A_{n-1,2k}(1/y, xy^2) - \frac{1/y}{1-1/y} A_{n-1,2k-1} = \frac{N_{n,k}}{D_{n,k}}$$

with $D_{n,k} = (1/y;xy)_{k+1}(x^{n-2k+1}y^{n-2k+2};xy)_k$. Then

$$\begin{aligned} N_{n,k} &= \begin{bmatrix} n-1 \\ k \end{bmatrix}_{xy} (1 - x^k y^{k-1}) - 1/y \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{xy} (1 - x^{n-k} y^{n-k+1}) \\ &= \begin{bmatrix} n-1 \\ k \end{bmatrix}_{xy} + x^{n-k} y^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{xy} - \frac{1}{y} \left(x^k y^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{xy} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{xy} \right) \\ &= (1 - 1/y) \begin{bmatrix} n \\ k \end{bmatrix}_{xy}, \end{aligned}$$

where the last step follows from (28). □

5. Combinatorial Characterizations and Refinements

5.1. **Characterization of Truncated Lecture Hall Partitions.** In this section we characterize truncated lecture hall partitions in terms of partitions into odd parts with certain restrictions. We first need a few steps.

Proposition 17. *Given n, t, j, l, the weight generating function for the number of partitions into odd parts in $\{1, 3, ..., 2n-1\}$ in which exactly t of the parts can be chosen from the set $\{2j+1, 2j+3, ..., 2l-1\}$ is*

$$\frac{q^{t(2j+1)}}{(q;q^2)_j (q^{2n-1};q^{-2})_{n-l}} \begin{bmatrix} l-j-1+t \\ t \end{bmatrix}_{q^2}.$$

Proof. Note that $1/(q;q^2)_j$ is the generating function for partitions into odd parts in the set $\{1, 3, \ldots, 2j-1\}$ and $1/(q^{2n-1};q^{-2})_{n-l}$ is the generating function for partitions into odd parts in the set $\{2l+1, \ldots, 2n-1\}$. So the denominator of the fraction is the generating function for partitions into odd parts which are not restricted.

Now count partitions into exactly t parts from the set $\{2j+1, 2j+3, ..., 2l-1\}$. Take off $2j+1$ from each part. This is counted by $q^{t(2j+1)}$. We are left with a partition into even parts in a $t \times 2(l-j-1)$ box, which is counted by $\begin{bmatrix} l-j-1+t \\ t \end{bmatrix}_{q^2}$. □

Corollary 1. *The generating function for the number of partitions into odd parts in which exactly t of the parts are greater than or equal to $2j+1$ is*

$$\frac{q^{t(2j+1)}}{(q;q^2)_j (q^2;q^2)_t}.$$

Applying Proposition 17 gives the following.

Proposition 18. *Let $R_{n,k}$ be the set of partitions into odd parts less than or equal to $2n-1$ where at most $\lfloor k/2 \rfloor$ parts can be chosen from the set $\{2\lceil k/2 \rceil + 1, 2\lceil k/2 \rceil + 3, \ldots, 2(n - \lfloor k/2 \rfloor) - 1\}$.*

$$R_{n,k}(q) \triangleq \sum_{\lambda \in R_{n,k}} q^{|\lambda|} = \frac{\sum_{i=0}^{\lfloor k/2 \rfloor} q^{i(2\lceil k/2 \rceil + 1)} \begin{bmatrix} n-k-1+i \\ i \end{bmatrix}_{q^2}}{(q;q^2)_{\lceil k/2 \rceil} (q^{2n-1}; q^{-2})_{\lfloor k/2 \rfloor}}. \tag{32}$$

□

Let $\bar{R}_{n,k} = R_{n,k} - R_{n,k-1}$ and let $\bar{R}_{n,k}(q)$ be the corresponding generating function. Recall that $\bar{L}_{n,k}$ is the set of lecture hall partitions in $L_{n,k}$ with k positive parts.

Proposition 19. *For $k > 0$,*

$$\bar{L}_{n,k}(q) \triangleq \sum_{\lambda \in \bar{L}n,k} q^{|\lambda|} = \frac{q^{\binom{k+1}{2}} \begin{bmatrix} n - \lceil k/2 \rceil \\ \lfloor k/2 \rfloor \end{bmatrix}_{q^2}}{(q;q^2)_{\lceil k/2 \rceil} (q^{2n-1}; q^{-2})_{\lfloor k/2 \rfloor}}$$

Proof. Let $x = y = q$ in Theorem 3. □

Theorem 5. (Characterization of Truncated Lecture Hall Partitions) *The number of truncated lecture partitions of N in $L_{n,k}$ is equal to the number of partitions of N into odd parts less than $2n$, with the following constraint on the parts: at most $\lfloor k/2 \rfloor$ parts can be chosen from the set*

$$\{2\lceil k/2 \rceil + 1, 2\lceil k/2 \rceil + 3, \ldots, 2(n - \lfloor k/2 \rfloor) - 1\}.$$

Proof. We must show that $L_{n,k}(q) \triangleq \sum_{m=0}^{k} \bar{L}_{n,m} = R_{n,k}(q)$. Since $L_{n,0}(q) = R_{n,0}(q) = 1$, it suffices to show that for $k > 0$, $\bar{L}_{n,k}(q) = \bar{R}_{n,k}(q)$. We use $\bar{L}_{n,k}(q)$ from Proposition 19, and now compute $\bar{R}_{n,k}(q) = R_{n,k}(q) - R_{n,k-1}(q)$.

For $k > 0$, first look at the case $k = 2s$. Using Proposition 18,

$$\bar{R}_{n,2s}(q) = \frac{\sum_{i=0}^{s} q^{i(2s+1)} \begin{bmatrix} n-2s-1+i \\ i \end{bmatrix}_{q^2} - (1 - q^{2(n-s)+1}) \sum_{i=0}^{s-1} q^{i(2s+1)} \begin{bmatrix} n-2s+i \\ i \end{bmatrix}_{q^2}}{(q;q^2)_s (q^{2n-1}; q^{-2})_s}.$$

Since the denominators of $\bar{R}_{n,2s}(q)$ and $\bar{L}_{n,2s}(q)$ agree, we focus on the numerator $E(n,s)$:

$$E(n,s) = \sum_{i=0}^{s} q^{i(2s+1)} \begin{bmatrix} n-2s-1+i \\ i \end{bmatrix}_{q^2} - (1 - q^{2(n-s)+1}) \sum_{i=0}^{s-1} q^{i(2s+1)} \begin{bmatrix} n-2s+i \\ i \end{bmatrix}_{q^2}.$$

Therefore, using the first identity of (28),

$$\begin{aligned}\sum_{i=0}^{s-1} q^{i(2s+1)}\left(\begin{bmatrix}n-2s-1+i\\ i\end{bmatrix}_{q^2}-\begin{bmatrix}n-2s+i\\ i\end{bmatrix}_{q^2}\right) &= -q^{2(n-2s)}\sum_{i=0}^{s-1} q^{i(2s+1)}\begin{bmatrix}n-2s+i-1\\ i-1\end{bmatrix}_{q^2}\\ &= -q^{2(n-s)+1}\sum_{i=0}^{s-2} q^{i(2s+1)}\begin{bmatrix}n-2s+i\\ i\end{bmatrix}_{q^2}.\end{aligned}$$

We then get:

$$\begin{aligned}E(n,s) &= q^{s(2s+1)}\begin{bmatrix}n-s-1\\ s\end{bmatrix}_{q^2}+q^{2(n-s)+1}q^{(s-1)(2s+1)}\begin{bmatrix}n-s-1\\ s-1\end{bmatrix}_{q^2}\\ &= q^{s(2s+1)}\begin{bmatrix}n-s\\ s\end{bmatrix}_{q^2}=q^{\binom{k+1}{2}}\begin{bmatrix}n-\lceil k/2\rceil\\ \lfloor k/2\rfloor\end{bmatrix}_{q^2}.\end{aligned}$$

Now when $k=2s+1$, applying Proposition 18 gives

$$\bar{R}_{n,2s+1}(q)=\frac{\sum_{i=0}^{s} q^{i(2s+3)}\begin{bmatrix}n-2s-2+i\\ i\end{bmatrix}_{q^2}-(1-q^{2s+1})\sum_{i=0}^{s} q^{i(2s+1)}\begin{bmatrix}n-2s-1+i\\ i\end{bmatrix}_{q^2}}{(q;q^2)_{s+1}(q^{2n-1};q^{-2})_s}.$$

Again we can focus on the numerator $O(n,s)$:

$$O(n,s)=\sum_{i=0}^{s} q^{i(2s+3)}\begin{bmatrix}n-2s-2+i\\ i\end{bmatrix}_{q^2}-(1-q^{2s+1})\sum_{i=0}^{s} q^{i(2s+1)}\begin{bmatrix}n-2s-1+i\\ i\end{bmatrix}_{q^2}.$$

Then, using the second identity of (28),

$$\begin{aligned}\sum_{i=0}^{s} q^{i(2s+1)}\left(q^{2i}\begin{bmatrix}n-2s-2+i\\ i\end{bmatrix}_{q^2}-\begin{bmatrix}n-2s-1+i\\ i\end{bmatrix}_{q^2}\right) &= -\sum_{i=1}^{s} q^{i(2s+1)}\begin{bmatrix}n-2s-2+i\\ i-1\end{bmatrix}_{q^2}\\ &= -q^{2s+1}\sum_{i=0}^{s-1} q^{i(2s+1)}\begin{bmatrix}n-2s-1+i\\ i\end{bmatrix}_{q^2}.\end{aligned}$$

Applying this to $O(n,s)$ gives the result:

$$O(n,s) = q^{2s+1}q^{s(2s+1)}\begin{bmatrix}n-s-1\\ s\end{bmatrix}_{q^2}=q^{\binom{k+1}{2}}\begin{bmatrix}n-\lceil k/2\rceil\\ \lfloor k/2\rfloor\end{bmatrix}_{q^2}.$$

□

Further refinements. We could also add another parameter. Let

$$R_{n,k}(z,q)\triangleq\sum_{\lambda\in R_{n,k}} z^{l(\lambda)}q^{|\lambda|}$$

where $l(\lambda)$ is the number of parts of λ. Using the same arguments we get that

$$R_{n,k}(z,q)=\frac{\sum_{i=0}^{\lfloor k/2\rfloor} z^i q^{i(2\lceil k/2\rceil+1)}\begin{bmatrix}n-k-1+i\\ i\end{bmatrix}_{q^2}}{(zq;q^2)_{\lceil k/2\rceil}(zq^{2n-1};q^{-2})_{\lfloor k/2\rfloor}}$$

and that

$$R_{n,k}(z,q)-R_{n,k-1}(z,q)=\bar{L}_{n,k}(xz,y/z).$$

Therefore

Theorem 6. *The number of truncated lecture partitions λ of N in $L_{n,k}$ such that $|\lambda_o| - |\lambda|_e = j$ is equal to the number of partitions of N into j odd parts less than $2n$, with the following constraint on the parts: at most $\lfloor k/2 \rfloor$ parts can be chosen from the set*

$$\{2\lceil k/2\rceil + 1, 2\lceil k/2\rceil + 3, \ldots, 2(n - \lfloor k/2 \rfloor) - 1\}.$$

5.2. **Finite Versions of Refinements of Euler's Theorem.** Euler's Theorem says that the number of partitions of N into distinct parts is equal to the number of partitions of N into odd parts. (A straightforward proof shows $(-q;q)_\infty = 1/(q;q^2)_\infty$.) The Lecture Hall Theorem is a finite version of that theorem: the number of partitions of N in L_n is equal to the number of partitions of N into odd parts less than $2n$. We will show how the truncated lecture hall results give finite versions of refinements of Euler's theorem that can be easily deduced from (slight modifications of) Sylvester's bijection [12].

Finite Version 1.

$$\sum_{m=0}^{k} q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(q^{n-m+1};q)_m}{(q^{2n-m+1};q)_m} = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{q^{i(2\lceil k/2\rceil+1)} \begin{bmatrix} n-k-1+i \\ i \end{bmatrix}_{q^2}}{(q;q^2)_{\lceil k/2\rceil}(q^{2n-1};q^{-2})_{\lfloor k/2\rfloor}}. \tag{33}$$

This is Theorem 5, which states that $L_{n,k}(q) = R_{n,k}(q)$, where $L_{n,k}(q)$ and $R_{n,k}(q)$ are given by Theorem 19 and Proposition 18, respectively.

Taking limits in (33), as $n \to \infty$, gives

$$\sum_{m=0}^{k} \frac{q^{\binom{m+1}{2}}}{(q;q)_m} = \sum_{i=0}^{\lfloor k/2\rfloor} \frac{q^{i(2\lceil k/2\rceil+1)}}{(q;q^2)_{\lceil k/2\rceil}(q^2;q^2)_i}.$$

Note that $q^{\binom{m+1}{2}}/(q;q)_m$ is the generating function for partitions into m distinct parts. Applying Corollary 1 to the right hand side then gives the following refinement of Euler's Theorem:

> **Refinement 1:** The number of partitions of N into at most k distinct parts is equal to the number of partitions of N into odd parts such that at most $\lfloor k/2\rfloor$ of the parts are greater than or equal to $2\lceil k/2\rceil + 1$.
>
> Theorem 5 itself is a further refinement.

Finite Version 2.

$$q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{n-k+1};q)_k}{(q^{2n-k+1};q)_k} = \frac{q^{\binom{k+1}{2}} \begin{bmatrix} n-\lceil k/2\rceil \\ \lfloor k/2\rfloor \end{bmatrix}_{q^2}}{(q;q^2)_{\lceil k/2\rceil}(q^{2n-1};q^{-2})_{\lfloor k/2\rfloor}}. \tag{34}$$

This is Proposition 19 combined with eq. (20), setting $u = v = 1$ in eq. (20).

Letting $n \to \infty$ in (34) gives

$$\frac{q^{\binom{k+1}{2}}}{(q;q^2)_k} = \frac{q^{\binom{k+1}{2}}}{(q;q^2)_{\lceil k/2\rceil}(q^2;q^{-2})_{\lfloor k/2\rfloor}}. \tag{35}$$

On the right-hand side of (35), if we use

$$q^{\binom{k+1}{2}} = q^{\lceil k/2\rceil - \lfloor k/2\rfloor} q^{\lfloor k/2\rfloor(2\lceil k/2\rceil+1)}$$

and apply Corollary 1, (35) can be read as the following refinement of Euler's theorem:

> **Refinement 2:** The number of partitions of N into exactly k distinct parts is equal to the number of partitions of $N - (\lceil k/2\rceil - \lfloor k/2\rfloor)$ into odd parts such that exactly $\lfloor k/2\rfloor$ of the parts are greater than or equal to $2\lceil k/2\rceil + 1$.

A combinatorial interpretation of (34) will give a further refinement of Refinement 2. To get this, note that the right-hand side of (34) can be written as

$$\left(\frac{q^{\lceil k/2\rceil - \lfloor k/2\rfloor}}{1 - q^{2n-2\lfloor k/2\rfloor+1}}\right) \frac{q^{\lfloor k/2\rfloor(2\lceil k/2\rceil+1)} \begin{bmatrix} n - \lceil k/2\rceil \\ \lfloor k/2\rfloor \end{bmatrix}_{q^2}}{(q;q^2)_{\lceil k/2\rceil}(q^{2n-1};q^{-2})_{\lfloor k/2\rfloor-1}}.$$

Using Proposition 17, the last quotient above is the generating function for the number of partitions into odd parts less than $2n$ with exactly $\lfloor k/2\rfloor$ parts in the set $\{2\lfloor k/2\rfloor + 1, \ldots, 2n - 2\lfloor k/2\rfloor + 1\}$. So (34) says:

> **Further refinement 2:** The number of partitions of N in $\bar{L}_{n,k}$ is equal to the number of partitions of $N - (\lceil k/2\rceil - \lfloor k/2\rfloor)$ into odd parts less than $2n$ with at least $\lfloor k/2\rfloor$ parts in $\{2\lfloor k/2\rfloor + 1, \ldots, 2n - 2\lfloor k/2\rfloor + 1\}$, but at most $\lfloor k/2\rfloor$ parts less than $2n - 2\lfloor k/2\rfloor + 1$.

Finite Version 3.

Using Theorem 3 with $x = zq$ and $y = q/z$, we get

$$\bar{L}_{n,k}(zq, q/z) = \frac{(zq^{2\lfloor k/2\rfloor+1})^{\lceil k/2\rceil} \begin{bmatrix} n - \lceil k/2\rceil \\ \lfloor k/2\rfloor \end{bmatrix}_{q^2}}{(zq;q^2)_{\lceil k/2\rceil}(zq^{2n-1};(q^2)^{-1})_{\lfloor k/2\rfloor}}.$$

Now let $n \to \infty$ and $\mathcal{D}_k$ be the set of partitions into k distinct parts and get

$$\sum_{\lambda \in \mathcal{D}_{2k-1} \cup \mathcal{D}_{2k}} z^{|\lambda_o| - |\lambda_e|} q^{|\lambda|} = \frac{z^k q^{k(2k-1)}}{(zq;q^2)(q^2;q^2)_k}.$$

We say that the *Durfee rectangle size* of a partition λ into odd parts is k, if $\lambda_k \geq 2k - 1$ and $\lambda_{k+1} \leq 2k - 1$.

> **Refinement 3:** The number of partitions λ of N into $2k - 1$ or $2k$ distinct parts with $|\lambda_o| - |\lambda_e| = j$ is equal to the number of partitions of N into j odd parts and Durfee rectangle size k.

5.3. **Interpretation of Anti-Lecture Hall Theorems.** We consider the limiting case for truncated anti-lecture hall compositions. Note that for fixed k, as $n \to \infty$, the set $A_{n,k}$ approaches P_k, the set of partitions into k nonnegative parts. From Theorem 4 we have

$$\sum_{\lambda \in A_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|} = \frac{\begin{bmatrix} n \\ \lfloor k/2\rfloor \end{bmatrix}_{xy}}{(x;xy)_{\lceil k/2\rceil}(x^{n-k+1}y^{n-k+2};xy)_{\lfloor k/2\rfloor}}.$$

Taking limits as $n \to \infty$ and substituting $x = zq$ and $y = q/z$, gives

$$\sum_{\lambda \in P_k} q^{|\lambda|} z^{|\lambda_o| - |\lambda_e|} = \frac{1}{(zq;q^2)_{\lceil k/2 \rceil} (q^2;q^2)_{\lfloor k/2 \rfloor}},$$

which is the well known "transpose theorem": The number of partitions λ of N into k nonnegative parts and $|\lambda_o| - |\lambda_e| = j$ is equal to the number of partitions of N with largest part less than or equal to k and j of the parts odd.

Similarly, consider the identity which combines Theorem 7, setting $u = v = 1$ with Theorem 4, setting $x = y = q$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1};q)_k}{(q^{2(n-k+1)};q)_k} = \frac{\begin{bmatrix} n \\ \lfloor k/2 \rfloor \end{bmatrix}_{q^2}}{(q;q^2)_{\lceil k/2 \rceil} (q^{2n-2k+3};q^2)_{\lfloor k/2 \rfloor}}. \tag{36}$$

Taking limits as $n \to \infty$ shows that (36) can be interpreted as a finite version of the transpose theorem.

6. Conclusion

We hopefully have demonstrated that basic hypergeometric series are a good tool to give proofs of various refined Lecture Hall Theorems. Furthermore, the Υ mapping plays a significant role in the development of 2-variable generating functions for anti-lecture hall compositions and truncated objects. Our study of the 2-variable generating function of the truncated objects leads to the following x, y-series identity. Using Equations (2) and (8), we have

Proposition 20.

$$\sum_{m=0}^{n} \left(x^{\lfloor m/2 \rfloor + 1} y^{\lfloor m/2 \rfloor} \right)^{\lceil m/2 \rceil} \frac{\begin{bmatrix} n - \lceil m/2 \rceil \\ \lfloor m/2 \rfloor \end{bmatrix}_{xy}}{(x;xy)_{\lceil m/2 \rceil} (x^n y^{n-1}; (xy)^{-1})_{\lfloor m/2 \rfloor}} = \prod_{i=1}^{n} \frac{1}{1 - x^i y^{i-1}}.$$

As a special case, substituting $x = a$ and $y = q/a$, we get the identity:

Proposition 21.

$$\sum_{m=0}^{n} \left(a q^{\lfloor m/2 \rfloor} \right)^{\lceil m/2 \rceil} \frac{\begin{bmatrix} n - \lceil m/2 \rceil \\ \lfloor m/2 \rfloor \end{bmatrix}_{q}}{(a;q)_{\lceil m/2 \rceil} (a q^{n-1}; q^{-1})_{\lfloor m/2 \rfloor}} = 1/(a;q)_n.$$

In future work we will consider truncated versions of the (k, l)-lecture hall partitions of [5].

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