Effective Development of Flexible Systems in Multidisciplinary Optimization

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As the complexity of multidisciplinary system design has substantially increased, so too has the need for incorporating tradeoffs into the design process. By incorporating such tradeoffs, the optimal system performance of each objective is sacrificed to increase the overall range of the system's functionality. Flexible systems have been defined as systems designed to maintain a high level of performance through real time change in their configuration when operating conditions or requirements change in a predictable or unpredictable way. When effectively applied, flexible systems have the ability to achieve the optimal performance for all system objectives, while also meeting the different constraints faced by each discipline. This method of designing effective flexible systems focuses on determining the manner in which the design variables of a system change, as well as investigating the stability of a flexible system through the application of a state-feedback controller. This method is then applied to a three discipline – structural, aerodynamics, and handling - study involving the design of a Formula One racecar traversing a pre-defined racetrack.

I. Introduction

In this paper, a methodology for the effective design of flexible systems is introduced. The motivation for this work is that in the field of multidisciplinary optimization, flexibility can play an important role in reducing the duration of the design process while also increasing the overall performance of the system by meeting multiple criteria. The definition of flexible systems, as applied in this research, describes systems that are designed to maintain a high level of performance by changing their configuration to meet multiple functional requirements or a change in operating conditions. Current design practices force designers, or disciplinary design teams, to incorporate tradeoffs into the system in an effort to resolve issues involving conflicting objectives. To obtain a functional design, it therefore may not be possible for each discipline to obtain an optimal performance for its design objective or system component.

An approach to the effective design of flexible systems is presented in this paper that first focuses on the optimization of the overall system for the expected changes in operating conditions. At each operating condition, or objective, the optimal values of the design variables used by each discipline will likely vary from that of the previous objective. Development of a flexible system requires the understanding of this dynamic. To accomplish this, the optimal trajectory of the design variable changes is determined for the system based upon the known conditions within the operating environment. After identifying the required changes in design variables, it is necessary to develop a controller that will allow for effective trajectory tracking. This controller will accomplish two tasks: ensuring proper behavior of the system within a changing environment and verifying that the changes in the design variables over time are appropriate for the system considered. For this paper, the controller solution was found based upon an LQR approach by generalizing the process for a linear regulator problem to a linear tracking problem. Simulations of the system are run to serve as the analytical feasibility assessment of the flexible system. The incorporation of flexibility into the design process is discussed in greater detail in the next section.

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II. Flexible System Design

As the complexity of system design has substantially increased, so too has the need for incorporating tradeoffs into the design process. For example, in a case study around the design of NASA’s Next Generation Space Telescope, the TRW Space and Electronics Group used multiobjective optimization to create a spacecraft to meet the mission specifications. Organizing the project into Integrated Product Teams, the design options were then explored, with the most promising alternatives chosen through a series of tradeoffs and analyses. If no design solutions were found, the requirements for the project were changed slightly until a design could be achieved. Introducing flexibility into such an environment would allow for greater interaction between the thermal, dynamic, and optimal models used. This enhanced interaction would allow for an optimal flexible design to be obtained, preventing the need to change the mission requirements or make crucial tradeoffs to produce a feasible design. Also, previous research work has noted that performance tradeoffs may be the result of inferior design.

Mayflower Corporation is currently testing a variable motion engine, in which the compression ratio and the stroke of the engine may be varied through alterations of the lever-arm pivot point. By modifying these two variables, compression ratio and the capacity, it is possible to optimize the engine to meet a particular running condition. By creating these optimal conditions within the cylinder, it is estimated that fuel consumption may be reduced by nearly 40% with a corresponding 50% reduction in emissions. Meanwhile, researchers at Xerox’s Palo Alto Research Center (PARC) are developing a new breed of robots that are flexible enough to adapt to their changing surroundings or applications.

Examining flight in nature, NASA is using the effortless flight of birds to provide inspiration for research involving a proposed smart wing that is capable of reconfiguring, or morphing, its shape while in flight. This morphing ability provides an unmanned aerial vehicle (UAV) the ability to gain altitude using lift from mountains or rising thermal currents, loiter for extended periods for landing site analysis, traverse long distances with minimal loss in altitude or energy, or provide a stable platform to photograph geological structures. Each of these tasks, however, requires a different airfoil shape or wing cross section for optimal performance. Elsewhere, a “smart” material has been utilized by the engineers at Penn State’s Center for Acoustics and Vibration to develop a high-torque rotary motor that can be configured into a wide range of formats. With a flat format similar to the thickness of a CD case, it is possible for the motor to reconfigure the curvature of the wing or fin surface of an airplane by driving changes in the camber of the aircraft wings or fins. This would essentially allow for a “shape-shifting” aircraft that could meet a wide range of different operating environments and mission parameters. Developing these added abilities for commercial and military aircraft, benefits would include increased fuel efficiency, improved ride quality, better maneuverability, increased safety, lower landing speeds, adaptation to shorter runways, and extensive versatility. The development of such ideas is in response to an ever increasing complexity found in engineering design; therefore the emergence of concepts associated with flexible systems within today’s industrial mindset is not a surprise.

As defined in Ref. 1, Fig. 1 depicts the general hierarchy for the design of flexible systems. In this organization, flexible systems are actually a subset of open systems; systems that are capable of indefinite change, growth, and development over time.

![Figure 1. Hierarchy of Flexible Systems Design](image)

To best describe the methods through which flexibility can be obtained, flexible systems are defined as:

*Flexible systems – systems designed to maintain a high level of performance through real time adaptations in their configuration and/or through robust parameter settings when operating conditions or requirements change in a predictable or unpredictable way.*

American Institute of Aeronautics and Astronautics
While flexibility can be obtained through robustness, for the purposes of this paper, the focus will be on the implementation of adaptability into system designs.

Adaptability – Mode of achieving flexible systems where systems parameters (design variables) that can be changed and their range of change are identified to enhance performance of the system in predictable changes in the operating environment; they can be changed when the system is not in use (passive) or in real time (active).

Flexible systems are more apt to use active adaptability to enhance the performance of the system while it is in use. For the purposes of the research presented in this paper, active adaptability will be used in flexible system design.

Typically, each of the various requirements that a system must meet is represented in the form of an objective function. Through the incorporation of flexible systems, it becomes possible to design a system that can satisfy the optimality conditions for multiple objective functions. The Pareto frontier will provide the designer with information regarding the maximum performance of the system. As it is an assumption that the Pareto frontier can be determined for the investigated problem, it is therefore sound to assume that the corresponding points in the design space are also determined. This correlation is presented in Fig. 2, where the Pareto set in performance space is mapped to the design space. Therefore, by knowing the Pareto set, the extreme points both in the performance space and in the design space will be known. These extreme points define the configurations of the flexible system under consideration and represent the target level of flexibility.

Ideally, it is the goal of the designer to achieve the extreme points (dictated by A and B) of the Pareto frontier in the design space. For a static design problem, the designer is faced with the challenge of choosing a single Pareto optimal point or the final design; however incorporating flexibility eliminates this problem. Introducing adaptability, design variables are allowed to physically change their values and therefore the configuration of the system. However, this process of changing from one extreme point to another is not instantaneous. The process of a design variable adapting itself in discrete increments requires the designer to determine a path, or trajectory, of change for the system.

As the manner in which the system changes has direct implications on performance and the required amount of time for each modification to be completed, the path selected when changing configurations becomes inherently important. While the most direct, and obvious, path is a straight vector between the two points, there is the possibility of violating a system constraint. However, as seen in Fig. 3, changing in accordance with the path dictated by the Pareto frontier does not automatically result in the optimal trajectory of change.

The problem in Fig. 3 represents a purely mathematical multiobjective optimization problem. This example problem demonstrates the potential pitfall of following the trajectory dictated by the Pareto frontier, shown as path 1. Because both \( x_1 \) and \( x_2 \) oscillate along this path, simply moving along the frontier is unsatisfactory when designing a flexible system. Such a result is not only undesirable for implementing adaptability to the system, but also for the time that it will take the system to change configurations. Also, the development of a controller for such a trajectory would be complex as it is difficult to reduce error, and the physical cost of such a controller could make the incorporation of adaptability infeasible. However, path 2 was created taking the system constraints into consideration, while trying to reduce the oscillations in the trajectory. What is noticed is that while the dramatic changes in the design variables have been eliminated, resulting in a much more controllable adaptation, there is a slight performance decrease when compared to that of the Pareto frontier.
Therefore, when determining the optimal trajectory for the design variables when changing the performance of the system to meet different conditions, the issue of performance versus ease of control needs to be taken into account. The creation of the optimal trajectory is inherently important, as this information is used to develop the controller for the flexible system. The development of the control scheme is discussed in the next section.

III. Construction of a Control Law for Flexible Systems

Identification of the required changes in the design variables needed to meet the multiple system requirements leads to the development of a control scheme to allow for proper trajectory tracking of the design variables, and the verification of feasibility of the flexible system. A solution to the controller design problem is based upon a Linear Quadratic Regulator (LQR) by generalizing the process for a linear regulator problem to a linear tracking problem. For this type of problem, it is necessary to design a feedback control law, as described here from Ref. 10, that will cause the plant state to follow a given reference trajectory \( r(t) \). Linearizing the equations of motion for the system studied, the state equations are given as:

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t)
\]  

(1)

For this problem, the plant is desired to follow the reference trajectory \( r(t) \), such that the performance measure of the system is minimized to give a positive measure of the history of the final value of the tracking error in the form of:

\[
J = \frac{1}{2} \left[ x(t_f) - r(t_f) \right] ^T S \left[ x(t_f) - r(t_f) \right] + \frac{1}{2} \int_{t_0}^{t_f} \left[ \dot{x}(t) - r(t) \right] ^T Q(t) \left[ \dot{x}(t) - r(t) \right] + u(t) ^T R(t) u(t) dt
\]  

(2)

where \( S \) and \( Q \) are real symmetric positive semi-definite matrices, and \( R \) is a real symmetric positive definite matrix. The \( S \) matrix applies a cost to the final state deviation, \( Q \) applies a cost to the state deviation over the history of time, and \( R \) applies a cost to the magnitude of the input size.

For these matrices, the designer would ideally want to make the elements of these matrices as small as possible, while maintaining an acceptable level of deviation error in the changes of the design variables. Restricting the magnitude of the input allowed serves as a limiting control authority for the solution to the system. Managing the magnitude of each input allows for the designer to essentially dictate how great of a change each state variable is able to undergo for each discrete time step. For each of the three matrices – \( S \), \( Q \), and \( R \) – the designer selects the weighting value for the problem. Using this information, the cost function in Eq. 2 can be determined. For such a development, the final time \( t_f \) is fixed, and the final state \( x(t_f) \) is free.

From this derivation, and with \( \lambda \) as the vector of Lagrange multipliers, the Hamiltonian is determined to be:

\[
H = \frac{1}{2} \left[ x(t) - r(t) \right] ^T Q(t) \left[ x(t) - r(t) \right] + \frac{1}{2} u(t) ^T R(t) u(t) + \lambda ^T (t) \{ A(t)x(t) + B(t)u(t) \}
\]  

(3)

Based upon the necessary and sufficient conditions of optimality, the following equations are obtained:

\[
\dot{x}^*(t) = \frac{\partial H}{\partial x} = A(t)x^*(t) + B(t)u^*(t)
\]  

(4)
\[
\dot{x}^*(t) = \frac{\partial H}{\partial x} = -Q(t)x^*(t) - A^T(t)\dot{\lambda}^*(t) + Q(t)r(t)
\]
(5)
\[
0 = \frac{\partial H}{\partial u} = R(t)u^*(t) + B^T(t)\dot{\lambda}^*(t)
\]
(6)
Also, since \(\delta t = 0\), the following condition also holds true:
\[
\dot{\lambda}^*(t_f) = Sx^*(t_f) - Sr(t_f)
\]
(7)
From Eq. 6, it is possible to solve for the optimal control in terms of the costate to obtain:
\[
u^*(t) = -R^{-1}(t)B^T(t)\dot{\lambda}^*(t)
\]
(8)
Substituting Eq. 8 into Eq. 4 yields the coupled state and costate equations:
\[
\begin{bmatrix}
\dot{x}^*(t) \\
\dot{\lambda}^*(t)
\end{bmatrix} =
\begin{bmatrix}
A(t) & -B(t)R^{-1}(t)B^T(t) \\
-Q(t) & -A^T(t)
\end{bmatrix}
\begin{bmatrix}
x^*(t) \\
\dot{\lambda}^*(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
[Q(t)r(t)]
\end{bmatrix}
\]
(9)
whose general solution is given by:
\[
\begin{bmatrix}
x^*(t_f) \\
\dot{\lambda}^*(t_f)
\end{bmatrix} = \Phi(t_f,t_f)
\begin{bmatrix}
x^*(t) \\
\dot{\lambda}^*(t)
\end{bmatrix} +
\int_{t_f}^{t}
\Phi(t_f,\tau)
\begin{bmatrix}
0 \\
[Q(\tau)r(\tau)]
\end{bmatrix}
\]d\(\tau\)
(10)
where \(\Phi\) is the transition matrix of the state/costate system found in Eq. 9. The solution of the optimal control law provides the input necessary, as shown in Eq. 1.

The development of the optimal control law provides the designer with information regarding the feasibility and practicality of implementing the flexible system design for the reference trajectory created. For situations where the cost matrices are required to be large numbers to reduce the error of the tracking to an acceptable level, it may be necessary to reduce the back-off the requirements of the flexible system. This slight performance loss will be compensated by the development of a practical flexible system design. To demonstrate the design and application of flexibility to a system, a case study is introduced regarding the design of a Formula One racecar traversing a straightaway.

IV. Case Study – Design of a Formula One Racecar

In the design of a racecar, the difference between winning a race and not winning comes down to the ability of a driver to get the most out of his or her vehicle. The core vehicle design is aimed at an optimal compromise that allows the driver to repeatedly turn fast lap times at a particular race track. Simulations are now used not only to tune the vehicle on the weekend of the race, but also in the design phase where parameters that are not adjustable must be set and optimized. The basic configuration of the car, such as the center of gravity, suspension systems, and roll stiffness, remain constant. A particular vehicle design will be optimal at a certain speed and radius of the curvature. However, Formula One racetracks do not have constant radius corners and do not consist of only a few turns, yet the design team must choose a single vehicle configuration come race day.

Now, consider a flexible racecar design that is able to optimize its performance as a function of the current track conditions – ignoring current racing restrictions. Whether on a straightaway, a large turn, or a hairpin turn, the car could adjust variables such as the center of gravity, roll stiffness, and aerodynamic downforce (via wings and airfoils). The objective of this problem is to design a flexible racecar that has the ability to adapt itself in order to minimize the time along a portion of a racetrack consisting of two corners of different radii and a connecting straightaway. To describe the motion along such a course, the vehicle enters the straightaway at the maximum steady-state cornering velocity from the previous corner. It then accelerates as hard as possible until, at the last possible instant, it brakes as hard as possible so that at the end of the straightaway the vehicle’s velocity is equal to the maximum steady-state cornering velocity of the upcoming corner. Figure 4 depicts an example of the racetrack segment used in this case study.
A. Model Development

The amount of detail which can be modeled in a computer-based simulation of an automobile is almost limitless. While many of the details are unnecessary in the preliminary design stage, it is necessary to correctly model the basics of the vehicle design. A brief explanation of the racecar model is presented in Ref. 11. The design variables chosen for this model represent three potential disciplines working on the vehicle. The variables corresponding to the longitudinal center-of-gravity location, roll stiffness distribution, and the aerodynamic downforce distribution, are also the primary parameters that affect a vehicle’s performance.\(^{12}\) Figure 5 illustrates a basic representation of the vehicle model under consideration.

The fore/aft distance of the vehicle’s center of gravity behind the front axle, divided by the vehicle’s wheelbase, represents the weight distribution. Aerodynamic downforce distribution is the division of individual aerodynamic downforce acting at the front and rear axles. This force is created by the overall vehicle shape and the inverted airfoils. Roll stiffness distribution signifies the amount of resistance to vehicle roll the front axle provides relative to the total resistance provided by the front and rear tires. Each of these design variables is normalized between 0 and 1, as follows in Eq. 11-13.

\[
a' = \frac{a}{l} = \frac{a}{a+b}
\]

\[
K' = \frac{K_F}{K_F + K_R}
\]

\[
C' = \frac{C_{iF}}{C_{iF} + C_{iR}}
\]

The detail present in this model is sufficient enough for the analysis of the basic design concept for the optimization of the vehicle. Analysis of the vehicle design when in a turn of given radius is solely done in the condition of steady-state cornering. Using iterative solution techniques, the constant velocity at peak cornering – represented by maximum lateral acceleration – on a skidpad is found. The performance of this vehicle model on variable radius skidpads has been extensively studied in Ref. 13, and a more detailed look at the construction of the vehicle can be found in Ref. 14. The analysis behind the simulation of a race car on a straightaway between corners of two different radii is developed in Ref. 15. The performance on the straightaway is composed of an accelerating and braking phase, as seen in Fig. 6, to match the restrictions placed upon the system by the radius of the two corners. Therefore, one of the main obstacles of this analysis is the uncertainty regarding the location of the transition point between the acceleration and braking phases, as it is a function of the curve radii, the length of the straightaway, and the parameters of the vehicle. In this figure, \(V_f\) corresponds to the maximum steady-state velocity.
of the first radius, $V_2$ the maximum steady-state velocity of the second radius, and $V_T$ the terminal velocity of the vehicle along the straightaway.

B. Problem Development and Solution

The multiobjective problem developed by this analysis is defined as the minimization of the time it takes to traverse each section of the proposed racetrack, subject to the constraints placed on the initial and final values of the vehicle’s velocity as determined by the maximum attainable steady-state cornering speed for a given configuration. This optimization problem can be written in standard form as:

\[
\text{Minimize:} \\
F_1 = \text{Time to Traverse Radius } R_1 \text{ in seconds} \quad (14) \\
F_2 = \text{Time to Traverse a Straightaway of Length } L, \text{ in seconds} \quad (15) \\
F_3 = \text{Time to Traverse Radius } R_2, \text{ in seconds} \quad (16)
\]

\text{Subject To:}

The initial velocity of the straightaway, $V_1$, as dictated by the maximum steady-state cornering velocity of turn 1 \quad (17)

The final velocity of the straightaway, $V_2$, as dictated by the maximum steady-state cornering velocity of turn 2 \quad (18)

\[0.45 \leq a' \leq 0.65 \quad \text{(Normalized Center of Gravity Position)} \quad (19)\]

\[0.4 \leq K' \leq 0.7 \quad \text{(Normalized Roll Stiffness)} \quad (20)\]

\[0.25 \leq C' \leq 0.45 \quad \text{(Normalized Lift Coefficient)} \quad (21)\]

While the normalized ranges of the three design variables initially had an upper and lower bound of one and zero, respectively, the ranges used in Eqs. 19-21 are selected to reflect realistic results. For this problem, the first turn in the proposed racetrack has a radius of 140 feet. This radius, from which $V_1$ is found, merges into a 1250 foot straightaway. The vehicle then traverses the straightaway, turning into the second corner with a 400 foot radius. Figure 7 represents the three-objective performance space view of the Pareto frontier. From the results of the Pareto frontier, the optimal configuration for each objective can also be identified. The values of these points are listed in Table 1. This problem was solved using a grid search over the established bounds of the design variables, given the ease of integrating the iterative approach required to analyze a single evaluation of each evaluation.

<table>
<thead>
<tr>
<th>Endpoint</th>
<th>$a'$</th>
<th>$K'$</th>
<th>$C'$</th>
<th>$F_1$ (sec)</th>
<th>$F_2$ (sec)</th>
<th>$F_3$ (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A – 140 ft. Radius</td>
<td>0.65</td>
<td>0.63</td>
<td>0.32</td>
<td>11.156</td>
<td>7.746</td>
<td>14.627</td>
</tr>
<tr>
<td>B – 1250 ft. Straightaway</td>
<td>0.45</td>
<td>0.40</td>
<td>0.45</td>
<td>11.428</td>
<td>6.514</td>
<td>15.628</td>
</tr>
<tr>
<td>C – 400 ft. Radius</td>
<td>0.65</td>
<td>0.48</td>
<td>0.31</td>
<td>11.175</td>
<td>7.801</td>
<td>14.531</td>
</tr>
</tbody>
</table>
C. Development of Control Law

In order to incorporate flexibility into the system to account for the different operating objectives, two scenarios must be investigated. The first scenario involves changing the vehicle from the optimal configuration of the first radius (point A) to the optimal configuration of the straightaway (point B). The second scenario involves changing the configuration of the vehicle from the optimal configuration of the straightaway (point B) to that of the second, upcoming corner (point C). As discussed previously, a design variable adapts itself in discrete increments to attain the values of the Pareto extreme points. To obtain an optimal configuration during the period that the design variables are changing, the path of change would be that of the Pareto frontier. By moving along the Pareto frontier, the system retains a degree of optimality, in that it never takes on the configuration of a dominate design. However, the path dictated by the Pareto frontier may not be the most efficient or effective solution to the incorporating flexibility, and may therefore be potentially eliminated as a desired solution.

By determining alternative paths in the design space and then mapping them accordingly to the performance space, it is possible to develop a more effective and efficient flexible system. The most obvious deviation from the Pareto frontier path is a linear vector between the two Pareto extreme points in the design space. However, it should taking such a route could result in the violation of system constraints, thereby eliminating this path as a possible solution. Therefore, a path must be constructed in the design space that takes into account both efficiency of the adaptation and the constraints of the overall system. To determine the characteristics of these alternative paths, the following sub-optimization problem is presented for the design of flexible systems.

Minimize:

\[ F_j (\mathbf{x}) + \sum_{i=1}^{n} (x_i^q - x_i^B)^2 \]

Subject To:

\[ (x_i^q - \Delta x_i) \leq x_i \leq (x_i^q + \Delta x_i) \text{ for } i=1,n \]

\[ l_i \leq x_i \leq u_i \text{ for } i=1,n \]

\[ g_k \leq 0 \text{ for } k=1,m \]

This sub-optimization problem essentially places a hyperbox around the current design point in the design space, and search for the location within this box that reduces the objective function selected. Equation 23 establishes the boundary of the hyperbox for each instance of the sub-problem. The solution to this sub-problem is then repeated after updating the current design point in the design space until reaching the final system configuration. By
minimizing the values of the selected objective function for all iterations within the fabricated hyperbox, it is possible to develop an efficient path of change that closely approximates the Pareto frontier. These five cases are identified in Table 2, and the plots for each path are shown in the design space for the two different configuration changes. Cases 1 and 2 come from the Pareto frontier and the linear vector connecting the two Pareto extreme points. Cases 3-5 were created using the optimization sub-problem defined by Eqs. 22-25 with different boundaries used to create the hyperbox, resulting in different paths through the design space as seen in Fig. 8.

<table>
<thead>
<tr>
<th>Path Number</th>
<th>Path Identification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>Pareto Path</td>
</tr>
<tr>
<td>Case 2</td>
<td>Linear Path</td>
</tr>
<tr>
<td>Case 3</td>
<td>Sub-Optimization Case 1</td>
</tr>
<tr>
<td>Case 4</td>
<td>Sub-Optimization Case 2</td>
</tr>
<tr>
<td>Case 5</td>
<td>Sub-Optimization Case 3</td>
</tr>
</tbody>
</table>

Having developed the possible paths of change for the system, it is necessary to ensure that the proposed paths are both feasible and practical. The states for the system are defined as the position and velocity of the vehicle, and the value of the three design variables. In developing the state-space model, the changes in configuration occur only when the vehicle is on the straightaway, and there is no modeling of the “transition” from steady-state cornering to the straightaway, or visa-versa. This allows for the elimination of lateral forces from the state-space model, simplifying the model while not detracting from the effectiveness of the analysis.

<table>
<thead>
<tr>
<th>States</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_F</td>
<td>x Position of Vehicle in World Frame</td>
</tr>
<tr>
<td>u</td>
<td>Longitudinal Velocity</td>
</tr>
<tr>
<td>a'</td>
<td>Normalized Center of Gravity Position</td>
</tr>
<tr>
<td>K'</td>
<td>Normalized Roll Stiffness Distribution</td>
</tr>
<tr>
<td>C'</td>
<td>Normalized Aerodynamic Downforce Distribution</td>
</tr>
</tbody>
</table>

To determine the equation of motion for the vehicle in terms of the state variables, it is necessary to first study the forces acting on the system. The first force is a longitudinal weight transfer that is given by Eq. 26.

\[
\Delta F = \frac{hmA_f}{\ell}
\]  

(26)

The forward and rear aerodynamic forces on the vehicle are represented as:

\[
F_{aero-x} = \frac{\rho C_a A u^2}{2 C_{L_{0,\text{tot}}}}
\]  

(27)
\[ F_{z \text{aero}} = -\frac{\rho (C_{l_{\text{total}}} - \frac{C'}{C_{l_{\text{total}}}}) A u^2}{2} \]  

(28)

where: \( C_{l_{\text{TOTAL}}} \) is the summation of the front and rear lift coefficients
\( \rho \) is the atmospheric density
\( A \) is the frontal reference area of the vehicle

The drag force experienced by the vehicle is written as:
\[ F_{\text{aero}} = \left( \frac{\text{abs} \left( \frac{C'}{C_{l_{\text{total}}}} \right)}{6} \right) + \left( \frac{\text{abs} \left( C_{l_{\text{total}}} - \frac{C'}{C_{l_{\text{total}}}} \right)}{3} \right) u^2 \]  

(29)

Since the vehicle is traveling down a straightaway in one dimension, the wheel loads of the vehicle are left-right symmetric. Eqs. 30 and 31 represent the front and rear wheel loads experienced by the vehicle.

\[ F_{z_{\text{fl}}} = F_{z_{\text{al}}} = (1 - a') \frac{m}{2} \left( \frac{h m A}{\ell} + \frac{1}{2} \left( \frac{\rho C' A u^2}{2C_{l_{\text{total}}}} \right) \right) \]  

(30)

\[ F_{z_{\text{rr}}} = F_{z_{\text{ar}}} = a' \left( \frac{m}{2} \right) \left( \frac{h m A}{\ell} + \frac{1}{2} \left( \frac{\rho (C_{l_{\text{total}}} - \frac{C'}{C_{l_{\text{total}}}}) A u^2}{2} \right) \right) \]  

(31)

By varying the center of gravity by breaking the vehicle up into a chassis mass and an “upper body mass”, it becomes possible to maneuver a portion of the vehicle’s mass in order to change the center of gravity. To do this, a new state variable, \( X \), is introduced, which represents the location of the upper mass with respect to the front of vehicle. For a standard racecar, a nominal value of \( a' \) is 0.59524. For this flexible racecar, we will consider the chassis to represent 75% of the total mass of the vehicle. The upper body (the moveable portion of the racecar) will consist of the remaining 25% of the vehicle mass. Using these values, \( a' \) can be expressed in terms of \( X \) by Eq. 32.

\[ a' = 0.75(\text{cg of Chassis}) + 0.25(\text{cg of Upper Mass}) \]  

(32)

Assuming negligible friction between the upper mass and the chassis mass, the equation of motion for the upper mass can be represented by the standard, second-order differential equation:

\[ m_U \ddot{X}(t) + c_U \dot{X}(t) + k_U X(t) = F_X(t) \]  

(33)

The new listing of states used in the state-space model is shown in Table 4.

<table>
<thead>
<tr>
<th>States</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_F</td>
<td>x Position of Vehicle in World Frame</td>
</tr>
<tr>
<td>u</td>
<td>Longitudinal Velocity</td>
</tr>
<tr>
<td>K'</td>
<td>Normalized Roll Stiffness Distribution</td>
</tr>
<tr>
<td>C'</td>
<td>Normalized Aerodynamic Downforce Distribution</td>
</tr>
<tr>
<td>X</td>
<td>Position of the Upper Mass</td>
</tr>
<tr>
<td>\dot{X}</td>
<td>Velocity of the Upper Mass</td>
</tr>
</tbody>
</table>

Table 4. Final State Listing for the State-Space Model

Modeling of the equations of motion for the roll stiffness distribution and the aerodynamic downforce distribution is completed using a first-order differential equation. This representation is shown in Eqs. 34 and 35.

\[ \dot{C'}(t) + \frac{1}{\tau_C} C'(t) = F_C(t) \]  

(34)

\[ \dot{K'}(t) + \frac{1}{\tau_K} K'(t) = F_K(t) \]  

(35)
However, the resulting acceleration equation is nonlinear in nature, and needs to be linearized with respect to each state variable and input, such that it can be used in the state-space form, as shown in Eq. 36.

\[
\begin{pmatrix}
\dot{x}_r \\
\dot{u}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\tau_c} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_r \\
u
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_r \\
u
\end{pmatrix} + \begin{pmatrix}
A_x \\
F_x
\end{pmatrix}
\]

where the values of the partial derivatives are evaluated at each time step from the values of the reference trajectory to update the A and B state-space matrices.

**D. Results Analysis and Race Simulation**

For this problem, the results of the five flexible systems are compared to the results of three static racecars. The design of the static racecars was done in such a way that the vehicles are optimal for one of the three objectives studied in this problem. For example, Car A is designed such that its configuration is optimal for the first radius. However, since the vehicle is statically designed, it must also traverse the straightaway and the second radius without changing configurations. Because of this, the vehicle will not experience optimal performance for these two objectives. Cars D-H represent the racecars that are designed to be flexible systems. For these vehicles, the incorporation of flexibility allows for the attainment of the optimal configuration for the two corners. As seen in Table 5, the flexible vehicles each complete the two corners of different radii in the minimum possible time required.

The result of the optimal control law solved in the previous section determines the path of change for the design variables for a flexible system, allowing for the simulation of the vehicle on the straightaway. Beginning at the optimal configuration of the first radius, the vehicle changes to the optimal configuration of the straightaway, travels down the straightaway at this configuration for a variable amount of time, and then changes again to the configuration needed to be optimal for the second radius. From the subset of flexible racecars, the change of the design variables for Car D is dictated by the information from the Pareto frontier. Similarly, the change in the design variables for Car E comes from the linear vector connecting the two optimal configurations under consideration. Cars F-H represent flexible systems where the manner in which the design variables change is dictated by the results of the sub-optimization problem. For these cases, the solution to each iteration of the sub-optimization problem is a function of the boundaries placed upon the hyperbox. As each of these flexible systems differs in how their design variables change along the straightaway, they potentially have a different configuration at a given time. As the vehicle is modeled at steady-state cornering in this model, the flexible systems only change configuration when on the straightaway. Based upon this fact, different paths for the design variables lead to different times required to traverse the straightaway. As the flexible systems are at an optimal configuration for the two corners in this problem, the major difference in overall performance comes from the different time needed to complete the straightaway. The simulated times for the three objects in this problem are shown in Table 5.

<table>
<thead>
<tr>
<th>Car</th>
<th>Configuration</th>
<th>F1 (sec)</th>
<th>F2 (sec)</th>
<th>F3 (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car A</td>
<td>Optimal Configuration for Radius 1</td>
<td>11.156</td>
<td>7.746</td>
<td>14.627</td>
</tr>
<tr>
<td>Car B</td>
<td>Optimal Configuration for Radius 2</td>
<td>11.175</td>
<td>7.801</td>
<td>14.531</td>
</tr>
<tr>
<td>Car C</td>
<td>Optimal Configuration for Straightaway</td>
<td>11.428</td>
<td>6.514</td>
<td>15.628</td>
</tr>
<tr>
<td>Car D</td>
<td>Flexible System – Pareto Frontier</td>
<td>11.156</td>
<td>6.753</td>
<td>14.531</td>
</tr>
<tr>
<td>Car E</td>
<td>Flexible System – Linear Path</td>
<td>11.156</td>
<td>6.678</td>
<td>14.531</td>
</tr>
<tr>
<td>Car F</td>
<td>Flexible System – Sub-Optimization Case 1</td>
<td>11.156</td>
<td>6.546</td>
<td>14.531</td>
</tr>
<tr>
<td>Car G</td>
<td>Flexible System – Sub-Optimization Case 2</td>
<td>11.156</td>
<td>6.642</td>
<td>14.531</td>
</tr>
<tr>
<td>Car H</td>
<td>Flexible System – Sub-Optimization Case 3</td>
<td>11.156</td>
<td>7.503</td>
<td>14.531</td>
</tr>
</tbody>
</table>

Simulation of an entire race was completed using a racetrack of 10 times around a skidpad of 140 foot radius, 20 times around a skidpad of 400 foot radius, and 30 times down a straightaway of length 1250 feet. The results of this race are shown in Fig. 10. From this figure, it can be seen that all but one of the racecars designed to incorporate
flexibility would finish faster than the three static configurations tested. Also, Car F and Car G, provide the best performance of all the vehicles analyzed in this study. This result, in that Cars D and E are slower, demonstrates that the paths dictated by the Pareto frontier and the linear vector through the design space do not always provide the best flexible system performance. However, flexible systems in general do perform noticeably better than their static counterparts.

V. Conclusions

This paper presents a procedure for incorporating flexible systems into a multiobjective optimization problem, presenting design teams with the ability to eliminate the need for making tradeoffs between objectives. By making the design variables of a system adaptable, it is possible to change the configuration of a system, thereby altering the system’s performance. In this work, three different ways for determining the path of change for the design variables are introduced. The first method involves following the path dictated by the Pareto frontier solution to the multiobjective problem, while the second method uses the information from the linear vector in the design space that connects the two optimal configurations being investigated. However, the linear path must be investigated to ensure that the path of change does not violate any system constraints placed on the problem. The third method applies an iterative optimization sub-problem that contains the original system constraints. For this sub-problem, a hyperbox is created in the design space whose size is dictated by the boundary constraints selected by the designer. The space within this hyperbox is searched to locate the minimum value of the sub-problem’s objective function. The solution to this problem is the next point in the path of change for the system, and the next iteration begins. Next, the stability of the system when changing configurations according to each proposed path is analyzed through the application of a linear-tracking problem. The optimal control law developed by this analysis serves to help the design teams ensure that the proposed flexible systems are both stable and practical.

Applying this approach to the design of a Formula One racecar demonstrates the potential benefits of designing flexible systems. For a proposed racetrack – consisting of two corners and a straightaway – the multiobjective problem was solved to identify the optimal configuration for each of the three objectives. Developing the different potential paths of change for the design variables was completed using the three different approaches. The equations of motion of the vehicle traveling along the straightaway were then used in the stability analysis of the flexible system. As the vehicle’s operation in the corners was modeled under steady-state cornering, the changes in configuration of the vehicle occur only when the vehicle is traveling along the straightaway. When compared to their static counterparts, almost all of the flexible vehicles were successful in attaining an increased overall performance. Evaluating the different flexible vehicles, it was seen that the changes in configuration developed under the optimization sub-problem actually outperformed the cases where the paths were dictated by either the Pareto frontier or the linear vector. This result demonstrates that the most effective flexible system designs do not necessarily come from the paths determined by the Pareto frontier or the linear vector. The significantly increased
performance of the flexible systems over their static counterparts provides proof towards the argument that flexible system design could become a major area of interest for further multidisciplinary problems.

A source of future work involves examining cases where it may be impossible, or too great of an engineering requirement, to make a design variable adaptable. In these cases, an alternative approach is needed. By designing for robustness, a design variable will remain constant but will be chosen at a value that would be optimal for a wide range of operating conditions or objectives. Further study of the relationships and interactions when applying adaptability and robustness to a system is essential for the future development of more complex flexible systems. There may also be cases where it may be impossible, or implausible to achieve the optimal performance. Also, the work presented in this thesis is theoretical in nature. Through computer simulations, the practicality and benefits of flexible systems can be determined, but the need exists to actually construct a flexible system for testing.

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References


